

Abstract

In this paper, we study viscosity approximation methods in reflexive Banach spaces. Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping $J: X \rightarrow X^*$, C a nonempty closed convex subset of X , h_n , where $n \geq 1$ a sequence of contractions on C and T_n , $n = 1, 2, 3, \dots, N$, for $N \in \mathbb{N}$, a finite family of commuting nonexpansive mappings on C . We show that under appropriate conditions on σ_n the explicit iterative sequence τ_n defined by

$$\tau_{n+1} = \sigma_n h_n(\tau_n) + (1 - \sigma_n) T_n \tau_n, \quad n \geq 1, \quad \tau_1 \in C$$

where $\sigma_n \in (0, 1)$ converges strongly to a common fixed point $\tau \in \bigcap_{k=1}^N F_{T_k}$. We consequently show that the results is true for an infinite T_n , $n = 1, 2, 3, \dots$ of commuting nonexpansive mapping on C

1 Introduction

Let X denote a real Banach space with dual X^* , and let J be the normalized duality mapping $J: X \rightarrow 2^{X^*}$ defined by $Jx = \{h \in X^* : \langle x, h \rangle = \|x\| \|h\|, \|x\| = \|h\|\}$, $\forall x \in X$. The single-valued duality mapping will be denoted by j ; and F_T given by $F_T = \{x \in X; Tx = x\}$. We denote the strong convergence of the sequence $\{x_n\}$ in X to $x \in X$ by $x_n \rightarrow x$ and the weak convergence by $x_n \rightharpoonup x$.

Let X denote a real Banach and space C a closed convex nonempty subset of X . Given a mapping $T: C \rightarrow C$, T is called φ -Lipschitzian if there exists a constant $\varphi > 0$ such that

$$\|Tx - Ty\| \leq \varphi \|x - y\|, \quad \forall \text{ pair } x, y \in X. \quad (1.1)$$

T is a nonexpansive if for every pair $x, y \in X$, $\|Tx - Ty\| \leq \|x - y\|$. T is said to be a contraction on C if a constant $\alpha \in (0, 1)$ exists such that for every pair $x, y \in X$, $\|Tx - Ty\| \leq \alpha \|x - y\|$. We denote by $\prod_C$ the set of all contractions on C .

In [1], H. K Xu proposed the iterative scheme $\{x_n\}$ given by

$$X_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \quad n \geq 1, \quad x_1 \in C \quad (1.2)$$

where $f \in \prod_C$, α_n is a sequence in $(0, 1)$ and T is a nonexpansive mapping on C , and proved the strong convergence of x_n to a fixed point of T .

Again in [2], Chang proved that as $n \rightarrow \infty$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n, \quad n \geq 1, \quad x_1 \in C \quad (1.3)$$

converges to a common fixed point of the nonexpansive mappings T_n , $n = 1, 2, \dots, N$ in a Banach space. The purpose of this paper is to prove the strong convergence of the sequence

$$\tau_{n+1} = \sigma_n h_n(\tau_n) + (1 - \sigma_n) T_n \tau_n, \quad n \geq 1, \quad \tau_1 \in C \quad (1.4)$$

in a reflexive Banach space X which admits a weakly sequentially continuous duality mapping $J: X \rightarrow X^*$ to a common fixed point of the family T_n , $n = 1, 2, \dots, N$. This generalizes and improves several recent results. Particularly, it extends and improves Theorem 4.2 of [1] and Theorem 1 of [2].

2 Preliminaries

Lemma 2.1

(Xu, [1]) Let X be a Banach space and C a bounded closed convex subset of X . Let $T: C \rightarrow C$ be a nonexpansive mapping on C and $h \in \prod_C$. Given a real number $\kappa \in (0, 1)$ define a mapping $T_\kappa: C \rightarrow C$ by $T_\kappa x = \kappa h(x) + (1 - \kappa)Tx$ $\forall x \in C$. Then T_κ is a contraction on C .

Proof

Let $x, y \in C$. Then

$$\begin{aligned} \|T_\kappa x - T_\kappa y\| &= \|\kappa h(x) + (1 - \kappa)Tx - [\kappa h(y) + (1 - \kappa)Ty]\| \\ &= \|\kappa[h(x) - h(y)] + (1 - \kappa)[Tx - Ty]\| \\ &\leq \kappa\alpha\|x - y\| + (1 - \kappa)\|x - y\| \\ &= (\kappa\alpha + (1 - \kappa))\|x - y\| \\ &= (1 - \kappa(1 - \alpha))\|x - y\| \end{aligned}$$

Since $1 - \kappa(1 - \alpha) \in (0, 1)$, T_κ is a contraction. ■

Theorem 2.2

Let X be a reflexive Banach space with dual X^* and bidual X^{**} . Let C be a nonempty bounded closed convex subset of X . Let h_n be a sequence of contractions on C such that $h_n \in C^* \quad \forall \quad n \geq 1$ and $h_n(x) \leq h_{n+1}(x)$, $\forall \quad n \geq 1, \quad x \in X$. If h_n converges pointwise on C to a contraction h then the convergence is uniform.

Proof

Let $f_n(x) = h(x) - h_n(x)$ for each $n \in \mathbb{N}$. Then f_n is a sequence of contractions on the compact set C such that $f_n(x) \geq f_{n+1}(x) \geq 0$ for all $x \in X$ and $n \in \mathbb{N}$.

Moreover,

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \{h(x) - h_n(x)\} = 0 \quad (2.1)$$

Let $M_n = \sup\{f_n(x) : x \in C\}$ and let $\varepsilon > 0$ be given.

Let

$$E_n = \{x \in C : f_n(x) < \varepsilon\} = f_n^{-1}((-\infty, \varepsilon))$$

Then E_n is open for each n and $E_n \subset E_{n+1}$ since $f_n(x) \geq f_{n+1}(x)$.

Since for each $x \in C$, $\lim_{n \rightarrow \infty} f_n(x) = 0$ there exists $n \in \mathbb{N}$ such that $f_n(x) < \varepsilon$ which implies $x \in E_n$. Thus

$\bigcup_{n=1}^{\infty} E_n$ is an open cover for C and $\bigcup_{n=1}^{\infty} E_n = C$.

Since C is compact, it has a finite subcover and in view of the fact that $E_n \subset E_{n+1}$, the largest of these also covers C . Hence there is $N \in \mathbb{N}$ such that $E_N = C$ and this means that $f_N(x) < \varepsilon$ for all $x \in C$ and $n \geq N$. Thus $M_N \leq \varepsilon$ and since $M_n \geq 0$, $\lim_{n \rightarrow \infty} M_n(x) = 0$. This indicates that the sequence f_n converges uniformly to 0 on C and therefore the sequence of contractions h_n converges uniformly to h on C . ■

Lemma 2.3

(Chang, [2]) Let X be a Banach space and let C be a bounded closed convex subset of X . Let T_k , $k = 1, 2, \dots, N$ be a finite family of commuting nonexpansive self mappings defined on C .

Then $\bigcap_{k=1}^N F_{T_k} = F_T$ where $T = T_1 T_2 T_3 \cdots T_{N-1} T_N$

Proof

Let $x \in \bigcap_{k=1}^N F_{T_k}$. Then $x \in F_{T_k}$, for $k = 1, 2, \dots, N$.

Hence $T_k x = x$ for each $k = 1, 2, \dots, N$. Thus $x \in F_T$ since $Tx = T_1 T_2 T_3 \cdots T_{N-1} T_N x = x$ implying that $\bigcap_{k=1}^N F_{T_k} \subset F_T$.

Next, suppose $x \in F_T$. Then $T_1 T_2 T_3 \cdots T_{N-1} T_N x = x$.

Letting $T = T_1$ gives $Tx = T_1 x = x$. Thus $x \in F_{T_1}$.

Again letting $T = T_1 T_2$ implies that $x = Tx = T_1 T_2 x = T_2 T_1 x = T_2 x$. Hence $x \in F_{T_2}$.

Next, if $T = T_1 T_2 T_3$ then we have $x = Tx = T_1 T_2 T_3 x = T_3 T_2 T_1 x = T_3 T_2 x = T_3 x$ and this implies $x \in F_{T_3}$.

Finally, for $N \in \mathbb{N}$ such that $N \geq 1$, let $T = T_1 T_2 T_3 \cdots T_{N-1}$ and assume that $T = T_1 T_2 T_3 \cdots T_{N-1} x = x$ implies $x \in F_{T_k}$ for each $k = 1, 2, \dots, N-1$.

Now, if $T = T_1 T_2 T_3 \cdots T_{N-1} T_N$ then

$$x = Tx = T_1 T_2 T_3 \cdots T_{N-1} T_N x = T_N T_1 T_2 T_3 \cdots T_{N-1} x = T_N x$$

Therefore $x \in F_{T_N}$. Thus, by induction if $T_1 T_2 T_3 \cdots T_{N-1} T_N x = x$ then $T_k x = x$ for each $k = 1, 2, \dots, N$ and so $x \in \bigcap_{k=1}^N F_{T_k}$.

This follows that $F_T \subset \bigcap_{k=1}^N F_{T_k}$ and the conclusion is that $\bigcap_{k=1}^N F_{T_k} = F_T$. ■

Lemma 2.4

Let X be a Banach space and let C be a bounded closed convex subset of X . Let T_k , $k = 1, 2, \dots, N$ be a finite family of commuting nonexpansive self mappings defined on C . Let $\bigcap_{k=1}^N F_{T_k} = F_T$, where $T = T_1 T_2 T_3 \cdots T_{N-1} T_N$. Then the mapping $T : C \rightarrow C$ is nonexpansive.

Proof

Let $x, y \in C$. Then $Tx = T_1 T_2 T_3 \cdots T_{N-1} T_N x$, $Ty = T_1 T_2 T_3 \cdots T_{N-1} T_N y$ and $\|Tx - Ty\| = \|T_1 T_2 T_3 \cdots T_{N-1} T_N x - T_1 T_2 T_3 \cdots T_{N-1} T_N y\|$

Next we show that P_N is true $\forall N \in \mathbb{N}$.

Let $P_N : \|T_1 T_2 T_3 \cdots T_{N-1} T_N x - T_1 T_2 T_3 \cdots T_{N-1} T_N y\| \leq \|x - y\| \forall x, y \in C$

By hypothesis,

$$\|T_1 x - T_1 y\| \leq \|x - y\| \forall x, y \in C. \quad (2.2)$$

Therefore P_N is true when $N = 1$

Assuming that for some $k \in \mathbb{N}$, P_k is true we get

$$\|T_1 T_2 T_3 \cdots T_{k-1} T_k x - T_1 T_2 T_3 \cdots T_{k-1} T_k y\| \leq \|x - y\| \forall x, y \in C.$$

Finally by commutation of the nonexpansive maps;

$$\|T_1 T_2 T_3 \cdots T_k T_{k+1} x - T_1 T_2 T_3 \cdots T_k T_{k+1} y\| \leq \|T_{k+1} x - T_{k+1} y\| \leq \|x - y\| \forall x, y \in C$$

Thus $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ proving that T is nonexpansive. ■

Lemma 2.5

(Xu, [1]) Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers, $\{\beta_n\}$ a sequence in $(0,1)$ and $\{\varphi_n\}$ a sequence in $\mathbb{R}$ such that $\alpha_{n+1} \leq (1 - \beta_n)\alpha_n + \varphi_n$, $n \geq 0$, where $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sum_{n=1}^{\infty} |\varphi_n| < \infty$. Then $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof

$\forall n \geq 1$, $\alpha_n \geq 0$ which implies $0 \leq \alpha_{n+1} \leq (1 - \beta_n)\alpha_n + \varphi_n$.

Thus

$$\lim_{n \rightarrow \infty} \alpha_{n+1} \leq \lim_{n \rightarrow \infty} \{(1 - \beta_n)\alpha_n + \varphi_n\} \quad (2.3)$$

Since $\beta_n \in (0,1) \forall n \geq 1$, $\lim_{n \rightarrow \infty} \beta_n = 0$ and therefore $\lim_{n \rightarrow \infty} \beta_n \alpha_n = 0$.

Again since $\sum_{n=1}^{\infty} \varphi_n < \infty$, it's sequence of partial sums is bounded and $\lim_{n \rightarrow \infty} \varphi_n = 0$.

Now, $\forall \beta_n \in (0,1)$, $1 - \beta_n \in (0,1)$. Letting $\lambda_n = 1 - \beta_n$ gives $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$.

Therefore $0 \leq \lim_{n \rightarrow \infty} \alpha_{n+1} \leq 0$ and $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. ■

Lemma 2.6

(Yisheng, [3]) Let C be a nonempty closed convex subset of a reflexive Banach space X which satisfies Opial's condition. Suppose also that $T : C \rightarrow C$ is nonexpansive and $\{x_n\}$ is a sequence in C such that $x_n - Tx_n \rightarrow 0$. Then $x = Tx$

Proof

Let $x_n \rightharpoonup x$ and let $\lim_{n \rightarrow \infty} \sup \|x_n - Tx_n\| = 0$.

Now, since $\{x_n\}$ is bounded there exists $r > 0$ such that $\{x_n\} \subset \mathcal{A} := C \cap B_r$, where B_r is a closed ball of X with centre at 0 and radius r . Thus $\mathcal{A}$ is a nonempty bounded closed convex subset of C .

For an arbitrary $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ such that $\|x_n - Tx_n\| \leq \varepsilon$, $\forall n \geq n_0$.

Let $\{\tau_n\}$ be a sequence in X such that $\tau_n \rightarrow \tau$ and $\|\tau_n - x_n\| < \frac{1}{n}$.

Again since $T : C \rightarrow C$ is nonexpansive and by extension Lipschitzian with Lipschitz constant ϕ ,

$$\|Tx_n - T\tau_n\| \leq \phi\|\tau_n - x_n\|$$

. Since

$$\|T\tau_n - \tau_n\| < \varepsilon \quad \forall \quad n, \quad \lim_{n \rightarrow \infty} \sup \|T\tau_n - \tau_n\| < \varepsilon \quad (2.4)$$

Finally,

$$\begin{aligned} \|Tx_n - x_n\| &\leq \|Tx_n - T\tau_n\| + \|T\tau_n - \tau_n\| + \|\tau_n - x_n\| \\ \implies \|Tx_n - x_n\| &\leq L\|x_n - \tau_n\| + \|T\tau_n - \tau_n\| + \|\tau_n - x_n\| \\ \implies \|Tx_n - x_n\| &\leq (1 + L)\|\tau_n - x_n\| + \|T\tau_n - \tau_n\| \end{aligned}$$

By virtue of (2.4) and the fact that $\|\tau_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$, $\|Tx_n - x_n\| \leq \varepsilon$

Again, by virtue of $\lim_{n \rightarrow \infty} \sup \|Tx_n - x_n\| = 0 \quad \forall \quad n$,

$$\|Tx_n - x_n\| \leq \varepsilon \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{and } x = Tx.$$

■

Lemma 2.7

(Chang, [2]) Let X be a Banach space with dual X^* . Let $J : X \rightarrow 2^{X^*}$ defined by

$$J(x) = \{f \in X^* : (x, f) = \|x\|\|f\|, \|x\| = \|f\|\}, \quad \forall \quad x \in X$$

be the normalized duality mapping on X .

Then $\forall x \in X, \forall j(x) \in J(x)$ and $\forall j(x+y) \in J(x+y)$, the following subdifferetial inequalities (i) and (ii) hold in X .

$$(i) \quad \|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y) \rangle$$

$$(ii) \quad \|x\|^2 + 2\langle y, j(x) \rangle \leq \|x+y\|^2$$

Proof

(i) $\forall x, y \in E$ and $\forall j(x+y) \in J(x+y)$,

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, j(x+y) \rangle \\ &= \langle x, j(x+y) \rangle + \langle y, j(x+y) \rangle \\ &\leq \|x\|\|x+y\| + \langle y, j(x+y) \rangle \end{aligned}$$

Now, $\forall a, b \in \mathbb{R}, \exists a \geq 0$ and $b \geq 0$, $ab \leq \frac{1}{2}(a^2 + b^2)$

Therefore

$$\begin{aligned} \|x+y\|^2 &\leq \frac{1}{2}(\|x\|^2 + \|x+y\|^2) + \langle y, j(x+y) \rangle \\ \implies \|x+y\|^2 &\leq \|x\|^2 + 2\langle y, j(x+y) \rangle \end{aligned}$$

■

(ii) $\forall x, y \in X$ and $\forall j(x) \in J(x)$,

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \langle x, x+y \rangle + \langle y, x+y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \langle j(x), y \rangle + \langle y, j(x) \rangle + \|y\|^2 \\ &= \|x\|^2 + 2\langle y, j(x) \rangle + \|y\|^2 \end{aligned}$$

Therefore

$$\|x\|^2 + 2\langle y, j(x) \rangle = \|x+y\|^2 - \|y\|^2 \quad (2.5)$$

From (2.5),

$$\|x\|^2 + 2\langle y, j(x) \rangle \leq \|x+y\|^2$$

■

Lemma 2.8

(Xu, [1]) Let X be a real Banach space with dual X^* and let J be as in Lemma (3.8). If C is a closed convex subset of X and $h : C \rightarrow C$ is a contraction with coefficient $\alpha \in (0, 1)$, then $(1 - \alpha)\|x - y\|^2 \leq \langle (I - h)x - (I - h)y, j(x - y) \rangle \quad \forall x, y \in C$ and $(I - h)$ is said to be strongly monotone where I is the identity operator.

Proof

Since h is a contraction with contractive constant $\alpha \in (0, 1)$, $\|h(x) - h(y)\| \leq \alpha\|x - y\|$
Now, $\forall$ pair $x, y \in C$ and any $j(x - y) \in J(x - y)$,

$$\begin{aligned} \langle (I - h)x - (I - h)y, j(x - y) \rangle &= \langle x - hx - y + hy, j(x - y) \rangle \\ &= \langle x - y - (hx - hy), j(x - y) \rangle \\ &= \langle x - y, j(x - y) \rangle - \langle hx - hy, j(x - y) \rangle \\ &\geq \|x - y\|^2 - \|hx - hy\|\|x - y\| \\ &\geq \|x - y\|^2 - \alpha\|x - y\|\|x - y\| \\ &= \|x - y\|^2 - \alpha\|x - y\|^2 \\ &= (1 - \alpha)\|x - y\|^2 \end{aligned}$$

Therefore $(1 - \alpha)\|x - y\|^2 \leq \langle (I - h)x - (I - h)y, j(x - y) \rangle$. ■

3 Main Results

Theorem 3.1

Let X be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping $J : X \rightarrow X^*$, C a bounded closed convex nonempty subset of X , h_n , where $n \in \mathbb{N}$ a sequence of contractions on C such that $h_n(\tau) \leq h_{n+1}(\tau)$, $\forall n, \tau \in C$, and T_n , $n = 1, 2, 3, \dots, N$ a finite family of commuting nonexpansive mappings on C such that $\bigcap_{n=1}^N F_{T_n} \neq \emptyset$ and satisfies the condition $\bigcap_{n=1}^N F_{T_n} = F(T_1 T_2 T_3 \dots T_{N-1} T_N) = F_T$, where $T = T_1 T_2 T_3 \dots T_{N-1} T_N$.

Suppose also that for each $h_n \in \prod_C$, $h(p) \neq p$, $\forall p \in \bigcap_{n=1}^N F_{T_n}$ and σ_n is a real sequence in $(0, 1)$ that satisfies the following conditions;

$$K_1 : \lim_{n \rightarrow \infty} \sigma_n = 0$$

$$K_2 : \sum_{n=0}^{\infty} \sigma_n = \infty$$

$$K_3 : \|\sigma_{n+1} - \sigma_n\| < \infty$$

Then the sequence τ_n defined by

$$\tau_{n+1} = \sigma_n h_n(\tau_n) + (1 - \sigma_n) T_n \tau_n, \quad \tau_1 \in C, \quad n \geq 1 \quad (3.1)$$

satisfying the condition

$$K_4 : \|\tau_n - T \tau_n\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

converges strongly to $\tau^* \in \bigcap_{n=1}^N F_{T_n}$ such that τ^* is the unique solution of the variational inequality $\langle (I - h)\tau^*, j(\tau^* - \tau) \rangle \leq 0, \forall \tau \in \bigcap_{n=1}^N F_{T_n}$.

Proof

From Theorem 2.2, $h_n \rightarrow h$ uniformly on C . Thus (3.1) becomes

$$\tau_{n+1} = \sigma_n h(\tau_n) + (1 - \sigma_n) T_n \tau_n \quad (3.2)$$

To show that $\{\tau_{n+1}\}$ is bounded we choose $p \in \bigcap_{n=1}^N F_{T_n}$ to compute $\|\tau_{n+1} - p\|$.

It follows that

$$\|\tau_{n+1} - p\| = \|\sigma_n h(\tau_n) + (1 - \sigma_n) T_n \tau_n - p\|$$

$$\begin{aligned}
&= \|\sigma_n(h(\tau_n) - p) + (1 - \sigma_n)(T_n\tau_n - p)\| \\
&\leq \sigma_n\|h(\tau_n) - p\| + (1 - \sigma_n)\|T_n\tau_n - p\| \\
&\leq \sigma_n(\|h(\tau_n) - h(p) + h(p) - p\|) + (1 - \sigma_n)\|\tau_n - p\| \\
&\leq \sigma_n\|h(\tau_n) - h(p)\| + \sigma_n\|h(p) - p\| + (1 - \sigma_n)\|\tau_n - p\| \\
&= \sigma_n\alpha\|\tau_n - p\| + \sigma_n\|h(p) - p\| + (1 - \sigma_n)\|\tau_n - p\| \\
&= (1 - (1 - \alpha)\alpha_n)\|\tau_n - p\| + \sigma_n\|h(p) - p\| \\
&\leq \max\{\|\tau_n - p\|, \frac{1}{1 - \alpha}\|h(p) - p\|\}
\end{aligned}$$

By induction,

$$\|\tau_n - p\| \leq \max\{\|\tau_1 - p\|, \frac{1}{1 - \alpha}\|h(p) - p\|\}, n \geq 1.$$

Thus $\{\tau_n\}$ is bounded and so are $\{T\tau_n\}$ and $\{h(\tau_n)\}$ where $T = T_1T_2T_3 \cdots T_{N-1}T_N$.

Now,

$$\begin{aligned}
\|\tau_{n+1} - \tau_n\| &= \|\sigma_nh(\tau_n) + (1 - \sigma_n)T\tau_n - \sigma_{n-1}h(\tau_{n-1}) - (1 - \sigma_{n-1})T\tau_{n-1}\| \\
&= \|(1 - \sigma_n)(T\tau_n - T\tau_{n-1}) + (\sigma_n - \sigma_{n-1})(h(\tau_{n-1}) - T\tau_{n-1}) + \sigma_n(h(\tau_n) - h(\tau_{n-1}))\| \\
&\leq (1 - \sigma_n)\|\tau_n - \tau_{n-1}\| + K|\sigma_n - \sigma_{n-1}| + \sigma_n\alpha\|\tau_n - \tau_{n-1}\|, \\
&= (1 - (1 - \alpha)\alpha_n)\|\tau_n - \tau_{n-1}\| + K|\sigma_n - \sigma_{n-1}|
\end{aligned}$$

where $K = \sup_{n \geq 1} \|h(\tau_{n-1}) - T\tau_{n-1}\| > 0$. Thus

$$\|\tau_{n+1} - \tau_n\| = (1 - (1 - \alpha)\alpha_n)\|\tau_n - \tau_{n-1}\| + K|\sigma_n - \sigma_{n-1}|$$

Lemma 2.5 together with conditions K_2 and K_3 ensure that $\|\tau_{n+1} - \tau_n\| \rightarrow 0$.

Again,

$$\begin{aligned}
\|\tau_n - T\tau_n\| &= \|\tau_n - \tau_{n+1} + \tau_{n+1} - T\tau_n\| \\
&\leq \|\tau_n - \tau_{n+1}\| + \|\tau_{n+1} - T\tau_n\| \\
&= \|\tau_n - \tau_{n+1}\| + \|\sigma_nh(\tau_n) + (1 - \sigma_n)T\tau_n - T\tau_n\| \\
&= \|\tau_n - \tau_{n+1}\| + \sigma_n\|h(\tau_n) - T\tau_n\| \rightarrow 0 \text{ as } n \rightarrow \infty
\end{aligned}$$

Thus $\|\tau_n - T\tau_n\| \rightarrow 0$ as $n \rightarrow \infty$ and condition K_4 is proved.

Next we show that $\limsup_{n \rightarrow \infty} \langle \tau^* - h(\tau^*), j(\tau_n - \tau^*) \rangle \geq 0 \forall \tau^* \in F_T = \bigcap_{n=1}^N F_{T_n}$. To do this we take a subsequence $\{\tau_{n_k}\}$ of $\{\tau_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle \tau^* - h(\tau^*), j(\tau_n - \tau^*) \rangle = \limsup_{k \rightarrow \infty} \langle \tau^* - h(\tau^*), j(\tau_{n_k} - \tau^*) \rangle$$

Now, since X is reflexive and τ_n is bounded, we may assume that

$$\tau_{n_k} \rightharpoonup \tau^* \text{ as } k \rightarrow \infty.$$

Therefore by K_4 and Lemma 2.6, $\tau^* \in F_T = \bigcap_{n=1}^N F_{T_n}$.

By the fact that the duality mapping j is weakly sequentially continuous from X to X^* , we obtain

$$\limsup_{n \rightarrow \infty} \langle \tau^* - h(\tau^*), j(\tau_n - \tau^*) \rangle = \langle \tau^* - h(\tau^*), j(\tau^* - \tau^*) \rangle \geq 0 \quad (3.3)$$

Finally, $\tau_n \rightarrow \tau^*$ is shown below.

By (Lemma 2.7 ii) we have,

$$\begin{aligned}
\|\tau_{n+1} - \tau^*\|^2 &= \|\sigma_nh(\tau_n) + (1 - \sigma_n)T_n\tau_n - \tau^*\|^2 \\
&= \|(1 - \sigma_n)(T_n\tau_n - \tau^*) + \sigma_n(h(\tau_n) - \tau^*)\|^2 \\
&\leq (1 - \sigma_n)^2\|(T_n\tau_n - \tau^*)\|^2 + 2\sigma_n\langle h(\tau_n) - \tau^*, j(\tau_{n+1} - \tau^*) \rangle \\
&\leq (1 - \sigma_n)^2\|\tau_n - \tau^*\|^2 + 2\sigma_n\langle h(\tau_n) - h(\tau^*) + h(\tau^*) - \tau^*, j(\tau_{n+1} - \tau^*) \rangle \\
&\leq (1 - \sigma_n)^2\|\tau_n - \tau^*\|^2 + 2\sigma_n\langle h(\tau_n) - h(\tau^*), j(\tau_{n+1} - \tau^*) \rangle + 2\sigma_n\langle h(\tau^*) - \tau^*, j(\tau_{n+1} - \tau^*) \rangle \\
&\leq (1 - \sigma_n)^2\|\tau_n - \tau^*\|^2 + 2\sigma_n\alpha\|\tau_n - \tau^*\|\|\tau_{n+1} - \tau^*\| + 2\sigma_n\langle h(\tau^*) - \tau^*, j(\tau_{n+1} - \tau^*) \rangle
\end{aligned}$$

$$\begin{aligned}
\|\tau_{n+1} - \tau^*\|^2 &\leq (1 - \sigma_n)^2 \|\tau_n - \tau^*\|^2 + \alpha \sigma_n \{\|\tau_n - \tau^*\|^2 + \|\tau_{n+1} - \tau^*\|^2\} \\
&\quad + 2\sigma_n \langle h(\tau^*) - \tau^*, j(\tau_{n+1} - \tau^*) \rangle \\
(1 - \alpha \sigma_n) \|\tau_{n+1} - \tau^*\|^2 &\leq (1 - 2\sigma_n + \sigma_n^2) \|\tau_n - \tau^*\|^2 + \alpha \sigma_n \{\|\tau_n - \tau^*\|^2 \\
&\quad + 2\sigma_n \langle h(\tau^*) - \tau^*, j(\tau_{n+1} - \tau^*) \rangle\}
\end{aligned} \tag{3.4}$$

Since $j(-x) = -j(x) \forall x \in X$, (3.3) becomes

$$\lim_{n \rightarrow \infty} \sup \langle h(\tau^*) - \tau^*, j(\tau^* - \tau_n) \rangle \leq 0 \tag{3.5}$$

If $\varphi_n = \max\{\langle h(\tau^*) - \tau^*, j(\tau^* - \tau_n) \rangle, 0\} \geq 0, \forall n \geq 0$ then (3.5) ensures that for any given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $\langle h(\tau^*) - \tau^*, j(\tau^* - \tau_n) \rangle < \varepsilon, \forall n \geq N_1$.

Therefore $0 \leq \varphi_n < \varepsilon \forall n \geq N_1$ and since ε is arbitrary $\varphi_n \rightarrow 0$ as $n \rightarrow \infty$.

Thus

$$\langle h(\tau^*) - \tau^*, j(\tau_{n+1} - \tau^*) \rangle \leq \varphi_n.$$

Therefore by (3.4),

$$(1 - \alpha \sigma_n) \|\tau_{n+1} - \tau^*\|^2 \leq (1 - 2\sigma_n + \sigma_n^2) \|\tau_n - \tau^*\|^2 + \alpha \sigma_n M + 2\sigma_n \varphi_n,$$

where $M = \sup_{n \geq 0} \|\tau_n - \tau^*\|^2$

$$\|\tau_{n+1} - \tau^*\|^2 \leq \frac{1 - 2\sigma_n + \sigma_n^2}{1 - \alpha \sigma_n} \|\tau_n - \tau^*\|^2 + \frac{1}{1 - \alpha \sigma_n} (\alpha \sigma_n M + 2\sigma_n \varphi_n).$$

Now, since $\alpha \in (0, 1)$ and $\sigma_n \rightarrow 0$ as $n \rightarrow \infty$ there exists $N_2 \in \mathbb{N}$ such that $1 - \alpha \sigma_n > \frac{1}{2}, \forall n \geq N_2$.

Hence $\frac{1}{1 - \alpha \sigma_n} < 2$ and $\frac{1}{1 - \alpha \sigma_n} (\alpha \sigma_n M + 2\sigma_n \varphi_n) < 2(\alpha \sigma_n M + 2\sigma_n \varphi_n)$.

Again, let $\frac{1 - 2\sigma_n + \sigma_n^2}{1 - \alpha \sigma_n} = \frac{1 - 2\sigma_n + \alpha \sigma_n}{1 - \alpha \sigma_n} = \frac{1 - \sigma_n(2 - \alpha)}{1 - \alpha \sigma_n} = 1 - \frac{2\alpha_n(1 - \alpha)}{1 - \alpha \sigma_n} \leq 1 - 2\alpha_n(1 - \alpha)$.

It follows from above analysis that

$$\|\tau_{n+1} - \tau^*\|^2 \leq \{1 - 2\alpha_n(1 - \alpha)\} \|\tau_n - \tau^*\|^2 + 2\sigma_n(\alpha M + 2\varphi_n), \forall n \geq N_2.$$

Thus, by Lemma 2.5, $\|\tau_n - \tau^*\| \rightarrow 0$ as $n \rightarrow \infty$ and consequently $\tau_n \rightarrow \tau^* \in \cap_{n=1}^N F_{T_n}$. ■

We next consider an infinite family $T_1, T_2, T_3, \dots$ of nonexpansive commuting maps on C . For the purpose of achieving our objective, we group these mappings into equivalent classes each of size N where $N \in \mathbb{N}$. Thus the iteration of study becomes

$$\tau_{n+1} = \sigma_n h_n(\tau_n) + (1 - \sigma_n) T_n \tau_n, \tau_1 \in C, n \geq 1 \text{ where } T_n = T_n \pmod{N}.$$

Lemma 3.2

Let X be a nonempty set. Then all equivalent classes of X are disjoint and X is the union of its equivalent classes.

Proof

Let A be a nonempty subset of X . Then A is an equivalent class if there is an element $x \in A$ such that $\forall y \in A, x \sim y$. Thus, A consists precisely of elements which are equivalent to x .

Let A, B be any two equivalence classes of X and suppose $A \cap B \neq \phi$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$.

This implies $x \sim y \forall y \in A$ and $x \sim z \forall z \in B$. Therefore $y \sim x$ and $x \sim z$. Hence by transitivity $y \sim z$. This gives a contradiction since A and B are two distinct equivalent classes. Thus $A \cap B = \phi$. Finally, since $A \subset X, B \subset X$ and $A \cap B = \phi$ for every pair of subsets of X , X is the union of its equivalent classes.

Theorem 3.3

Let X be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping $J : X \rightarrow X^*, C$ a bounded closed convex nonempty subset of $X, h_n, n \in \mathbb{N}$, a sequence of contractions on C such that $h_n(\tau) \leq h_{n+1}(\tau), \forall n, \tau \in C$, and $T_n, n = 1, 2, 3, \dots$, an infinite family of commuting nonexpansive self mappings on C such that $\cap_{n=1}^N F_{T_n} \neq \phi$ and satisfies the condition $\cap_{n=1}^N F_{T_n} =$

$F(T_1 T_2 T_3 \dots T_{N-1} T_N) = F_T$, where $T = T_1 T_2 T_3 \dots T_{N-1} T_N$.

Suppose also that for each $h_n \in \prod_C, h(p) \neq p, \forall p \in \cap_{n=1}^N F_{T_n}$ and σ_n is a real sequence in $(0,1)$ that satisfies the following conditions;

$$K_1 : \lim_{n \rightarrow \infty} \sigma_n = 0.$$

$$K_2 : \sum_{n=0}^{\infty} \sigma_n = \infty$$

$$K_3 : \sum_{n=0}^{\infty} |\sigma_{n+1} - \sigma_n| < \infty$$

Then the sequence $\tau_{n+1} = \sigma_n h_n(\tau_n) + (1 - \sigma_n) T_n \tau_n, \tau_1 \in C, n \geq 1$, satisfying the condition

$$K_4 : \|\tau_n - T \tau_n\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

converges strongly to $\tau^* \in \cap_{n=1}^N F_{T_n}$ such that τ^* is the unique solution of the variational inequality

$$\langle (I - h)\tau^*, j(\tau^* - \tau) \rangle \leq 0, \forall \tau \in \cap_{n=1}^N F_{T_n}$$

Proof

The proof is done by partitioning the infinite family $T_n, n = 1, 2, 3, \dots$ into equivalence classes of size N , for a positive integer N . Thus, all the assumptions made for the finite number of nonexpansive maps in Theorem 1 hold for each class. Again, since equivalent classes are disjoint, there are no spillovers into other classes. Therefore any result that is true for one class will also hold for other classes. Consequently, the result is apriori true from Theorem 1.

Conclusion

Halpern, Lions, Wittmann Bauschke, Xu [1] and Chang [2] considered iterative sequences which were special cases of the sequence

$$\tau_{n+1} = \sigma_n h_n(\tau_n) + (1 - \sigma_n) T_n \tau_n, \tau_1 \in C, n \geq 1 \quad (3.6)$$

1. If X is a Hilbert space, $h_n = u \in C \forall n \geq 1$ and $N = 1$ then (3.6) is reduced to

$$\tau_{n+1} = \sigma_n u + (1 - \sigma_n) T \tau_n, \tau_1 \in C, n \geq 1 \quad (3.7)$$

which was studied by Halpern, Lions and Wittmann.

2. In a Hilbert space X , if $h_n = u \in C \forall n \geq 1$ and $T_n, n = 1, 2, 3, \dots, N$ is a finite family of nonexpansive mappings on C with $\cap_{k=1}^N F_{T_k} \neq \emptyset$, then (3.6) is reduced to

$$\tau_{n+1} = \sigma_n u + (1 - \sigma_n) T_n \tau_n, \tau_1 \in C, n \geq 1 \quad (3.8)$$

which was considered by Bauschke.

3. If X is a uniformly smooth Banach space, C a nonempty closed convex subset of X , $T : C \rightarrow C$ is a nonexpansive mapping with $F_T \neq \emptyset$ and $h_n = h \in \prod_C \forall n \geq 1$, then (3.6) is reduced to

$$\tau_{n+1} = \sigma_n h(\tau_n) + (1 - \sigma_n) T \tau_n, \tau_1 \in C, n \geq 1 \quad (3.9)$$

which was studied by Xu [1].

4. Again, if X is a uniformly smooth Banach space, C a nonempty closed convex subset of X , $T : C \rightarrow C$ is a nonexpansive mapping with $F_T \neq \emptyset$ and $h_n = h \in \prod_C \forall n \geq 1$, $T_n, n = 1, 2, 3, \dots, N$ is a finite family of nonexpansive self mappings on C with $\cap_{k=1}^N F_{T_k} \neq \emptyset$, then (3.6) is reduced to the special case

$$\tau_{n+1} = \sigma_n h(\tau_n) + (1 - \sigma_n) T \tau_n, \tau_1 \in C, n \geq 1 \quad (3.10)$$

which was studied by Chang [3].

In summary, the iterative sequence (3.6) is a more general sequence which contains (3.7), (3.8), (3.9) and (3.10) as special cases.

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