

The Domination Number of $P_m \times \vec{P}_n$

Abstract. A mixed graph $G_M(V, E, A)$ is a graph containing unoriented edges (set E) as well as oriented edges (set A), referred to as arcs. In this paper we calculate the domination number of the Cartesian product of a path P_m with directed path $\vec{P}_n$ (mixed-grid graph $P_m \times \vec{P}_n$) for $8 \leq m \leq 10$ and arbitrary n .

Keywords: graph, directed graph, Cartesian product, path, directed path, mixed graph, mixed-grid graph, dominating set, domination number.

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1. Introduction.

All graphs and digraphs are assumed to be loopless and without duplicate edges or arcs. A mixed graph $G_M(V, E, A)$ is a graph containing unoriented edges (set E) as well as oriented edges (set A), referred to as arcs. This notion was first introduced in [1].

Let $G = (V_1, E)$ be a graph and $D = (V_2, A)$ be a digraph. The Cartesian product $G \times D$ is the mixed-graph with vertex set $V(G \times D) = V_1(G) \times V_2(D)$ and edge (arc) set is $((u_1, v_1), (u_2, v_2)) \in E(G \times D)$ if and only if either $v_1 = v_2$ and $(u_1, u_2) \in E(G)$ or $u_1 = u_2$ and $(v_1, v_2) \in A(D)$. A subset S of the vertex set $V(G \times D)$ is a dominating set of $G \times D$ if for each vertex $v \in G \times D$ there exists a vertex $u \in S$ such that (u, v) is an edge (arc) of $G \times D$. The domination number of $G \times D$, $\gamma(G \times D)$, is the cardinality of the smallest dominating set of $G \times D$.

Let P_m be a path with vertex set $V(P_m) = \{1, 2, \dots, m\}$, and edge set $E(P_m) = \{(i, i+1) : 1 \leq i \leq m-1\}$, and let $\vec{P}_n$ be a directed path with vertex set $V(\vec{P}_n) = \{1, 2, \dots, n\}$, and arc set $A(\vec{P}_n) = \{(i, i+1) : 1 \leq i \leq n-1\}$. Then for Cartesian product P_m and $\vec{P}_n$ is mixed-grid graph $P_m \times \vec{P}_n$ with $V(P_m \times \vec{P}_n) = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$, such that there is an arc from (i, j) to (p, q) if and only if $i = p$ and $q - j = 1$ and is an edge from (i, j) to (p, q) if and only if $j = q$ and $p - i = 1$. The i 'th row of $V(P_m \times \vec{P}_n)$ is $R_i = \{(i, j) : j = 1, 2, \dots, n\}$, and the j 'th column $K_j = \{(i, j) : i = 1, 2, \dots, m\}$. If S is a dominating set for $P_m \times \vec{P}_n$, then we denote $W_j = S \cap K_j$. Let $s_j = |W_j|$, where the sequence $(s_1, s_2, \dots, s_n)$ is called a dominating sequence corresponding to S . For $1 \leq j \leq n$, the vertices of j -column are dominated by vertices of j -column or $(j-1)$ -column. The vertices of the first column are dominated only by vertices of K_1 . Also, for $1 \leq i \leq m$, the vertices of i -row are dominated by vertices of i -row,

(i-1)-row or (i+1)-row. The vertices of the first row are dominated only by vertices of R_1 or R_2 . Thus the following is true:

$$\gamma(P_m \times P_n) \leq \gamma(\vec{P}_m \times \vec{P}_n) \leq \gamma(\vec{P}_n \times \vec{P}_n).$$

For finding domination number of grid graphs $P_m \times P_n$, Jacobson and Kinch in [2], were calculated the domination number of cartesian product of undirected paths P_m and P_n for $m=1, 2, 3, 4$. The cases $m=5, 6$ were calculated by Chang and Clark [3]. Also, Chang et al. [4], established the upper bounds of cartesian product of undirected paths P_m and P_n for $5 \leq m \leq 10$ and arbitrary n . In [5], Gravier and Mollard given un upper and lower bounds of general cartesian product of two undirected paths. Goncalves et al., [6] proved Chang's conjecture saying that for every $16 \leq n \leq m$, $\gamma(P_m \times P_n) = \lfloor (n+2)(m+2)/5 \rfloor - 4$.

For domination number of directed grid graphs, Liu et al. [7], they studied the domination number of $\vec{P}_m \times \vec{P}_n$ for $m=2, 3, 4, 5, 6$ and arbitrary n . Also, in [8] the author studied the

domination number of $\vec{P}_m \times \vec{P}_n$ for arbitraries m and n . Also, in [9] we have the following

results: $\gamma(P_1 \times \vec{P}_n) = \gamma(\vec{P}_n) = \left\lceil \frac{n}{2} \right\rceil$. $\gamma(P_2 \times \vec{P}_n) = n$. $\gamma(P_3 \times \vec{P}_n) = n$. $\gamma(P_4 \times \vec{P}_n) = \left\lceil \frac{3n}{2} \right\rceil$.

$$\gamma(P_5 \times \vec{P}_n) = \left\lceil \frac{3n}{2} \right\rceil + 1. \quad \gamma(P_6 \times \vec{P}_n) = 2n. \quad \gamma(P_7 \times \vec{P}_n) = 2n + 2.$$

2. Main results.

In this section we calculate the domination number of the Cartesian product of a path P_m and a directed path $\vec{P}_n$ for $m=8, 9, 10$ and arbitrary n .

Observation 2.1. Since for each vertex $(i, j) \in V(P_m \times \vec{P}_n)$ has two undirected degrees in $V(K_j)$, one outdegree in $V(K_{j+1})$ and one indegree from $V(K_{j-1})$, then can it dominates at must four vertices of $P_m \times \vec{P}_n$ with itself. Thus implies that $\gamma(P_m \times \vec{P}_n) \geq mn/4$. $\square$

Observation 2.2. Let S be a dominating set of $P_m \times \vec{P}_n$. Since the vertices of the first column are dominated only by vertices of K_1 . Also, for $2 \leq j \leq n$, the vertices of j -column are dominated by vertices of j -column or $(j-1)$ -column. Then the following are holds:

$$1. s_1 \geq \lceil m/3 \rceil.$$

$$2. s_j + 3s_{j+1} \geq m \text{ for all } j = 1, \dots, n. \quad \square$$

Lemma 2.1. There is a minimum dominating set S for $P_m \times \vec{P}_n$ with dominating sequence $(s_1, s_2, \dots, s_n)$ such that for all $j = 1, 2, \dots, n$, $\lceil m/5 \rceil \leq s_j \leq \lceil m/2 \rceil$, where $m > 1$.

Proof. Let S be a minimum dominating set for $P_m \times \vec{P}_n$ with dominating sequence $(s_1, s_2,$

$\dots, s_n$). Assume that for some j , s_j is large. Then we modify S by moving some vertices from column j to column $j+1$, such that the resulting set is still dominating set for $P_m \times \vec{P}_n$ (because each vertex in $S \cap K_j$ is dominates only vertices from K_j and K_{j+1}). For $1 \leq i \leq m$ and $1 \leq j \leq n$, let $W = S \cap \{(i, j), (i+1, j), (i+2, j), (i+3, j)\}$. If $|W| \geq 3$, then we have three cases:

Case 1. If $\{(1, j), (2, j)\} \subseteq S$ or $\{(m-1, j), (m, j)\} \subseteq S$. Then we can moving $(1, j)$ to $(1, j+1)$ or (m, j) to $(m, j+1)$. Furthermore, S is still dominating set of $P_m \times \vec{P}_n$.

Case 2. $|W| = 4$, then we put $S_1 = (S - W) \cup \{(i, j), (i+1, j+1), (i+2, j+1), (i+3, j)\}$.

Case 3. $|W| = 3$, then we have two sub cases

SubCase 3.1. $|W| = 3$ and $W = S \cap \{(i, j), (i+1, j), (i+2, j), (i+3, j)\} = \{(i, j), (i+1, j), (i+2, j)\}$ or $W = \{(i+1, j), (i+2, j), (i+3, j)\}$. Two cases are similar by symmetry, for the first we put $S_1 = (S - W) \cup \{(i, j), (i+2, j), (i+1, j+1)\}$ and for the second we put $S_1 = (S - W) \cup \{(i, j), (i+3, j), (i+2, j+1)\}$.

SubCase 3.2. $|W| = 3$ and $W = S \cap \{(i, j), (i+1, j), (i+2, j), (i+3, j)\} = \{(i, j), (i+1, j), (i+3, j)\}$ or $W = \{(i, j), (i+2, j), (i+3, j)\}$. Also, two cases are similar by symmetry. Then we change S , respectively as follows: $S_1 = (S - W) \cup \{(i, j), (i+3, j), (i+1, j+1)\}$, $S_1 = (S - W) \cup \{(i, j), (i+3, j), (i+2, j+1)\}$, see Figure 1 for cases 2, 3. We repeat this process if necessary eventually leads to a dominating set with required properties. Also, we get S_1 is a dominating set for $P_m \times \vec{P}_n$ with $|S| = |S_1|$. Thus, we can assume that every four consecutive vertices of the j 'th column include at most two vertices of S . This implies that $s_j \leq \lceil m/2 \rceil$, for all $1 \leq j \leq n$.

To prove the lower bound, we suppose that $|K_{j-1} \cap D|$ is be a maximum, i.e., $s_{j-1} = \lceil m/2 \rceil$. Then for each five vertices in K_j must include at last one vertex from S , otherwise K_j contain three successive vertices from S . This contradiction with the upper bounds. Thus we get $s_j \geq \lceil m/5 \rceil$ for all $1 \leq j \leq n$. $\square$

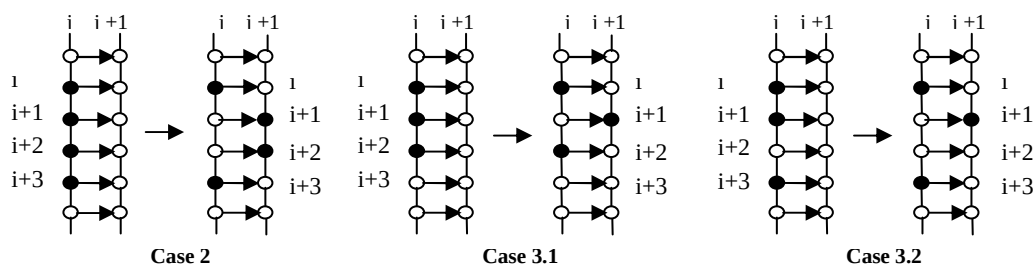


Figure 1. Modify S

By Lemma 2.1, always we have a minimum dominating set S with dominating sequence $(s_1, s_2, \dots, s_n)$, such that $\lceil m/5 \rceil \leq s_j \leq \lceil m/2 \rceil$, for all $j = 1, 2, \dots, n$. So, for all the next, we

consider a minimum dominating set of $P_m \times \vec{P}_n$ with dominating sequence $(s_1, \dots, s_n)$ with $\lceil m/5 \rceil \leq s_j \leq \lceil m/2 \rceil$, for all $j = 1, 2, \dots, n$.

Proposition 2.1.

1. $\gamma(P_m \times \vec{P}_n) \geq \gamma(P_r \times \vec{P}_n) + \gamma(P_{m-r-2} \times \vec{P}_n)$.
2. $\gamma(P_m \times \vec{P}_n) \leq \gamma(P_r \times \vec{P}_n) + \gamma(P_{m-r} \times \vec{P}_n)$.

Proof. For 1 and 2, the proofs are easy.

Proposition 2.2. $\gamma(P_8 \times \vec{P}_n) \leq \left\lceil \frac{5n}{2} \right\rceil + 1$.

Proof. Let S be a set defined as follows:

$$S = \{(1, 1), (3, 1), (5, 1), (7, 1)\} \cup \{(3, 2j), (7, 2j) : 1 \leq j \leq \lfloor n/2 \rfloor\} \\ \cup \{(1, 2j+1), (5, 2j+1), (8, 2j+1) : 1 \leq j \leq \lfloor (n-1)/2 \rfloor\}.$$

We have $|S| = n$, also S is a dominating set of $P_8 \times \vec{P}_n$ (see Figure 2, for $\gamma(P_8 \times \vec{P}_{11})$). Thus

$$\gamma(P_8 \times \vec{P}_n) \leq 4 + 3\lfloor n/2 \rfloor + 2\lfloor (n-1)/2 \rfloor = \lceil 5n/2 \rceil + 1.$$

□

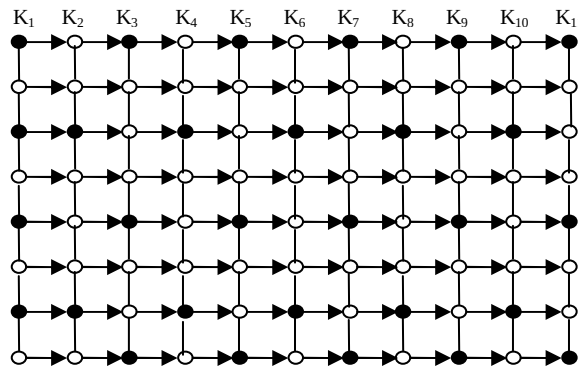


Figure 2. A dominating set of $P_8 \times P_{11}$.

Proposition 2.3.

1. The case $(s_1, s_2, s_3) = (3, 3, 2)$ is not possible.
2. The case $(s_1, s_2, s_3) = (4, 2, 2)$ is not possible.
3. There are four possibilities for $(s_j, s_{j+1}) = (2, 2)$.
4. The case $(s_j, s_{j+1}, s_{j+2}) = (2, 2, 2)$ is not possible.
5. The case $(s_j, s_{j+1}, s_{j+2}, s_{j+3}) = (2, 2, 3, 2)$ is not possible.

Proof. 1 and 2 by drawing.

3. For $(s_j, s_{j+1}) = (2, 2)$ we have four cases are:

- a. $\{(1, j), (2, j), (4, j+1), (7, j+1)\}$, b. $\{(1, j), (5, j), (3, j+1), (7, j+1)\}$,
- c. $\{(1, j), (8, j), (3, j+1), (6, j+1)\}$, d. $\{(4, j), (5, j), (2, j+1), (7, j+1)\}$,

4. For all cases in 3, we get $(s_j, s_{j+1}, s_{j+2}) = (2, 2, 2)$ is not possible.
5. By Completed drawing of cases $(s_j, s_{j+1}) = (2, 2)$, we results that $(s_j, s_{j+1}, s_{j+2}, s_{j+3}) = (2, 2, 3, 2)$ is not possible.

Proposition 2.4.

1. $s_1 + s_2 \geq 6$.
2. $s_1 + s_2 + s_3 \geq 9$.
3. $s_1 + s_2 + s_3 + s_4 \geq 11$.
4. $s_1 + s_2 + s_3 + s_4 + s_5 \geq 14$.

Proof. 1. We have $s_1 \geq 3$ and $2 \leq s_j \leq 4$. If $s_1 = 4$, then $s_2 \geq 2$ and so $s_1 + s_2 \geq 6$. Let $s_1 = 3$, then needs $|\{(1,1), (2,1), (7,1), (8,1)\} \cap S| = 2$, at the same time $\{(1,1), (2,1), (7,1), (8,1)\} \cap S \neq \{(1, 1), (8, 1)\}$. Suppose $\{(1, 1), (2, 1), (7, 1), (8, 1)\} \cap S = \{(1, 1), (7, 1)\}$ or $\{(2, 1), (7, 1)\}$ where two cases $\{(1, 1), (2, 1), (7, 1), (8, 1)\} \cap S = \{(1, 1), (7, 1)\}$ or $\{(2, 1), (8, 1)\}$ are similar by symmetry. If $\{(1, 1), (2, 1), (7, 1), (8, 1)\} \cap S = \{(1, 1), (7, 1)\}$, then we need $(4, 1) \in S$ and $s_2 \geq 3$. Thus we get $s_1 + s_2 \geq 6$. Let $\{(1, 1), (2, 1), (7, 1), (8, 1)\} \cap S = \{(2, 1), (7, 1)\}$, then $(4, 1) \in S$ or $(5, 1) \in S$, the two cases are similar by symmetry. Assume that $(5, 1) \in S$, then $s_2 \geq 3$. Thus we get $s_1 + s_2 \geq 6$.

2. From 1 we have $s_1 + s_2 \geq 6$. If $s_1 + s_2 \geq 7$, then $s_1 + s_2 + s_3 \geq 9$ {because $s_j \geq 2$ for all $j = 1, \dots, n$ }. Suppose $s_1 + s_2 = 6$, then immediately from Proposition 2.3 (1 and 2). $\square$

3. It clear from 2 and $s_j \geq 2$ for all $j = 1, \dots, n$.

4. From 3, $s_1 + s_2 + s_3 \geq 9$. If $s_1 + s_2 + s_3 \geq 10$, then finish {because $s_j \geq 2$ }. Let $s_1 + s_2 + s_3 = 9$ and suppose that $\sum_{j=1}^5 s_j < 14$. Then we must have $s_4 = s_5 = 2$, By Proposition 2.3(3), we have four cases for $(s_4, s_5) = (2, 2)$. But for all cases we get $s_1 + s_2 + s_3 \geq 10$, and this a contradiction. Finally we get $\sum_{j=1}^5 s_j \geq 14$. $\square$

Theorem 2.1. $\gamma(P_8 \times \vec{P}_n) = \left\lceil \frac{5n}{2} \right\rceil + 1$.

Proof. By Lemma 2.1, $2 \leq s_j \leq 4$. Observation 2.2, gets $s_1 \geq 3$ and $s_j + 3s_{j+1} \geq m$. Which implies that, if $s_j = 2$ is $s_{j+1} \geq 2$ and if $s_j = 3$ is $s_{j+1} \geq 2$. By Proposition 2.3, for each four columns including 10 vertices from S . We consider four cases:

Case a. $n \equiv 0 \pmod{4}$. By Proposition 2.4, $s_1 + s_2 + s_3 \geq 9$ and $s_1 + s_2 + s_3 + s_4 \geq 11$ {because $s_j \geq 2$ for all $j = 1, \dots, n$ }. Then

$$\gamma(P_8 \times \vec{P}_n) = \sum_{j=1}^n s_j \geq \sum_{j=1}^4 s_j + \sum_{j=5}^n s_j \geq 11 + 10(n-4)/4 = (5n+2)/2.$$

Case b. $n \equiv 1 \pmod{4}$. By Proposition 2.4, $\sum_{j=1}^5 s_j \geq 14$. Thus

$$\sum_{j=1}^n s_j \geq \sum_{j=1}^5 s_j + \sum_{j=6}^n s_j \geq 14 + 10(n-5)/4 = (5n+3)/2.$$

Case c. $n \equiv 2 \pmod{4}$. By Proposition 2.4, $s_1 + s_2 \geq 6$. Also, gets

$$\gamma(P_8 \times \vec{P}_n) = \sum_{j=1}^n s_j \geq \sum_{j=1}^2 s_j + \sum_{j=3}^n s_j \geq 6 + 10(n-2)/4 = (5n+2)/2.$$

Case d. $n \equiv 3 \pmod{4}$. By Proposition 2.4, $s_1 + s_2 + s_3 \geq 9$. Then

$$\gamma(P_8 \times \vec{P}_n) = \sum_{j=1}^n s_j \geq \sum_{j=1}^3 s_j + \sum_{j=4}^n s_j \geq 9 + 10(n-3)/4 = (5n+3)/2.$$

For all the cases, we get $\gamma(P_8 \times \vec{P}_n) \geq \left\lceil \frac{5n}{2} \right\rceil + 1$.

Finally, Proposition 2.2 together with the last result gets

$$\gamma(P_8 \times \vec{P}_n) = \left\lceil \frac{5n}{2} \right\rceil + 1.$$

□

Proposition 2.5.

$$\gamma(P_9 \times \vec{P}_n) \leq 3n : n \leq 3.$$

$$\gamma(P_9 \times \vec{P}_n) \leq \left\lceil \frac{5n}{2} \right\rceil + 2 : n \geq 4.$$

Proof. For $n \leq 3$, let $S_1 = \{(2, j), (5, j), (8, j) \text{ for } j = 1, \dots, n\}$.

If $n \geq 4$, we define S_2 as follows:

$$S_2 = \{(3, 1), (7, 1)\} \cup \{(1, 2j-1), (5, 2j-1), (9, 2j-1) : 1 \leq j \leq \lceil n/2 \rceil\} \cup \{(3, 2j), (7, 2j) : 1 \leq j \leq \lfloor n/2 \rfloor\}.$$

The set S_1 is a dominating set of $P_9 \times \vec{P}_n$ for $n \leq 3$ with $|S_1| = 3n$.

Also, S_2 is a dominating set of $P_9 \times \vec{P}_n$ for $n \geq 4$ with $|S_2| = 2 + 3 \lceil n/2 \rceil + 2 \lfloor n/2 \rfloor = \lceil 5n/2 \rceil + 2$.

(see Figure 3, for $\gamma(P_9 \times \vec{P}_{13})$). Thus

$$\gamma(P_9 \times \vec{P}_n) \leq 3n : n \leq 3. \tag{1}$$

$$\gamma(P_9 \times \vec{P}_n) \leq 2 + 3 \lceil n/2 \rceil + 2 \lfloor n/2 \rfloor = \lceil 5n/2 \rceil + 2. \tag{2} \square$$

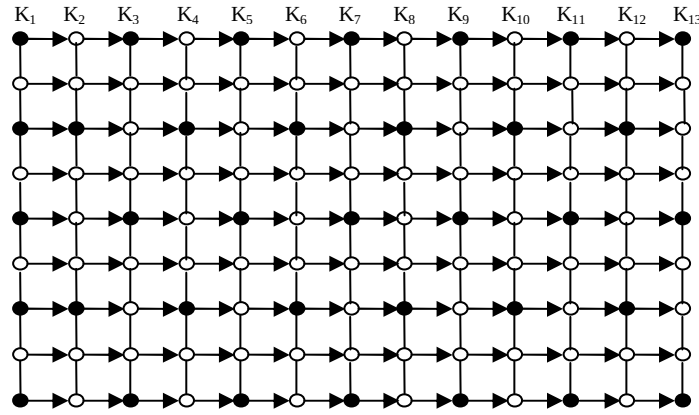


Figure 3. A dominating set of $P_9 \times P_{13}$.

Proposition 2.6.

1. If $s_2 = 2$ then $s_1 = 5$.
2. If $s_3 = 2$ then $s_1 + s_2 \geq 8$.
3. If $(s_1, s_2, \dots, s_j) = (3, 3, \dots, 3)$ then $s_{j+1} \geq 3$ for $j \geq 2$.
4. If K_j the first column, such that $s_j = |K_j \cap S| = 2$, then $\sum_{d=1}^{j-1} s_d \geq 3(j-1)+1$ for $j > 1$.

Proof. 1 and 2, immediately by drawing.

3. There is one possible for $s_1 = 3$ is $K_1 \cap S = \{(2, 1), (5, 1), (8, 1)\}$. This implies that $s_2 \geq 3$. If $s_2 = 3$ then $K_2 \cap S = \{(2, 2), (5, 3), (8, 3)\}$. Furthermore, if $s_1 = s_2 = \dots = s_j = 3$, then $K_j \cap S = \{(2, j), (5, j), (8, j)\}$. Thus gets $s_{j+1} \geq 3$.

4. Immediately from 3. □

Theorem 2.2.

$$\gamma(P_9 \times \vec{P}_n) = 3n : n \leq 3.$$

$$\gamma(P_9 \times \vec{P}_n) = \left\lceil \frac{5n}{2} \right\rceil + 2 : n \geq 4.$$

Proof. From Observation 2.2 and Lemma 2.1, we have $s_1 \geq 3$ and $2 \leq s_j \leq 5$. Furthermore, if $s_j = 2$ then $s_{j-1} \geq 3$, i.e., $(s_j, s_{j+1}) = (2, 2)$ is not possible. This implies that, if $s_j = 2$ then

$$\sum_{d=j}^{j+r} s_d \geq \left\lfloor \frac{5(r+1)}{2} \right\rfloor \text{ for } r \geq 1. \text{ We consider two cases:}$$

Case a. If $s_j \geq 3$ for $j = 1, \dots, n$, then $\gamma(P_9 \times \vec{P}_n) = \sum_{j=1}^n s_j \geq 3n$ for $n \leq 3$. Also for $n \geq 4$ is

$$\gamma(P_9 \times \vec{P}_n) \geq 3n \geq \left\lceil \frac{5n}{2} \right\rceil + 2. \text{ Then, (1) and (2) together with last results, gets the required.}$$

Case b. $s_j = 2$ for some $j \geq 2$. Suppose K_j is the first column, such that $s_j = 2$. We consider the following subcases:

SubCase b.1. $j = 2$. By Proposition 2.6, $s_1 = 5$ and $s_3 \geq 3$. Then $\gamma(P_9 \times \vec{P}_n) = \sum_{j=1}^n s_j \geq 3n$ where $n \leq 3$. For $n \geq 4$, by Proposition 2.6(4), we have

$$\gamma(P_9 \times \vec{P}_n) \geq \sum_{j=1}^n s_j = s_1 + \sum_{j=2}^n s_j \geq 5 + \left\lfloor \frac{5(n-1)}{2} \right\rfloor = \left\lceil \frac{5n}{2} \right\rceil + 2.$$

SubCase b.2. $j = 3$. By Proposition 2.6, $s_1 + s_2 \geq 8$. Thus $\gamma(P_9 \times \vec{P}_n) = \sum_{j=1}^n s_j \geq 3n$ where $n = 3$. For $n \geq 4$, Proposition 2.6(4), implies

$$\gamma(P_9 \times \vec{P}_n) \geq \sum_{j=1}^n s_j = s_1 + s_2 + \sum_{j=3}^n s_j \geq 8 + \left\lfloor \frac{5(n-2)}{2} \right\rfloor \geq \left\lceil \frac{5n}{2} \right\rceil + 2.$$

SubCase b.3. $j \geq 4$. From Proposition 2.6(4), we get

$$\gamma(P_9 \times \vec{P}_n) \geq \sum_{d=1}^n s_d = \sum_{d=1}^{j-1} s_d + \sum_{d=j}^n s_d \geq 3(j-1) + 1 + \left\lfloor \frac{5(n-j+1)}{2} \right\rfloor \quad \{\text{because } \gamma(P_m \times \vec{P}_n) \text{ is a natural number}\}, \text{ then } \gamma(P_9 \times \vec{P}_n) \geq \left\lceil \frac{5n}{2} \right\rceil + 2.$$

Thus, for all cases, gets

$$\gamma(P_9 \times \vec{P}_n) \geq \left\lceil \frac{5n}{2} \right\rceil + 2.$$

The last result together with (2), gets

$$\gamma(P_9 \times \vec{P}_n) = \left\lceil \frac{5n}{2} \right\rceil + 2.$$

□

Proposition 2.7. $\gamma(P_{10} \times \vec{P}_n) \leq \begin{cases} 3n+1: n \equiv 0 \pmod{2}, \\ 3n+2: n \equiv 1 \pmod{2}. \end{cases}$

Proof. Let S be a set defined as follows:

$$\begin{aligned} S = & \{(1,1), (4,1), (7,1), (10,1), (3,2), (5,2), (8,2)\} \\ & \cup \{(1, 2j+1), (5, 2j+1), (6, 2j+1), (10, 2j+1) : 1 \leq j \leq \lceil n/2 \rceil - 1\} \\ & \cup \{(3, 2j), (8, 2j) : 2 \leq j \leq \lfloor n/2 \rfloor\}. \end{aligned}$$

S is a dominating set of $P_{10} \times \vec{P}_n$ with $|S| = 7 + 4(\lceil n/2 \rceil - 1) + 2(\lfloor n/2 \rfloor - 1) = 3n + 1$ for n is even and $|S| = 3n + 2$ for n is odd (see Figure 4, for $\gamma(P_{10} \times \vec{P}_{12})$). Thus

$$\gamma(P_{10} \times \vec{P}_n) \leq \begin{cases} 3n+1: n \equiv 0(\text{mod } 2), \\ 3n+2: n \equiv 1(\text{mod } 2). \end{cases}$$

□

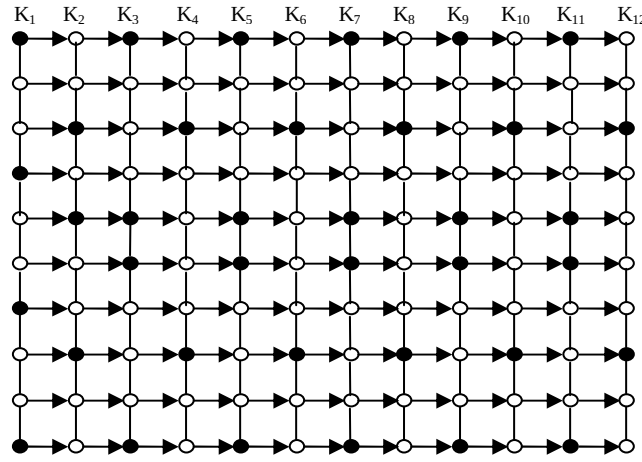


Figure 4. A dominating set of $P_{10} \times P_{12}$.

Proposition 2.8.

1. If $s_2 = 2$ then $s_1 = 6$.
2. If $s_2 = 3$ then $s_1 \geq 4$.

Proof. 1 and 2, immediately by drawing.

Theorem 2.3.

$$\begin{aligned} \gamma(P_{10} \times \vec{P}_n) &= 3n+1: n \equiv 0(\text{mod } 2), \\ 3n+1 \leq \gamma(P_{10} \times \vec{P}_n) &\leq 3n+2: n \equiv 1(\text{mod } 2), \end{aligned}$$

Proof. By Observation 2.2 and Lemma 2.1, we have $s_1 \geq 4$ and $2 \leq s_j \leq 5$. Furthermore, $(s_j, s_{j+1}) = (3, 2)$ is not possible {Observation 2.2(2)}. Then by Proposition 2.8, gets

$\gamma(P_{10} \times \vec{P}_n) \geq 3n+1$. Proposition 2.7 together with the last result, produces

$$\begin{aligned} \gamma(P_{10} \times \vec{P}_n) &= 3n+1: n \equiv 0(\text{mod } 2). \\ 3n+1 \leq \gamma(P_{10} \times \vec{P}_n) &\leq 3n+2: n \equiv 1(\text{mod } 2). \end{aligned}$$

In another way, by Proposition 2.1 we get $\gamma(P_{10} \times \vec{P}_n) \geq 2\gamma(P_4 \times \vec{P}_n) = 2\left\lceil \frac{3n}{2} \right\rceil$. Thus

$$\begin{aligned} \gamma(P_{10} \times \vec{P}_n) &\geq 2\gamma(P_4 \times \vec{P}_n) = 2\left\lceil \frac{3n}{2} \right\rceil = 3n+1 \text{ for } n \equiv 1(\text{mod } 2). \text{ Then Proposition 2.7, including} \\ 3n+1 \leq \gamma(P_{10} \times \vec{P}_n) &\leq 3n+2: n \equiv 1(\text{mod } 2). \end{aligned}$$

□

3. Conclusion

In this paper, we find the domination numbers of the Mixed-graph $P_m \times \vec{P}_n$ for $m = 8, 9, 10$ and arbitrary n . As a future work, we would like to work on the bounds of $\gamma(P_m \times \vec{P}_n)$ for arbitrariness m and n .

References.

- [1] Y.N. Sotskov, V.S. Tanaev, Chromatic polynomial of a mixed graph, *Vestsi Akademii Navuk BSSR, Seriya Fiz.-Mat Navuk* 6 (1976) 20-23 (in Russian).
- [2] M. Jacobson, and L. Kinch, On the domination number of products of graphs I: *Ars Combinatoria*, 18(1983), 33-44.
- [3] T. Chang., and W. Clark., The domination number of the $5 \times n$ and $6 \times n$ grid graphs, *J. of Graph Theory*, 17(1)(1993), 81-107.
- [4] T. Chang., and W. Clark., and E. Hare., Domination number of complete grid graphs, I., *Ars Combinatoria*, 38(1994), 97-111.
- [5] S. Gravier and M. Modlard., On domination number of Cartesian product of paths complete., *Discrete applied mathematics* 80 (1997), 2497-250.
- [6] D. Goncalves, A. Pinlou, M. Rao and S. Thomasse, The Domination Number of Grids, *SIAM J. Discrete Math.* 25(3) (2011), 1443–1453.
- [7] J. Liu, X. Zhang, X. Chen, J. Meng., On domination Number of Cartesian Product of directed paths, *J. Combinatorial Optimization*, 22(4) (2011) 651-662.
- [8] R. Shaheen. On the domination number of Cartesian products of two directed paths., *Int. J. Contemp. Math. Sciences*, Vol. 7, 2012, no. 36, 1785 – 1790.
- [9] R. Shaheen. On Domination Number of Mixed-Grid Graph., submitted to *Mathematical Communications*.