

DEFINE

$$I = (D \text{ IN THE PAPER}) = \left(\pi - \frac{1+\epsilon}{e}, \pi + \frac{1+\epsilon}{e} \right)$$

COMPUTE:

$$L = \frac{1}{1+\epsilon} \sup_{x,y \in I} \left| \frac{x+y}{2!} - \frac{x^3+x^2y+xy^2+y^3}{4!} + \frac{x^5+x^4y+\dots+y^5}{6!} - \dots \right|$$

$$L_0 = \frac{1}{1+\epsilon} \sup_{x \in I} \left| \frac{x+\pi}{2!} - \frac{x^3+x^2\pi+x\pi^2+\pi^3}{4!} + \frac{x^5+x^4\pi+\dots+\pi^5}{6!} - \dots \right|$$

THEN MUST HAVE
 $L_0 \leq L$.

MOREOVER, IF
 $L \leq \gamma$

OR

$$L_0 \leq \gamma.$$

THEN, THE CONDITIONS "h" IN THE ATTACHED PAPER WILL DO BETTER THAN .24708. THAT IS BETTER THAN THE BEST CONDITIONS GIVEN IN YOUR PAPER. CARRY THE COMPUTATIONS AND REPORT ON THESE FINDINGS IN THE REVISION OF YOUR PAPER.

Semilocal Convergence Newton Method Applied to Kepler Equation: New Results

Abstract

The idea of this paper is to consider as a starting point the value $E_0 = \pi$, also proposed by Charles y Tatum [Charles (1998)], in the theorems of semilocal convergence of Kantorovich, Gutiérrez, α -teoría of Smale and the α -teoría of Wang-Zhao, to compare the convergence conditions obtained. Once set E_0 , They should calculate the parameters listed in the statement of these theorem. So, we will generalize the study in [Dilone-Gutierrez (2011)] for the case $E_0 = M$. The choice of starting point for the Newton Method as $E_0 = M$. It is quite common in the mathematical literature, but not the only. For example, in Avendano [Avendano (2014)], It is a collection of classic starting points.

In the Table 1 we do a comparison of the increasing functions of the Kantorovich, Gutiérrez, Theorem, α -teoría of Smale and the α -teoría of Wang-Zhao, applied to the Kepler equation, depending on the points $E_0 = M$ y $E_0 = \pi$, and their respective parameters δ . In the Table 2 the convergence conditions of the theorems mentioned above are compared depending of the starting points, of the parameters δ and the maximum value of e theorems for which guarantee convergence of the Newton Method to a solution of the Kepler equation. Numerics calculations were obtained by applying Mathematica V10.

Keywords: Newton method, semilocal convergence theorem, Kepler equation

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

1 Introduction

In this article we analyze the behavior of the famous Kepler's equation,

$$f(E) = E - e \sin E - M, \quad (1.1)$$

where eccentricity $e \in (0, 1)$ and the mean anomaly $M \in [0, \pi]$ are known parameters. We have selected this equation as a function test in the analysis of various semilocal convergence results for the Newton-Raphson method, which when is applied to (1.3) gives the sequence

$$E_{n+1} = E_n - \frac{f(E_n)}{f'(E_n)}, \quad n \geq 0. \quad (1.2)$$

The idea of this paper is to consider as a starting point the value $E_0 = \pi$, also proposed by Charles and Tatum [Charles (1998)]. So we prove different semilocal convergence theorems for Newton's method. In particular, we consider the results given by Kantorovich [Kantorovich (1948)], Gutiérrez [Gutierrez (1995)], α -theory of Smale [Smale (1986)] an α -theory of Wang-Zhao [Wang (1997)]. In addition, we obtain new results of existence and uniqueness of solution for the equation (1.3). In this way, we generalize the study in [Dilone-Gutierrez (2011)] for the case $E_0 = M$.

We present now the semilocal convergence theorems that we use in this paper. Although they can be stated in a Banach space setting, we show here the version for real valued functions. The details of the demonstration of the previous theorems can be found in [Dilone (2009)].

In this article we analyze the behavior of the famous Kepler's equation,

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$$E_{n+1} = E_n - \frac{f(E_n)}{f'(E_n)}, \quad n \geq 0. \quad (1.4)$$

The choice of starting point for the Newton method as $E_0 = M$ is quite common in the mathematical literature, but it is not the only one. For example, in Avendano [Avendano (2014)], we can find a collection of different starting points for Newton's method applied to Kepler's equation. The idea of this paper is to consider as a starting point the value $E_0 = \pi$, also proposed by Charles and Tatum [Charles (1998)]. So we prove different semilocal convergence theorems for Newton's method. In particular, we consider the results given by Kantorovich [Kantorovich (1948)], Gutiérrez [Gutierrez (1995)], α -theory of Smale [Smale (1986)] an α -theory of Wang-Zhao [Wang (1997)]. In addition, we obtain new results of existence and uniqueness of solution for the equation (1.3). In this way, we generalize the study in Dilone-Gutierrez (2011) for the case $E_0 = M$.

The main idea of the paper is to construct the corresponding majoring function for the mentioned theories for the case $E_0 = \pi$ and then to give conditions on the eccentricity e that guarantee that these majoring functions have real roots. In this way, we establish conditions for the solution for Kepler's equation in terms of eccentricity e .

We present now the semilocal convergence theorems that we use in this paper. Although they can be stated in a Banach space setting, we show here the version for real valued functions. The details of the demonstration of the previous theorems can be found in Dilone [Dilone (2009)].

Theorem 1.1 (Kantorovich Theorem's). *Let us consider $f : I \rightarrow \mathbb{R}$, where I is an open interval in $\mathbb{R}$, a differentiable function in I . Let us assume, without loss of generality that $f(x_0) \neq 0$ and*

- i) $x_0 \in I$.
- ii) $f'(x_0) \neq 0$.
- iii) $\left| \frac{f(x_0)}{f'(x_0)} \right| \leq \beta$.
- iv) $\left| \frac{f'(x) - f'(y)}{f'(x_0)} \right| \leq \gamma |x - y|, \quad \forall x, y \in I$.
- v) $h = \gamma\beta \leq 1/2$.

vi) $t^* = \frac{1 - \sqrt{1 - 2h}}{2\beta}$ it is such that $(x_0 - t^*, x_0 + t^*) \subset I$.

Then, the Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0, \quad (1.5)$$

is well defined and $\{x_n\}_{n=0}^{\infty}$ converges a $x^* \in I$, where $f(x^*) = 0$. In addition, this solution is located in the range

$$[x_0 - t^*, x_0 + t^*]. \quad (1.6)$$

and it is unique in

$$(x_0 - t^{**}, x_0 + t^{**}). \quad (1.7)$$

where

$$t^{**} = \frac{1 + \sqrt{1 - 2h}}{2\beta}.$$

Theorem 1.2 (Gutiérrez's theorem or Kantorovich's theorem under strong conditions). Let us consider $f : I \rightarrow \mathbb{R}$, where I is an open interval in $\mathbb{R}$, a differentiable function in I . Let us assume, without loss of generality, that $f(x_0) \neq 0$ and that

- i) $\left| \frac{f(E_0)}{f'(E_0)} \right| \leq a$.
- ii) $\left| \frac{f''(E_0)}{f'(E_0)} \right| \leq b$.
- iii) $\left| \frac{f''(E) - f''(E_0)}{f'(E_0)} \right| \leq K|E - E_0|$.

If the majorizing polynomial

$$\phi_G(t) = a - t + \frac{b}{2}t^2 + \frac{K}{6}t^3, \quad (1.8)$$

has two positive real roots t^* and t^{**} then the Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0, \quad (1.9)$$

well defined and $\{x_n\}_{n=0}^{\infty}$ converges a $x^* \in I$, where $f(x^*) = 0$. In addition, this solution is located in the range

$$[x_0 - t^*, x_0 + t^*]. \quad (1.10)$$

and it is unique in

$$(x_0 - t^{**}, x_0 + t^{**}). \quad (1.11)$$

Theorem 1.3 (Smale α -theory). Let us consider $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ a continuous and differentiable function in an interval D where the following conditions hold:

- i) $\left| \frac{f(x_0)}{f'(x_0)} \right| \leq \beta$, $f'(x_0) \neq 0$.
- ii) $\frac{1}{k!} \left| \frac{f^{(k)}(x_0)}{f'(x_0)} \right| \leq \gamma^{k-1}$, for $k \geq 2$.
- iii) $\alpha = \beta\gamma \leq 3 - 2\sqrt{2}$.

If the majorizing function

$$\phi_S(t) = \beta - t + \sum_{k \geq 2} \gamma^{k-1} t^k = \beta - t + \frac{\gamma t^2}{1 - \gamma t}, \quad \text{for } 0 \leq t < \frac{1}{\gamma}, \quad (1.12)$$

has two positive real roots t^* and t^{**} then the Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0, \quad (1.13)$$

is well defined and $\{x_n\}_{n=0}^{\infty}$ converges a $x^* \in I$, where $f(x^*) = 0$. In addition, this solution is located in the range

$$[x_0 - t^*, x_0 + t^*]. \quad (1.14)$$

and it is unique in

$$(x_0 - t^{**}, x_0 + t^{**}). \quad (1.15)$$

Theorem 1.4 (Wang-Zhao α -theory). Let us consider $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ a continuous and differentiable function in an interval D where the following conditions hold:

$$i) \quad f'(E_0) \neq 0.$$

$$ii) \quad \beta(E_0, f) = \left| \frac{f(E_0)}{f'(E_0)} \right|.$$

$$iii) \quad \frac{1}{k!} \left| \frac{f^{(k)}(E_0)}{f'(E_0)} \right| \leq \gamma_k, \quad k \geq 2.$$

If the majorizing function

$$\phi_W(t) = \beta - t + \sum_{k \geq 2} \gamma_k t^k. \quad (1.16)$$

has two positive real roots t^* and t^{**} , then the Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0, \quad (1.17)$$

is well defined and $\{x_n\}_{n=0}^{\infty}$ converges a $x^* \in I$, where $f(x^*) = 0$. In addition, this solution is located in the range

$$[x_0 - t^*, x_0 + t^*]. \quad (1.18)$$

and it is unique in

$$(x_0 - t^{**}, x_0 + t^{**}). \quad (1.19)$$

2 Main Results

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Applying the conditions of the semilocal convergence theorems of Kantorovich (1.1), Gutiérrez (1.2), Smale (1.3) and Wang-zhao (1.4) to the Kepler's equation, we have obtained, from $E_0 = \pi$, the following results on the existence and uniqueness of solution in terms of the eccentricity e .

Theorem 2.1 (Kantorovich's conditions). If $e \leq 0.247808$, then Kepler's equation has a solution. This solution is located in the range

$$[\pi - t^*, \pi + t^*]. \quad (2.1)$$

and it is unique in

$$(\pi - t^{**}, \pi + t^{**}), \quad (2.2)$$

where $\delta = 1/(1 + e)$ and

$$t^* = \frac{1 - \sqrt{1 - \pi e \delta^2}}{e \delta},$$

$$t^{**} = \frac{1 + \sqrt{1 - \pi e \delta^2}}{e \delta},$$

are the roots of the majorizing polynomial

$$\phi_K(t) = \frac{1}{2}e\delta t^2 - t + \pi\delta.$$

In addition, the Newton-Raphson method, defined by

$$E_{n+1} = E_n - \frac{f(E_n)}{f'(E_n)}, \quad n \geq 0, \quad (2.3)$$

starting in $E_0 = \pi$, converges to this solution.

Proof. We take $E_0 = \pi$ in Kantorovich's theorem to check the following conditions:

- i) $f'(E_0) \neq 0$.
- ii) $\left| \frac{f(E_0)}{f'(E_0)} \right| \leq \beta$.
- iii) $\left| \frac{f'(E) - f'(y)}{f'(E_0)} \right| \leq \gamma(E - y), \quad \forall E, y \in I$.
- iv) $h = \gamma\beta \leq \frac{1}{2}$.

In this case, we see that

$$\begin{aligned} f(E) &= E - e \sin E - M \Rightarrow f(E_0) = \pi - M \Rightarrow |f(E_0)| \leq \pi. \\ f'(E) &= 1 - e \cos E \Rightarrow f'(E_0) = 1 + e \\ f'(y) &= 1 - e \cos y. \\ \left| \frac{f(E_0)}{f'(E_0)} \right| &\leq \frac{\pi}{1+e} = \beta = \pi\delta. \\ \left| \frac{f'(E) - f'(y)}{f'(E_0)} \right| &= \frac{e}{1+e} |\cos E - \cos y| \leq \frac{e}{1+e} \left(\frac{\pi}{1+e} - y \right) \Rightarrow \gamma = \frac{e}{1+e} = e\delta \end{aligned}$$

so the majorizing polynomial is given by (2.3). Then

$$h = \beta\gamma = \pi e \delta^2 \leq \frac{1}{2} \Rightarrow \frac{\pi e}{(1+e)^2} \leq \frac{1}{2} \Rightarrow e \leq 0.247808.$$

The rest of the proof follows immediately from the theorem of Kantorovich. $\square$

Theorem 2.2 (Gutiérrez's conditions). *If $e \leq 0.129927$ then Kepler's equation has a solution. This solution is located in the range*

$$[\pi - t^*, \pi + t^*]. \quad (2.4)$$

and it is unique in

$$(\pi - t^{**}, \pi + t^{**}), \quad (2.5)$$

where t^* and t^{**} are the positive roots of the majoring polynomial

$$\phi_G(t) = \pi\delta - t + \frac{1}{6}e\delta t^3, \quad (2.6)$$

whit $\delta = 1/(1+e)$. In addition, the Newton-Raphson method, defined by (2.3), starting in $E_0 = \pi$, converges to this solution.

Proof. We take $E_0 = \pi$ in Gutiérrez's theorem to obtain the following conditions:

- i) $\left| \frac{f(E_0)}{f'(E_0)} \right| \leq a$.
- ii) $\left| \frac{f''(E_0)}{f'(E_0)} \right| \leq b$.

$$\text{iii)} \quad \left| \frac{f''(E) - f''(E_0)}{f'(E_0)} \right| \leq K|E - E_0|$$

and to calculate the parameters a , b y K . As in the previous result, we have

$$\left| \frac{f(E_0)}{f'(E_0)} \right| \leq \frac{\pi}{1+e} = a. \quad (2.7)$$

$$(2.8)$$

In addition,

$$\begin{aligned} f''(E) &= e \sin E \Rightarrow f''(E_0) = e \sin \pi \Rightarrow f''(E_0) = 0. \\ |f''(E) - f''(E_0)| &= e|\sin E - \sin E_0| \leq e|E - \pi|. \end{aligned}$$

then

$$\left| \frac{f''(E_0)}{f'(E_0)} \right| \leq 0 = b. \quad (2.9)$$

$$\left| \frac{f''(E) - f''(E_0)}{f'(E_0)} \right| \leq \frac{e}{1+e}|E - E_0| \Rightarrow K = \frac{e}{1+e}. \quad (2.10)$$

Substituting (2.7), (2.9) and (2.10) in (1.8) it is obtained (2.6) Let's analyze the polynomial (2.6).

$$\begin{aligned} \phi'_G(t) &= -1 + \frac{1}{2}e\delta t^2. \\ \phi'_G(t) &= 0 \Leftrightarrow -1 + \frac{1}{2}e\delta t^2 = 0 \Leftrightarrow t = \pm \sqrt{\frac{2}{e\delta}}. \end{aligned} \quad (2.11)$$

$$(2.12)$$

Let us consider $\hat{t} = \sqrt{2/(e\delta)}$, then

$$\phi''_G(\hat{t}) = e\delta \sqrt{\frac{2}{e\delta}} > 0$$

and $\phi_G(t)$ has a local minium at $\hat{t}$. Note that

$$\phi_G(\hat{t}) = \pi\delta - \sqrt{\frac{2}{e\delta}} + \frac{1}{6}e\delta \left(\sqrt{\frac{2}{e\delta}} \right)^3 = \pi\delta - \frac{2\sqrt{2}}{3} \sqrt{\frac{1}{e\delta}} \quad (2.13)$$

So the polynomial $\phi_G(t)$ has positive roots if $\phi_G(\hat{t}) < 0$, that is

$$\pi\delta - \frac{2\sqrt{2}}{3} \sqrt{\frac{1}{e\delta}} < 0 \Leftrightarrow \frac{1}{1+e} < \frac{2}{\sqrt[3]{(3\pi)^2 e}} \Leftrightarrow e \leq 0.129927$$

Accordingly, when $e \leq 0.129927$, Kepler's equation (1.4) has a solution and the rest of the test is a direct application of Theorem 1.2. $\square$

Theorem 2.3 (Smale's conditions). *If $e \leq 0.142599$, then Kepler's equation has a solution. This solution is located in the range*

$$[\pi - t^*, \pi + t^*], \quad (2.14)$$

and it is unique in

$$(\pi - t^{**}, \pi + t^{**}), \quad (2.15)$$

where t^* and t^{**} are the positive roots of the majorizing function

$$\phi_S(t) = \pi\delta - t + \frac{e\delta t^2}{2 - e\delta t}, \quad (2.16)$$

whit $\delta = 1/(1+e)$. In addition, the Newton-Raphson method, defined by (2.3), starting in $E_0 = \pi$, converges to this solution.

Proof. We take to $E_0 = \pi$ in Smale's α -theorem. So we must find the parameters β , γ and α that appear in the following conditions:

- i) $f'(E_0) \neq 0$.
- ii) $\left| \frac{f(E_0)}{f'(E_0)} \right| \leq \beta$.
- iii) $\frac{1}{k!} \left| \frac{f^{(k)}(E_0)}{f'(E_0)} \right| \leq \gamma^{k-1}$, $k \geq 2$.
- iv) $\alpha = \beta\gamma \leq 3 - 2\sqrt{2}$.

As in Theorem 2.1 we have

$$\left| \frac{f(E_0)}{f'(E_0)} \right| \leq \frac{\pi}{1+e} = \beta.$$

Furthermore

$$\begin{aligned} f''(E) &= e \sin \pi & \Rightarrow & f''(E_0) = 0. \\ f'''(E) &= e \cos \pi & \Rightarrow & |f'''(E_0)| \leq e. \\ & \vdots & & \\ |f^k(E_0)| &\leq \begin{cases} 0, & k = 2n, \forall n \in \mathbb{N}; \\ e, & k = 2n + 1, \forall n \in \mathbb{N}. \end{cases} \end{aligned}$$

Then we must find a constant γ such that

$$\left| \frac{1}{k!} \frac{f^k(E_0)}{f'(E_0)} \right|^{\frac{1}{k-1}} \leq \left[\frac{1}{k!} \frac{e}{1+e} \right]^{\frac{1}{k-1}} \leq \gamma. \quad (2.17)$$

Let us introduce the sequence

$$x_k = \left[\frac{1}{k!} \frac{e}{1+e} \right]^{\frac{1}{k-1}}, \quad k \geq 2.$$

As $e \in [0, 1]$, $\{x_k\}$ is a decreasing monotone sequence that converges to 0. So we can choose

$$\gamma = x_2 = \frac{e}{2(1+e)}.$$

Now we must prove

$$\alpha = \beta\gamma \leq \frac{\pi e}{2(1+e)^2} \leq 3 - 2\sqrt{2}. \quad (2.18)$$

This inequality is true for $e \leq 0.142599$, so the hypothesis of Smale's theorem 1.3 are satisfied and the rest of the proof is a consequence of this theorem. $\square$

Theorem 2.4 (Wang-Zhao's condition). *If $e \leq 0.018826$, then Kepler's equation has a solution. This solution is located in the range*

$$[\pi - t^*, \pi + t^*]. \quad (2.19)$$

and it is unique in

$$(\pi - t^{**}, \pi + t^{**}), \quad (2.20)$$

where t^* and t^{**} are the positive roots of the majorizing polynomial

$$\phi_{W-Z}(t) = \delta e \exp(t) - t(1 + e\delta) + \delta(\pi - e), \quad (2.21)$$

with $\delta = 1/(1+e)$. In addition, the Newton-Raphson method, defined by (2.3), starting in $E_0 = \pi$, converges to this solution.

Proof. Once again, we take $E_0 = \pi$ in the α -theorem of Wang-Zhao to check the following conditions:

- i) $f'(E_0) \neq 0$.
- ii) $\left| \frac{f(E_0)}{f'(E_0)} \right| \leq \beta$.
- iii) $\frac{1}{k!} \left| \frac{f^{(k)}(E_0)}{f'(E_0)} \right| \leq \gamma_k, \quad k \geq 2$.

We proceed as in the previous theorem to obtain the parameters $\beta = \pi\delta$ and $\gamma_k = e\delta/k!$. So the corresponding majorizing function is

$$\begin{aligned} \phi_{W-Z}(t) &= \pi\delta - t + \sum_{k \geq 2} \gamma_k t^k = \pi\delta - t + \sum_{k \geq 2} \frac{e}{k!} \delta t^k. \\ &= \pi\delta - t + e\delta \sum_{k \geq 2} \frac{t^k}{k!}. \\ &= \delta[e \exp(t) + (\pi - e)] - t(1 + e\delta). \end{aligned} \quad (2.22)$$

Let's analyze (2.22)

$$\phi'_{W-Z}(t) = \delta[e \exp(t)] - (1 + e\delta). \quad (2.23)$$

So, the critical point of equation (2.23) is attained when

$$\phi'_{W-Z}(\hat{t}) = 0 \Leftrightarrow \hat{t} = \ln \left(1 + \frac{1}{e\delta} \right). \quad (2.24)$$

We check if there is a minimum or maximum. Note that

$$\phi''_{W-Z}(t) = e\delta \exp(t). \quad (2.25)$$

substituting (2.24) en (2.25) it follows that

$$\phi''_{W-Z}(\hat{t}) = e\delta \exp \left[\ln \left(1 + \frac{1}{e\delta} \right) \right] = e\delta + 1 > 0,$$

then, there is a minimum at $\hat{t}$ defined by (2.24).

Then the majorizing function $\phi_{W-Z}(t)$ has positive solutions if

$$\phi_{W-Z}(\hat{t}) = (1 + \pi\delta) - \ln \left(1 + \frac{1}{e\delta} \right) (1 + e\delta) < 0 \quad (2.26)$$

or equivalently, if $e \leq 0.018826$, where 0.018826 is the only positive solution of the equation

$$\exp \left(\frac{1 + e + \pi}{1 + 2e} \right) - \frac{1}{e} - 2 < 0.$$

The rest of the proof follows immediately. □

3 Numerical experiments for comparison of results from the theorems proved above

Table 1: Majorizing functions for different theories of semilocal convergence according to the starting points and the parameter δ .

Theory	$E_0 = M, \delta = \frac{e}{1-e}$	$E_0 = \pi, \delta = \frac{1}{1+e}$
Kantorovich	$\phi_K(t) = \frac{1}{2}\delta t^2 - t + \delta$	$\phi_K(t) = \frac{1}{2}e\delta t^2 - t + \pi\delta$
Gutiérrez	$\phi_G(t) = \delta - t + \frac{1}{2}\delta t^2 + \frac{1}{6}\delta t^3$	$\phi_G(t) = \pi\delta + \frac{e\delta t^3}{6}$
Smale's α -theory	$\phi_S(t) = \delta - t + \frac{\sqrt{\delta} t^2}{\sqrt{6-\sqrt{\delta} t}}$	$\phi_S(t) = \pi\delta - t + \frac{\sqrt[3]{e\delta} t^2}{2\sqrt[3]{3-\sqrt[3]{e\delta} t}}$
Wang-Zhao's α -theory	$\phi_{WZ}(t) = \delta \exp(t) - (1 + \delta)t$	$\phi_{WZ}(t) = e\delta \exp(t) - (1 + e\delta)t + \delta(\pi - e)$

Table 2: Values of δ and e that guarantee the existence of solution for Kepler's equation by applying different semilocal convergence techniques for Newton's method, according to the starting points

Theory	$E_0 = M$	$E_0 = \pi$	$E_0 = M$	$E_0 = \pi$
Kantorovich	$e \leq 0.4142$	$e \leq 0.2478$	$\delta = 0.707107$	$\delta = 0.801416$
Gutiérrez	$e \leq 0.3759$	$e \leq 0.1299$	$\delta = 0.6023$	$\delta = 0.885074$
Smale's α -theory	$e \leq 0.3594$	$e \leq 0.00397$	$\delta = 0.561069$	$\delta = 0.99616$
Wang-Zhao's α -theory	$e \leq 0.3678$	$e \leq 0.018826$	$\delta = 0.581977$	$\delta = 0.981527$

4 CONCLUSIONS

Given all the results obtained in the previous sections, both the starting point $E_0 = M$ and $E_0 = \pi$, it is concluded that Kepler's equation (1.3), the best results are deduced from the Kantorovich theory. In fact, regarding the values of the eccentricity e , Kantorovich theorem gives the highest value among the considered theorems. For more detail, in the table 2 the different theories discussed in this article are shown, and the corresponding values for the parameter δ and eccentricity e . In addition, it is found that for the starting point $E_0 = \pi$, eccentricity values that ensure convergence of Newton's method applied to the equation of Kepler, become more restrictive.

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