

The Riemann Zeta Function and its analytic continuation

Abstract

The objective of this dissertation is to study the zeta function in particular its analytic continuation, the functional equation and applications. We begin from the historical background, the definition of the zeta function, some important tools which lead us to the functional equation of the Riemann zeta function. Then, we prove four methods of the functional equation. In addition, the $\zeta(s)$ has generalizations, we present one of them the Dirichlet L-Function. Then, we move on to the zeros of $\zeta(s)$. At the end, link between the distribution of prime numbers and the Riemann zeta function.

1 Introduction

One of the oldest branches of mathematics is number theory. Many mathematicians have seen the significance of the complex number analysis and its application in number theory. The relationship between the number theory and a complex analysis is called Analytic Number Theory.

Our story begins in Euler's theorem, proved in 1737

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots$$

it is divergent. Then, Euler proposed the Euler product formula

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

Following that Riemann looked at the Euler's definition and considering complex argument. In 1859, Riemann extended the definition of Euler's zeta function from a real variable to a complex variable except for a simple pole at $s = 1$ with residue 1 by his paper "On The Number of Primes Less Than a Given Magnitude". Riemann forced the zeta function to be a meromorphic function. Furthermore, he proved the functional equation by using the analytic tools.

The Riemann zeta function is one of the most essential functions in mathematics. Its applications include many areas of study such as number theory and other sciences. This function was not created by Riemann. However, the zeta function is named after him because he developed the zeta function and proved the functional equation also, the significant relationship between the distribution of the prime number and the $\zeta(s)$.

We introduce some results. In order to present these results, we will use it frequently throughout this dissertation.

Definition 1.1. For $\text{Re}(s) > 1$ The zeta function is defined as

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Definition 1.2. For $\text{Re}(s) > 1$ The Euler product is defined as

$$\zeta(s) = \prod \left(1 - \frac{1}{p^s}\right)^{-1}$$

Definition 1.3. For $Re(s) > 0$ The Gamma function is defined as

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$$

Definition 1.4. For $Re(s) > 1$ the Xi function defined as

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

The aim of this dissertation is to study the Riemann zeta function, in particular its analytic continuation and functional equation and applications.

This dissertation is divided into eight sections. In the first section, we will briefly give the historical background particularly, the development of the zeta function. In the second section, we will recall some definitions and theorem which is related to the analytic continuation. Then, we will give some elementary asymptotic formula with out proof. In the third section, we will provide an important analytic tools which will help us to the prove the functional equation and play an important role in reaching our purpose. In the fourth section, we will state the definition of the zeta function in two ways as a Dirichlet series and Euler product. In the fifth section, we will prove the functional equation by fourth methods. Then, In the sixth section, the zeta function has generalizations we will present one of them the Dirichlet L-Function and the analytic continuation of the Dirichlet L-Function. In addition, proving the function equation for the Dirichlet L-Function. In the eighth section, we will introduce the zeros of the zeta function and the Dirichlet L-Function. At the end, we will give the application of the zeta function, the link between the zeta function and the distribution of the prime numbers.

2 Background

Commonly, the zeta function referred to the Riemann zeta function today. However, over the centuries, the zeta function has been studied in several various form. In the 14th century, the harmonic series, $\zeta(1)$ proved to be divergent. Then, in the first half 18th century, Euler was the first mathematician introduced the zeta function. Euler's zeta function is defined for any real number $s > 1$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$$

Euler merely defined $\zeta(s)$ on $\mathbb{R}$. It means that Euler gave birth to $\zeta(s)$. Then, in the 19th century, Riemann improved the zeta function by using a complex analysis. He extended the Euler definition from real variables to a complex variables. Additionally, He proved the analytic continuation of the zeta function, hence he obtained the functional equation. Riemann considered the zeta function as a complex function.

He published his paper in 1859 which is one of the most effective studies of the modern mathematics. Moreover, with the support of the complex analysis Riemann connected between the zeros of $\zeta(s)$ and the distribution of the prime number. However, before Riemann conjectures were proved or a results about prime, he proved the two key consequence:

(1) The zeta function can be meromorphic continued over the whole s -plane with a simple poles at $s = 1$, namely $\zeta(s) - \frac{1}{s-1}$ is an integral function.

(2) The zeta function satisfies that

$$\pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s) = \pi^{-\frac{1}{2}s-1} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

On the left hand side of this function is an even function on $s - \frac{1}{2}$, can be conclude the proprieties of the zeta function for $\sigma > 0$ from the proprieties for $\sigma > 1$ by the functional equation. Particularly, when $\sigma < 0$ the zeros of the zeta function are poles of $\Gamma\left(\frac{s}{2}\right)$ at negative even integers, these are called the trivial zeros. The stay of the plane, when the σ between 0 and 1 is called the critical strip. Furthermore, Riemann has some of remarkable conjectures (i) The zeta function has several zeros in the critical strip

(ii) $\xi(s)$ is an integral function defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

it has no pole when $\sigma \leq 1/2$ and this function is even function of $s - \frac{1}{2}$, also it has the product

$$\xi(s) = e^{A+Bs} \prod_{\lambda} \left(1 - \frac{s}{\lambda}\right) e^{s/\lambda}$$

such that A and B are absolutely constant and λ runs through the zeros of the zeta function in $0 \leq \sigma \leq 1$.

3 Analytic number theory

In this section, we recall one of the most significant of a complex analysis that will be help us to know the basic of the analytic number theory. Most of results taken from [3],[9] and [12]

3.1 Uniform convergence

Definition 3.1. If the partial sum $S_n(x) = \sum_{n=0}^{\infty} f_n(x)$, the series $\sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent in (a,b) , if every $\epsilon \geq 0$ such that ϵ is small. There exist a number n_1 depending on ϵ however, n_1 not depending on x

$$|S(x) - S_n(x)| < \epsilon$$

for $n > n_1$, all value of x in (a,b) .

Example: (i) For $x \in [a,b]$, the power series $\sum_{n=0}^{\infty} x^n$ is uniformly convergent if $-1 < a < b < 1$

(ii) Over any integral, the series $\sum_{n=0}^{\infty} \frac{\cos nx}{n^2}$ is uniformly convergent.

(iii) For $s \in [a,b]$, the Dirichlet series $\sum_{n=0}^{\infty} \frac{1}{n^s}$ is uniformly convergent if $1 < a < b$

3.2 Identity Theorem

If a region $\mathbb{R}$ has two functions which are analytic and these functions have the same values for all points within $\mathbb{R}$, both functions are identical everywhere within $\mathbb{R}$. Directly, we can use this consequence to extend functions from the real axis to the complex plane.

3.3 Analytic functions

Definition 3.2. Suppose $B \subset \mathbb{C}$ be open set and let $f : B \rightarrow \mathbb{C}$

(i) let $b_0 \in B$, then the function f is differentiable at b_0 if

$$f'(b_0) = \lim_{b \rightarrow b_0} \frac{f(b) - f(b_0)}{b - b_0}$$

exist.

(ii) f is analytic on B , it has derivative at all points in B .

- (iii) If the function f is analytic in a neighborhood of b_0 then f is analytic of b_0 .
- (iv) The entire is a function on the whole complex plane.

Remark: If the function f has derivative, then f is continuous. However, if f is a continuous function does not have to be differentiable.

Proposition 1. suppose $B \subset \mathbb{C}$ be open set and $f : B \rightarrow \mathbb{C}$, let f has a derivative at b_0 , hence f is continuous at b_0 .

3.4 Analytic Continuation

Definition 3.3. Suppose D_1 and D_2 are domains in the complex plane, the intersection of these domains $D_1 \cap D_2$ is the set of all common points between D_1 and D_2 . The union of these domains $D_1 \cup D_2$ is the set of all points in D_1 and D_2 .

Now, let $D_1 \cap D_2 \neq \emptyset$ also, it is connected. The functions f_1, f_2 are analytic over the domains D_1, D_2 , respectively, such that $f_1 = f_2$ on $D_1 \cap D_2$. Hence, we can call f_2 is an analytic continuation of f_1 on the second domain.

According to James and Ruel (1996) The essential theorem from the theory of complex variable is

Theorem 3.1. ([3] Theorem p.g. 84)

A function that is analytic in a domain D is uniquely determined over D by its values in a domain, or along a line segment, contained in D .

The definition of $F(b)$ analytic over $D_1 \cup D_2$

$$F(b) = \begin{cases} f_1(b) & \text{for } b \text{ is in } D_1 \\ f_2(b) & \text{for } b \text{ is in } D_2 \end{cases}$$

Since, $f_1 = f_2$ over $D_1 \cap D_2$, F is given by f_1 and f_2 over the domains D_1 and D_2 respectively. Based on theorem this is a holomorphic function.

Since F is analytic in the union D_1 and D_2 the function is uniquely determined by f_1 on a domain D_1 furthermore, it is uniquely determined by f_2 on a domain D_2 .

Thus, $F(b)$ is analytic continuation over the union of D_1 and D_2 of either f_1 or f_2 .

3.5 Some Elementary Asymptotic Formula

Definition 3.4. (The big oh notation)

If we have $h(x) > 0$ for all $x \geq a$, then $f(x)$ is big oh of $h(x)$ when $f(x) = O(h(x))$ and for $x \geq a$ $f(x)/h(x)$ is bounded. There exists a constant M greater than 0 such that $|f(x)| \leq Mh(x)$ Moreover, it satisfies this equation:

$$f(x) = g(x) + O(h(x))$$

we observe that $f(y) = O(h(y))$ for $l \geq a$ implies

$$\int_a^x f(y)dy = O\left(\int_a^x h(y)dy\right) \quad \text{for } x \geq a$$

Theorem 3.2. If $X \geq 1$ we have that

$$(a) \quad \sum_{n \leq x} \frac{1}{n} = \text{Log}x + C + O\left(\frac{1}{x}\right)$$

$$(b) \quad \sum_{n \leq x} \frac{1}{n^2} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}) \quad \text{if } s > 0 \quad s \neq 1$$

$$(c) \quad \sum_{n > x} \frac{1}{n^s} = O(x^{1-s}) \quad \text{if } s > 1$$

$$(d) \quad \sum_{n \leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha) \quad \text{if } \alpha \geq 0$$

Proof: See reference [1]

Note that the definition and theorem above is taken from [1]

4 Some Analytic Tools

As we mentioned before, Euler defined $\zeta(s)$ only for a real variable. However, Riemann developed this definition for real variable to a domain of complex variable and for achieve this aim Riemann studied into the analytic continuation of $\zeta(s)$. The functional equations are an important in analytic continuation. In order to prove functional equation we will provide some analytic tools and discuss some of their properties which are lead us to the prove functional equations. This section divides into six subsections, which start with The gamma function ,the theta function and the mellin transform. Then, the Dirichlet series and Abel summation (Partial summation). Finally, Fourier transform. The information taken from the references [1], [4] , [7] ,[12] [17] [18] and [19]

4.1 The Gamma Function

The Gamma function is one of an important function for analytic number theory. It is useful function when dealing with the theory surrounding zeta function. especially, it is extension of the factorial function to $Re(s) > 0$. Also, it is convergent.

Let we introduce this function :

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt \quad (4.1)$$

The first to denote it as $\Gamma(s)$ was Legendre but the first to introduce this function was Euler. However, Euler defined this function a bit different.

The Gamma function has some properties:

Proposition 2.

$$\Gamma(s+1) = s\Gamma(s)$$

Proof.

$$\begin{aligned} \Gamma(s+1) &= \int_0^{\infty} e^{-t} t^{(s-1)+1} dt = \int_0^{\infty} e^{-t} t^s dt \\ &= -e^{-t} t^s \Big|_0^{\infty} + s \int_0^{\infty} e^{-t} t^{s-1} dt = s\Gamma(s) \end{aligned}$$

For $\sigma > 0$ the term $(-e^{-t}s)|_0^\infty$ tend to zero. However, if we take $\Gamma(s)$ to the entire complex plane it needs the analytic continuation. we will present a nice properties that hold it.

□

Corollary 4.0.1. *The Gamma function has analytic continuation on complex number with simple pole at $s=0,-1,-2,-3,\dots$*

Definition 4.1. (Euler's gamma-function) the Euler gamma-function defined as

$$\frac{1}{\Gamma(s)} = s \exp(\gamma s) \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$$

such that γ is Euler's constant.

Theorem 4.1.

$$\Gamma(s) = \frac{1}{2} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^s \left(1 + \frac{s}{n}\right)^{-1} \quad (4.2)$$

Corollary 4.1.1.

$$\Gamma(n+1) = n! \text{ for } n \in \mathbb{N}$$

Proposition 3. The Gamma function $\Gamma(s)$ satisfies extension formula:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

Proof. By Euler gamma function

$$\frac{1}{\Gamma(s)} = s \exp(\gamma s) \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$$

$$\frac{1}{\Gamma(s)} \frac{1}{\Gamma(-s)} = -s^2 \exp(\gamma s) \exp(-\gamma s) \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n} \left(1 - \frac{s}{n}\right) e^{s/n}$$

Since we have

$$\Gamma(-s) = -\Gamma\left(\frac{1-s}{s}\right)$$

Then we can write that

$$\frac{1}{\Gamma(s)\Gamma(1-s)} = s \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right)$$

Using infinite product formula, we obtain that

$$\sin(\pi s) = s\pi \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right)$$

Hence

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

□

Corollary 4.1.2. ([7] Corollary 1)

$$\Gamma(s) = \lim_{n \rightarrow \infty} \frac{(n-1)!n^s}{(s(s+1)\dots(s+n)-1)}$$

Corollary 4.1.3. ([7] Corollary 4)

$$\Gamma(1/2) = \sqrt{\pi}$$

Corollary 4.1.4. (doubling formula) ([7] Corollary 5)

$$\Gamma(s)\Gamma(s + \frac{1}{2}) = 2^{-2s} \sqrt{\pi} \Gamma(2s)$$

Proposition 4. The integral of the gamma function is uniformly when $0 < a \leq \ell \leq b$, such that

$$\int_0^{\infty} t^{s-1} e^{-t} dt = \int_0^1 + \int_1^{\infty} = O\left(\int_0^1 t^{a-1} dt\right) + O\left(\int_1^{\infty} t^{b-1} e^{-t} dt\right) = O(1)$$

the oh big depend on ℓ Thus, for $\ell > 0$ this integral shows a continuous function.

Proposition 5. If s is complex, the gamma function 4.1 is uniformly convergent $R(s) \leq a > 0$, if $s = \sigma + it$. Hence, $|t^{s-1}| = t^{\sigma-1}$

4.2 The Theta function

Definition 4.2.

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \quad t > 0$$

Also, we can be written as

$$\theta(t) = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

Definition 4.3. If

$$w(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t} \text{ and } \theta(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

The relationship between $\theta(t)$ and $w(t)$ as $2w(t) = \theta(t) - 1$

Proposition 6. The theta function $\theta(t)$ satisfies the function equation:

$$\theta = \frac{1}{\sqrt{t}} \theta \left(\frac{1}{t} \right) \quad (4.3)$$

we will prove this proposition, However before we prove it we will introduce

The Poisson Summation Formula

If $\sum_{n=-\infty}^{\infty} f(t+n)$ uniformly converges for $0 \leq t \leq 1$, f is continuous and if $\sum_{n=-\infty}^{\infty} f(n)e^{2\pi nit}$ converges. Hence, $\sum_{n=-\infty}^{\infty} f(t+n) = \sum_{n=-\infty}^{\infty} \hat{f}(ne^{2\pi nit})$ where,

$$\hat{f}(n) = \int_{-\infty}^{\infty} f(t)e^{-2\pi nit} dt$$

Proof. Let $\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$ and $\sum_{n \in \mathbb{Z}} f(n) = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} f(y) e^{-2\pi iky} dy$

$$\begin{aligned} \theta(t) &= \sum_{n \in \mathbb{Z}} f(n) = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi y^2 t} e^{-2\pi iky} dy \\ &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi y^2 t - 2\pi iky} dy \\ &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi t[y^2 + 2i\frac{k}{t}y + i^2\frac{k^2}{t^2} - i^2\frac{k^2}{x^2}]} dy \\ &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi t(y + i\frac{k}{t})^2 - i^2\frac{k^2}{t^2}} dy \end{aligned}$$

It can be shown that

$$\begin{aligned} \theta(t) &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi k^2 \frac{1}{t}} e^{-\pi t(y + i\frac{k}{t})^2} dt \\ \theta(t) &= \sum_{k \in \mathbb{Z}} e^{-\pi k^2 \frac{1}{t}} \int_{-\infty}^{\infty} e^{-\pi t(y + i\frac{k}{t})^2} dt \end{aligned}$$

let $y + i\frac{k}{t} = z$ and $dy = dz$ $|\infty \rightarrow \infty + i\frac{k}{t}$ and $-\infty + i\frac{k}{t}$

$$\theta(t) = \sum_{k \in \mathbb{Z}} e^{-\pi k^2 \frac{1}{t}} \int_{-\infty + i\frac{k}{t}}^{\infty + i\frac{k}{t}} e^{-\pi t z^2} dt$$

We can say that

$$\int_{-\infty + i\frac{k}{t}}^{\infty + i\frac{k}{t}} e^{-\pi t z^2} dt = \int_{-\infty}^{\infty} e^{\pi t z^2} dt$$

So, we obtain that

$$\begin{aligned} \theta(t) &= \sum_{k \in \mathbb{Z}} e^{-\pi k^2 \frac{1}{t}} \int_{-\infty}^{\infty} e^{-\pi t z^2} dz = \sum_{k \in \mathbb{Z}} e^{-\pi k^2 \frac{1}{t}} \sqrt{\frac{\pi}{\pi t}} \\ &= \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \sum_{n \in \mathbb{Z}} e^{-\pi k^2 \frac{1}{t}} \sqrt{\frac{1}{t}} \end{aligned}$$

We know that

$$\theta(t) = \sum_{k \in \mathbb{Z}} e^{-\pi k^2 t}$$

It follow that

$$\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right)$$

□

Corollary 4.1.5. *suppose $\theta(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ then,*

$$\theta\left(\frac{1}{t}\right) = -\frac{1}{2} + \frac{1}{2} t^{\frac{1}{t}} + \sqrt{t} \theta(t) \quad \text{when } t > 0$$

4.3 Mellin Transform

The Mellin Transform is an integral transform defined as

$$M(f)(s) = \int_0^{\infty} f(t) t^{s-1} dt$$

Examples: (1) As we know the gamma function is defined as

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$$

directly, we show that

$$\Gamma(s) = M(f)(s)$$

such that

$$f(t) = e^{-t}$$

we obvious that this is analytic continuation on $(0, \infty)$.

(2) Suppose that $f(x) = \frac{1}{e^x - 1}$ and let $Re(s)$ greater than 1

By mellin transform we have that

$$M(f)(s) = \int_0^{\infty} f(x)x^{s-1} dt$$

Since

$$\frac{1}{e^x - 1} = \left(\frac{e^x}{e^x}\right) \left(\frac{e^{-x}}{1 - e^{-x}}\right) = \sum_{n=1}^{\infty} e^{-nx}$$

Then

$$M(f)(s) = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-nx} x^{s-1} dt$$

Setting $nx = t$ and $dt = ndx$

$$\begin{aligned} &= \sum_{n=1}^{\infty} \int_0^{\infty} \left(\frac{t}{n}\right)^{s-1} \frac{e^{-t}}{n} dt \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^{\infty} t^{s-1} e^{-t} dt \end{aligned}$$

Hence, from the definition of the zeta function and the gamma function we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^{\infty} t^{s-1} e^{-t} dt = \zeta(s)\Gamma(s)$$

Remark: We will relate the theta function and the zeta function. Specially, there is a relationship between the mellin function and the zeta function of the theta function. We will see later this relationship in the third method of the functional equation.

4.4 The Dirichlet series

Definition 4.4. A Dirichlet series is a statement of the form

$$f(n) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

such that $a_n \in C$ and $s = \sigma + it$

Theorem 4.2. *Let the series $\sum_{n=1}^{\infty} \frac{a_n}{n^{s_1}}$ is convergent, hence for every closed region consisted in the region $R(s) > R(s_1)$ the series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is uniformly convergent. Thus, for $R(s) > R(s_1)$ the function $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is analytic.*

Theorem 4.3. *Assume that in the region $\sigma_1 < \sigma < \infty$ for $\forall \sigma$*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

$$h(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

and

$$F(\sigma) = h(\sigma)$$

Hence, $a_n = b_n$ for $\forall n = 1, 2, 3, 4, \dots$

Theorem 4.4. *If*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and

$$h(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

for $\sigma = \sigma_1$, these functions are definitely convergent and $|a_n| \leq b_n$

Hence

$$\int_{-L}^L |f(\sigma_1 + it)|^2 dt \leq 2 \int_{-2L}^{2L} |h(\sigma_1 + it)|^2 dt$$

for any $L \geq 0$

4.5 Abel Summation (Partial Summation)

Theorem 4.5. *Suppose $\phi(x)$ is a function whose values belong to C and this function has continuous derivative on the interval $[a, b]$, let c_m be arbitrary in C and $C(x) = \sum_{a \leq m \leq x} c_m$*
Then,

$$\sum_{a \leq n \leq b} c_m \phi(n) = - \int_a^b C(x) \phi'(x) dx + C(b) \phi(b)$$

Theorem 4.6. (*Euler Summation Formula*)

Assume $f(x)$ be a complex valued and $f(x)$ has continuous derivative on $[a, b]$ then,

$$\sum_{a \leq n \leq b} \phi(n) = \int_a^b \phi(x) dx + \int_a^b \lambda_1(x) \phi'(x) dx + \lambda_1(a) \phi(a) - \lambda_1(b) \phi(b)$$

where, $\lambda_1(x) = x - [x] - \frac{1}{2}$

Theorem 4.7. From Theorem 4.5 , let $c_m = 1$ Then, $C[x] = [x] - [a]$

$$\begin{aligned} \sum_{a \leq n \leq b} \phi(n) &= ([b] - [a]) \phi(b) - \int_a^b ([x] - [a]) \phi'(x) dx \\ &= ([b] - [a]) \phi(b) + [a] (\phi(b) - \phi(a)) - \int_a^b [x] \phi'(x) dx \end{aligned}$$

We have $\lambda_1(x) = x - [x] - \frac{1}{2}$ also, we can be written as $[x] = x - \frac{1}{2} - \lambda_1(x)$

Therefore, we get

$$\begin{aligned} \sum_{a \leq n \leq b} \phi(n) &= ((b - \lambda_1(b) - (a - \lambda_1(a))) \phi(b) + (a - \frac{1}{2} - \lambda_1(a)) (\phi(b) - \phi(a)) \\ &\quad - \int_a^b (x - \frac{1}{2} - \lambda_1(x)) \phi'(x) dx \\ &= \lambda_1(a) \phi(a) - \lambda_1(b) \phi(b) + b \phi(b) - a \phi(b) + \lambda_1(a) \phi(b) + a \phi(b) \\ &\quad - a \phi(a) - \frac{1}{2} (\phi(b) - \phi(a)) - \lambda_1(a) \phi(b) - x \phi(x) \Big|_a^b + \frac{1}{2} \phi(b) - \phi(a) + \int_a^b \phi(x) dx + \int_a^b \lambda_1(x) \phi'(x) dx \end{aligned}$$

Hence, we obtain the required result, after canceling.

4.6 Fourier Transforms Functions

[10] If the function $F(x)$ is an even, then the Fourier integral represent as

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos xy \, dy \int_0^\infty \cos yt \, f(t) dt \quad (4.4)$$

this formula called the Fourier's cosine.

Similarly, If the function $F(x)$ is an odd, then we obtain that

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin xy \, dy \int_0^{\infty} \sin yt \, f(t) dt \quad (4.5)$$

this formula called the Fourier's sine. If we say

$$h(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \cos xt \, f(t) \, dt$$

Then, 4.4 provides:

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \cos xt \, h(t) \, dt$$

Hence, the functions $F(x)$ and $h(x)$ have a reciprocal relation between them. Likewise, from 4.5 we find that the reciprocal formula

$$g(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \sin xt \, f(t) \, dt$$

and

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \cos xt \, g(t) \, dt$$

4.6.1 Integration of Fourier Integral

By 4.4 we obtain that

$$\int_0^{\epsilon} f(x) \, dx = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \epsilon y}{y} \, dy \int_0^{\infty} \cos yt \, f(t) \, dt$$

it holds over the interval $(0, \infty)$ for any integral function $f(t)$

$$\begin{aligned} & \int_0^y \frac{\sin \epsilon y}{y} \, dy \int_0^{\infty} \cos yt \, f(t) \, dt \\ &= \int_0^{\infty} f(t) \, dt \int_0^y \frac{\sin \epsilon y \cos yt}{y} \, dy \end{aligned}$$

for every y and t the uniform convergent and inner product on the right is bounded.

Therefore, we obtain that

$$\frac{1}{2} \int_0^y \frac{\sin(\epsilon + t)y}{y} \, dy + \frac{1}{2} \int_0^y \frac{(\sin \epsilon - t)y}{y} \, dy$$

$$= \frac{1}{2} \int_0^y (\epsilon + t) \frac{\sin u}{u} du \pm \frac{1}{2} \int_0^y (\epsilon - t) \frac{\sin u}{u} du$$

Now, we write the sign of $\epsilon - t$ and

$$\int_0^U \frac{\sin u}{u} du$$

this formula is bounded function of U . Therefore, by Lebesgue's convergence theorem and when y goes to infinity under the integral sign

$$\int_0^\infty \frac{\sin \epsilon y \cos yt}{y} dy = \frac{\pi}{2}$$

5 The Riemann Zeta Function $\zeta(s)$

5.1 Definition of $\zeta(s)$:

The main definition in this dissertation is the definition of the zeta function. In this section, we will state the definition of the zeta function by two ways and we will prove these definitions. In addition, there are some properties of $\zeta(s)$ we will present these properties with proofs. Note that the information in this section had been taken from the references [6] and [7]

The Riemann zeta function defined by simple formula in two representations, the first representation of $\zeta(s)$ by the Dirichlet series

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } s = \sigma + it, \sigma > 1 \quad (5.1)$$

If $\ell > 1$ and $\sigma \geq \ell$, since $\left| \frac{1}{n^s} \right| = \left| \frac{1}{n^\sigma} \right|$ Then,

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| = \sum_{n=1}^{\infty} \left| \frac{1}{n^\sigma} \right| \leq \sum_{n=1}^{\infty} \left| \frac{1}{n^\ell} \right| < \infty$$

Hence, it is uniformly converges. Moreover, the Riemann zeta function is analytic for $\sigma > 1$.

5.2 The Euler Product:

Theorem 5.1. *The second way of $\zeta(s)$ due to by Euler in 1779 defined by*

$$\zeta(s) = \prod \left(1 - \frac{1}{p^s}\right)^{-1}$$

for $s = \sigma + it$, $\sigma > 1$ where, this product is taken over every prime number.

Proof. Suppose $X \geq 2$, let we have the function $\zeta_X(s)$ we defined it as

$$\zeta_X(s) = \prod_{p \leq X} \left(1 - \frac{1}{p^s}\right)^{-1} \quad (5.2)$$

with every factors on the right hand side, we can write the term by a geometric series

$$\frac{1}{1 - p^s} = \sum_{\ell=0}^{\infty} \frac{1}{p^{\ell s}}$$

such that every geometric series is convergent. Therefore, we can multiply term by term and substitute the right hand side with this equation:

$$\zeta_X(s) = \prod_{p \leq X} \sum_{\ell=0}^{\infty} \frac{1}{p^{\ell s}} \sum_{\ell_1=0}^{\infty} \dots \sum_{\ell_j=0}^{\infty} \frac{1}{(p_1^{\ell_1} \dots p_j^{\ell_j})^s} \quad (5.3)$$

such that $p_1 < p_2 < \dots < p_j$ and p_j represents all the prime numbers up to X .

By the fundamental theorem of arithmetic (unique factorization of integers). We see that every positive integer number $n \leq X$ can be represented as

$$n = p_1^{\ell_1} \dots p_j^{\ell_j}$$

when the $\ell_1, \dots, \ell_j$ are positive integers. As a result, can be taken the right side of 5.3 as

$$\sum_{n \leq X} \frac{1}{n^s} + \sum'_{n > X} \frac{1}{n^s} \quad (5.4)$$

where the ' is stands for summation over those natural numbers (n greater than X) whose primes divisors are all less than or equal X . We can provide an upper bound of this sum as

$$\left| \sum'_{n > X} \frac{1}{n^s} \right| \leq \sum'_{n > X} \frac{1}{n^\sigma} < \sum_{n > X} \frac{1}{n^\sigma} \leq \frac{1}{X^\sigma} + \int_X^\infty \frac{du}{u^\sigma}$$

$$= \frac{1}{X^\sigma} + \frac{u^{-\sigma+1}}{-\sigma+1} \Big|_X^\infty$$

$$= \frac{1}{X^\sigma} + \frac{X^{\sigma-1}}{\sigma-1} \leq \frac{\sigma}{\sigma-1} X^{\sigma-1}$$

The last formula provides an upper bound of the sum. Now, if we combine the definition of the zeta function $\zeta_X(s)$ and (5.2),(5.1),(5.4) we find that

$$\zeta_X(s) = \prod_{p \leq X} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n \leq X} \frac{1}{n^s} + O\left(\frac{\sigma}{\sigma-1} X^{1-\sigma}\right)$$

If X goes to $+\infty$ we obtain $X^{1-\sigma}$ goes to $+\infty$ as well. Since $\sigma > 1$ and thus, we proved that

$$\sum_{n \leq X} \frac{1}{n^s} = \prod_{p \leq X} \left(1 - \frac{1}{p^s}\right)^{-1}$$

□

Now, we present an important properties of Riemann zeta function and Analytic continuation.

Theorem 5.2. *The function $\zeta(s) - \frac{1}{s-1}$ extend to the right half plane for $\sigma > 0$ and this function is analytic when $\sigma > 0$.*

Proof. consider $f_n(s) = \zeta(s) - \frac{1}{s-1}$

Firstly, we will prove $f_n(s)$ is convergent then we will find that $f_n(s) = \zeta(s) - \frac{1}{s-1}$

By the equation 5.1 we have that

$$f_n(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s} dx$$

$$= \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx$$

Then

$$|f_n(s)| = \left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \right| \leq \max_{n \leq x \leq n+1} \left| \frac{1}{n^s} - \frac{1}{x^s} \right|$$

However, we know that

$$\frac{1}{n^s} - \frac{1}{x^s} = \int_n^x s u^{-s-1} du$$

Hence, we have

$$\left| \frac{1}{n^s} - \frac{1}{x^s} \right| \leq |s| \int_n^{n+1} u^{-\sigma-1} du \leq \frac{|s|}{\sigma} \left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right)$$

We obtain

$$\sum_{n=1}^{\infty} |f_n(s)| \leq \frac{|s|}{\sigma} \sum_{n=1}^{\infty} \left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right) = \frac{|s|}{\sigma}$$

Since, $f_n(s)$ is entire function. So, $\sum_{n=1}^{\infty} |f_n(s)|$ is converges. In addition, $\sum_{n=1}^{\infty} |f_n(s)|$ is analytic function for $\sigma > 1$.

$$\begin{aligned} \sum_{n=1}^{\infty} f_n(s) &= \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s} dx \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^s} dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \left(\frac{(n+1)^{-s+1} - n^{-s+1}}{-s+1} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{1}{s-1} \end{aligned}$$

By the equation 5.1 we get

$$= \zeta(s) - \frac{1}{s-1}$$

Thus, for $\sigma > 1$ $\zeta(s) = \sum_{n=1}^{\infty} f_n(s) + \frac{1}{s-1}$

Since, for $\sigma > 0$ $f_n(s)$ and $\frac{1}{s-1}$ are analytic, we show that $\sum_{n=1}^{\infty} f_n(s) + \frac{1}{s-1}$ is analytic continuation of the zeta function for $\sigma > 0$ with a simple pole at $s = 1$.

□

Theorem 5.3. *suppose*

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \quad (5.5)$$

such that $\xi(s)$ is analytic continuation on the entire plane also $\phi(s)$ satisfies

$$\xi(s) = \xi(1-s)$$

Proof. Using the definition of the zeta function and the gamma function. Since

$$\left(\frac{s}{2}\right) = \int_0^\infty x^{\frac{s}{2}-1} e^{-x} dx \quad (5.6)$$

and

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (5.7)$$

Applying (5.6) and (5.7) in (5.5) we find that

$$\begin{aligned} \xi(s) &= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{s}{2}} \left(\int_0^\infty x^{\frac{s}{2}-1} e^{-x} dx \right) \left(\sum_{n=1}^{\infty} \frac{1}{n^s} \right) \\ &= \sum_{n=1}^{\infty} \int_0^\infty \left(\frac{x}{n^2 \pi} \right)^{-s/2} e^{-x} dx \end{aligned}$$

Consider that $y = \frac{x}{\pi n^2}$ then $x = y\pi n^2$

$$\xi(s) = \sum_{n=1}^{\infty} \int_0^\infty y^{-\frac{s}{2}-1} e^{-\pi n^2 y} dy = \int_0^\infty y^{\frac{s}{2}-1} \psi(y) dy$$

Now, we divide the integral as

$$\xi(s) = \int_0^1 y^{\frac{s}{2}-1} \psi(y) dy + \int_1^\infty y^{\frac{s}{2}-1} \psi(y) dy$$

changing the variable $y = \frac{1}{u}$

then

$$\xi(s) = \int_0^1 u^{-\frac{s}{2}-1} \psi(y) dy + \int_1^\infty y^{\frac{s}{2}-1} \psi(y) dy$$

According to the corollary (4.1.2) we find that

$$\begin{aligned} \xi(s) &= \int_1^\infty u^{-\frac{s}{2}-1} \left(-\frac{1}{2} + \frac{1}{2} u^{\frac{1}{2}} + u^{\frac{1}{2}} \psi(u) \right) du + \int_1^\infty y^{\frac{s}{2}-1} \psi(y) dy \\ &= -\frac{1}{2} \int_1^\infty u^{-\frac{s}{2}-1} du + \frac{1}{2} \int_0^\infty u^{-\frac{s}{2}-\frac{1}{2}} du + \int_0^\infty u^{-\frac{s}{2}-1} \psi(u) du + \int_1^\infty y^{\frac{s}{2}-1} \psi(y) dy \\ &\quad -\frac{1}{s} + \frac{1}{s-1} + \int_0^\infty u^{\frac{s}{2}-1} \psi(u) du + \int_1^\infty y^{\frac{s}{2}-1} \psi(y) dy \end{aligned}$$

assume that

$$K(s) = \int_0^\infty y^{-\frac{s}{2}-1} \psi(y) dy + \int_1^\infty y^{\frac{s}{2}-1} \psi(y) dy$$

Therefore we obtain,

$$\xi(s) = -\frac{1}{s} + \frac{1}{s-1} + K(s)$$

$K(s)$ is entire function. In addition, since $\frac{1}{s}$ and $\frac{1}{s-1}$ are analytic then, $\xi(s) = -\frac{1}{s} + \frac{1}{s-1} + K(s)$ is analytic continuation of $\xi(s)$

□

6 Functional Equations and Analytic continuation

One of our aims in this dissertation is prove the functional equation of $\zeta(s)$. In this section, we will give fourth different proofs of the functional equation. The proofs had been taken from the references [4], [5], [7] and [11]. One can see more methods in [11].

6.1 First Method

Theorem 6.1. *The function $\zeta(s)$ is regular for all value of s except $s=1$, where there is a simple pole with residue 1, it satisfies the functional equation*

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{1}{2} s \pi \Gamma(1-s) \zeta(1-s) \quad (6.1)$$

Proof. We will prove this theorem depending on the summation formula

assume $\phi(x)$ be function which has continuous Differentiable on $[a, b]$ and $[x]$ denotes the greatest integer less than x by Euler summation formula we have

$$\begin{aligned} \sum_{a < m \leq b} \phi(m) &= \int_a^b \phi(x) dx + \int_a^b (x - [x] - \frac{1}{2}) \phi'(x) dx + (a - [a] - \frac{1}{2}) \phi(a) \\ &\quad - (b - [b] - \frac{1}{2}) \phi(b) \end{aligned} \quad (6.2)$$

with respect to $(a, b]$, this formula is clearly additive. It is enough to assume that $n \leq a < b \leq n+1$. then

$$\int_a^b (x - n - \frac{1}{2}) \phi'(x) dx = (b - n - \frac{1}{2}) \phi(b) - (a - n - \frac{1}{2}) \phi(a) - \int_a^b \phi(x) dx \quad (6.3)$$

Now, we take $\sigma > 1$, $a = 1$, b goes to infinite and add 1 to both side then we get:

$$\zeta(s) = s \int_1^\infty \frac{x - [x] + \frac{1}{2}}{x^{s+1}} dx + \frac{1}{s-1} + \frac{1}{2} \quad (6.4)$$

We know the numerator $x - [x] + \frac{1}{2}$ is bounded and when $\sigma > 0$ the integral in (6.4) is convergent and (this integral is uniformly convergent in any finite region to the right of $\sigma = 0$).

Hence, for $\sigma > 0$, it defines analytic function of s regular. The right hand side gives the analytic function $\zeta(s)$ up to $\sigma = 0$ and it is clear that a simple pole at $s=1$ with residue 1.

For $0 < \sigma < 1$ we have

$$\int_0^1 \frac{[x] - x}{x^{s+1}} dx = - \int_0^1 x^{-s} dx = \frac{1}{s-1}$$

and

$$\frac{s}{2} = \int_1^\infty \frac{dx}{x^{s+1}} = \frac{1}{2}$$

By (6.4) we can write that, for $0 < \sigma < 1$

$$\zeta(s) = \int_0^\infty \frac{[x] - x}{x^{s+1}} dx \quad (6.5)$$

We have seen the function (6.4) provides the analytic continuation for $\sigma > -1$

$$\text{If } f(x) = [x] - x + \frac{1}{2} \quad f_1(x) = \int_1^x f(y) dy$$

Since, $\int_n^{n+1} f(y) dy = 0 \quad \forall n \text{ integer}$, then $f_1(x)$ is also bounded. Thus,

$$\int_{x_1}^{x_2} \frac{f(x)}{x^{s+1}} dx = \left[\frac{f_1(x)}{x^{s+1}} \right]_{x_1}^{x_2} + (s+1) \int_{x_1}^{x_2} \frac{f_1(x)}{x^{s+2}} dx \quad (6.6)$$

which goes to zero as $x_1 \rightarrow \infty$ and $x_2 \rightarrow \infty$

If $\sigma > -1$, hence the integral (6.4) is convergent also,

$$s \int_0^1 \frac{[x] - x + \frac{1}{2}}{x^{s+1}} = \frac{1}{s-1} + \frac{1}{2} \text{ for } (\sigma < 0) \quad (6.7)$$

Therefore,

$$\zeta(s) = s \int_0^\infty \frac{[x] - x + \frac{1}{2}}{x^{s+1}} dx \text{ for } -1 < \sigma < 0 \quad (6.8)$$

The function $[x] - x + \frac{1}{2}$ has a Fourier series expansion

$$[x] - x + \frac{1}{2} = \sum_{n=1}^{\infty} \frac{\sin 2nx\pi}{\pi n} \quad (6.9)$$

If x is not integer, we will substitute the series (6.9) in equation (6.8) Then we get:

$$\begin{aligned} \zeta(s) &= \frac{s}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} \frac{\sin 2nx\pi}{x^{s+1}} dx \\ &= \frac{s}{\pi} \sum_{n=1}^{\infty} \frac{(2n\pi)^s}{n} \int_0^{\infty} \frac{\sin y}{y^{s+1}} dy \\ \zeta(s) &= \frac{s}{\pi} (2n\pi)^s - \Gamma(-s) \sin s\pi \zeta(1-s) \end{aligned} \quad (6.10)$$

This permitted for $-1 < \sigma < 0$, but, for $\sigma < 0$ the right hand side is analytic for all value of s . Therefore, it gives the analytic continuation (not just for $-1 < \sigma < 0$), this mean it provides the analytic continuation all over the plane.

Hence, it is enough to prove

$$\lim_{\lambda \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1}{n} \int_{\lambda}^{\infty} \frac{\sin 2n\pi x}{x^{s+1}} dx = 0 \quad (-1 < \sigma < 0)$$

to justify the term-wise integration.

Since, the series in (6.9) is bounded convergent, the term wise integration is absolutely justified over any finite range. Then

$$\begin{aligned} \int_{\lambda}^{\infty} \frac{\sin 2nx\pi}{x^{s+1}} dx &= \left[\frac{-\cos 2n\pi x}{2n\pi(x^{s+1})} \right]_{\lambda}^{\infty} - \frac{s+1}{2n\pi} \int_{\lambda}^{\infty} \frac{\cos 2n\pi x}{x^{s+2}} dx \\ &= O\left(\frac{1}{n\lambda^{\sigma+1}}\right) + O\left(\frac{1}{n} \int_{\lambda}^{\infty} \frac{dx}{x^{\sigma+2}}\right) = O\left(\frac{1}{n\lambda^{\sigma+1}}\right) \end{aligned}$$

Thus

$$\lim_{\lambda \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1}{n} \int_{\lambda}^{\infty} \frac{\sin 2n\pi x}{x^{s+1}} dx = 0$$

as result of (6.4) we obtain

$$\lim_{s \rightarrow 1} \left(\zeta(s) - \frac{1}{s-1} \right) = \int_1^{\infty} \frac{[x] - x + \frac{1}{2}}{x^2} dx + \frac{1}{2}$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \int_1^n \frac{[x] - x}{x^2} dx + 1 \\
 &= \lim_{n \rightarrow \infty} \sum_{m=1}^{n-1} m \int_m^{m+1} \left(\frac{dx}{x^2} - \log n + 1 \right) \\
 &= \lim_{n \rightarrow \infty} \sum_{m=1}^{n-1} \frac{1}{m+1} + 1 - \log n = \gamma
 \end{aligned}$$

Thus, we get

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$$

(near $s=1$)

□

Note:

We can write the function (6.1) as different ways, if we change s into $1 - s$ we obtain that

$$(1 - s) = 2^{1-s} \pi^{1-s} \cos \frac{1}{2} s \pi \Gamma(s) \zeta(s) \quad (6.11)$$

It could be written

$$\zeta(s) = \chi(s) \zeta(1 - s) \quad (6.12)$$

where

$$\chi(s) = 2^s \pi^{s-1} \sin \frac{1}{2} 2\pi \Gamma(1 - s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}$$

and

$$\chi(s) \chi(1 - s) = 1$$

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s)$$

it is exactly proved from (6.11) and (6.12) that

$$\xi(s) = \xi(1 - s)$$

$$\text{if } s = \frac{1}{2} + iz \quad \text{and } z \in C$$

$$\xi\left(\frac{1}{2} + iz\right) = \xi\left(1 - \left(\frac{1}{2} + iz\right)\right)$$

$$\Xi(z) = \left(\frac{1}{2} + iz\right)$$

$$\Xi(z) = \Xi(-z)$$

* Hardy applied similar argument in first method , However not to it self, to this function

$$(1 - 2^{1-s}) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

For $Re(s) > 0$, the Diriclet series is convergent. By the zero of other factor, the pole of Riemann zeta function at $s=1$ is cancelled.

Hardy's proof: one can see the proof in reference [11]

6.2 Second Method

The entire theory can be improved in another method, which is one of Riemann's ways. The fundamental formula is:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \quad \text{for } \sigma > 1 \quad (6.13)$$

Proof. to prove this formula, we have

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx \quad (6.14)$$

if we replace the variable $x = nx$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} &= \sum_{n=1}^{\infty} \int_0^{\infty} x^{s-1} e^{-nx} dx \\ &= \int_0^{\infty} \left[\sum_{n=1}^{\infty} e^{-nx} \right] x^{s-1} dx \quad \text{for } \sigma > 1 \end{aligned} \quad (6.15)$$

Since,

$$\sum_{n=1}^{\infty} \left[\int_0^{\infty} x^{s-1} e^{-nx} dx \right] \leq \sum_{n=1}^{\infty} \int_0^{\infty} x^{\sigma-1} e^{-nx} dx = \sum_{n=1}^{\infty} \frac{\Gamma(\sigma)}{n^{\sigma}} < \infty \quad (6.16)$$

this is definite convergent for $\sigma > 1$ Hence

$$\sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} = \int_0^{\infty} \frac{e^{-x}}{1 - e^{-x}} \cdot \left(\frac{e^x}{e^x} \right) x^{s-1} dx = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

$$\Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

By the definition of $\zeta(s)$ we find that

$$\Gamma(s) \zeta(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

Now, we assume the complex integral

$$I(s) = \int_c \frac{z^{s-1}}{e^z - 1} dz$$

such that the contour C containing of the positive real axis from ∞ to ρ for $0 < \rho < 2\pi$,
the circle $|z| = \rho$ and the positive real axis from ρ to ∞ then, on the circle $|z| = \rho$,

$$\begin{aligned} |z^{s-1}| &= |e^{(s-1) \log z}| = |e^{((\sigma-1)+it)(\log|z|-i \arg z)}| \\ &= e^{(\sigma-1)\log|z|-t \arg z} \\ &\leq |z|^{\sigma-1} e^{2\pi|t|} \\ |e^z - 1| &> A|z| \end{aligned}$$

Therefore, the integral round this circle goes to 0 as $\rho \rightarrow 0$ for $\sigma > 1$

Hence on letting $\rho \rightarrow 0$, we obtain

$$\begin{aligned} I(s) &= - \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx + \int_0^\infty \frac{(xe^{2\pi i})^{s-1}}{e^x - 1} dx \\ &= -\Gamma(s)\zeta(s) + e^{2\pi is}\Gamma(s)\zeta(s) \\ &= \Gamma(s)\zeta(s)(e^{2\pi is} - 1) \end{aligned}$$

by using the result

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

$$\begin{aligned} I(s) &= \frac{\zeta(s)}{\Gamma(1-s)} 2\pi i \frac{e^{2\pi is}}{e^{\pi is} - e^{-\pi is}} \\ &= \frac{\zeta(s)}{\Gamma(1-s)} 2\pi i e^{\pi is} \end{aligned}$$

Hence,

$$\zeta(s) = \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} \int_c \frac{z^{s-1}}{e^z - 1} dz$$

This integral is convergent in any finite region of the s -plane. Therefore, it defines an entire function. The zeta function defines as a meromorphic function over the whole plane. The poles of $\Gamma(1-s)$ are merely possible poles. On other words, $s = 1, 2, 3, \dots$. We see that $\zeta(s)$ is regular for all values of s except $s = 1$. Thus, the only possible pole is at $s = 1$

$$I(1) = \int_c \frac{dz}{e^z - 1} = 2\pi i \quad \text{and} \quad \Gamma(1-s) = -\frac{1}{s-1} + \dots$$

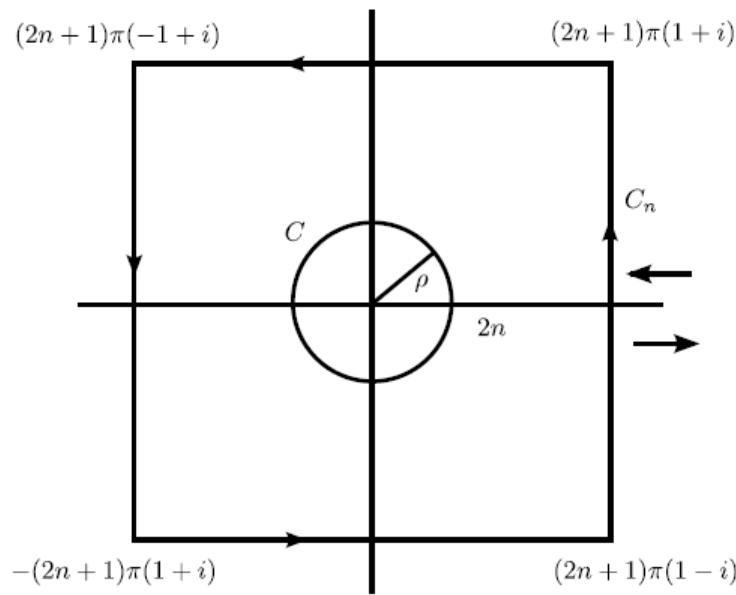
Therefore, the residue of $\zeta(s)$ at the pole is 1, if $s \in \mathbb{N}$ and since

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2}z + B_1 \frac{z^2}{2!} - B_2 \frac{z^3}{4!} + \dots$$

such that $(B_1, B_2, \dots, \text{Bernoulli numbers})$.

$$\text{Then, } \zeta(0) = -\frac{1}{2}, \quad \zeta(-2m) = 0, \quad \zeta(1-2m) = \frac{(-1)^m B_m}{2m}$$

such that $m = 1, 2, 3, \dots$



Let the integral $\int \frac{z^{s-1}}{e^z - 1} dz$ take the integral along the contour C_n as in the diagram, There is the integral between C and C_n which has poles at the points $\pm 2i\pi, \dots, \pm 2in\pi$ the residue at $-2mi\pi$ is $(2m\pi e^{\frac{3}{2}i\pi})^{s-1}$ and the residue at $2mi\pi$ is $(2m\pi e^{\frac{1}{2}i\pi})^{s-1}$. taking together

$$\begin{aligned} & (2m\pi e^{\frac{1}{2}i\pi})^{s-1} + (2m\pi e^{\frac{3}{2}i\pi})^{s-1} \\ &= (2m\pi)^{s-1} e^{i\pi(s-1)} 2 \cos \frac{1}{2}\pi(s-1) \\ &= -2(2m\pi)^{s-1} e^{i\pi s} \sin \frac{1}{2}\pi(s-1) \end{aligned}$$

By theorem of residue we have

$$I(s) = \int_{C_n} \frac{z^{s-1}}{e^z - 1} dz + 4\pi e^{i\pi s} \sin \frac{1}{2}\pi s \sum_{m=1}^n (2m\pi)^{s-1}$$

Now, consider $\sigma > 0$ and n goes to infinity, on the contours C_n , the function $\frac{1}{(e^z - 1)}$ is bounded.

$$|z^{s-1}| = O(|z|^{\sigma-1})$$

Therefore, the integral round C_n goes to zero and we get

$$\begin{aligned} I(s) &= 4 \pi i e^{i\pi s} \sin \frac{1}{2} \pi s \sum_{n=1}^{\infty} (2m\pi)^{s-1} \\ &= 4\pi i e^{i\pi s} \sin \frac{1}{2} (2\pi)^{s-1} \Gamma(1-s) \end{aligned}$$

* When $Re(s) < 0$ and therefore by analytic continuation for $\forall s$

Since we have

$$\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) = \frac{\sqrt{\pi}}{2^{s-1}} \Gamma(s)$$

Changing the variables from s to $1-s$ we obtain that

$$\Gamma\left(\frac{1-s}{2}\right) \Gamma\left(\frac{1-s+1}{2}\right) = \frac{\sqrt{\pi}}{2^{1-s-1}} \Gamma(s)$$

Then

$$\Gamma\left(1 - \frac{s}{2}\right) 2^{-s} = \sqrt{\pi} \Gamma(1-s)$$

Using this result

$$\Gamma(s) \Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

Therefore, we have

$$\Gamma\left(\frac{s}{2}\right) \Gamma\left(1 - \frac{s}{2}\right) = \frac{\pi}{\sin \frac{s\pi}{2}}$$

$$\Gamma(1-s) = 2^s \Gamma\left(1 - \frac{s}{2}\right) \Gamma\left(\frac{s}{2}\right)^{-1} \sqrt{\pi} \left(\sin \frac{\pi s}{2}\right)^{-1}$$

Hence

$$\pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s) = \pi^{-\frac{1}{2}s-1} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

□

6.3 Third Method

In this method we will prove this equation by the Gamma function

$$\pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s) = \pi^{-\frac{1}{2}s-1} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \quad (6.17)$$

The relationship between Gamma and Riemann

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt \quad (6.18)$$

We will replace from s to $\frac{s}{2}$

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} t^{\frac{s}{2}-1} e^{-t} dt$$

If $t = \pi n^2 x$ and $dt = \pi n^2 dx$, Then

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} (\pi n^2 x)^{\frac{s}{2}-1} e^{-\pi n^2 x} \pi n^2 dx$$

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} \pi^{\frac{s}{2}} n^s x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \frac{1}{n^s} = \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

We add sum to obtain the zeta function

$$\sum_{n=1}^{\infty} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \frac{1}{n^s} = \sum_{n=1}^{\infty} \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

Hence,

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^{\infty} x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx$$

since,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Hence,

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx$$

Now, we write

$$w(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx$$

and

$$\begin{aligned} \theta(x) &= \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} \\ &= 1 + 2 w(x) \end{aligned}$$

$$\begin{aligned} \int_0^{\infty} x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx &= \int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx \\ \int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx &= \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_0^1 x^{\frac{s}{2}-1} w(x) dx \end{aligned}$$

$$\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)$$

or

$$\begin{aligned} 2w(x) + 1 &= \frac{1}{\sqrt{x}} \left[2w\left(\frac{1}{x}\right) + 1 \right] \\ 2w(x) &= \frac{1}{\sqrt{x}} \left[2w\left(\frac{1}{x}\right) + 1 \right] - 1 \\ w(x) &= \frac{1}{\sqrt{x}} \left[\frac{2w\left(\frac{1}{x}\right) + 1}{2} \right] - \frac{1}{2} \end{aligned}$$

it follows that

$$\begin{aligned} w(x) &= \frac{1}{\sqrt{x}} w\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \\ \int_0^1 x^{\frac{s}{2}-1} w(x) dx &= \int_0^1 x^{\frac{s}{2}-1} \left(\frac{1}{\sqrt{x}} w\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \right) dx \\ &= \int_0^1 \left[x^{\frac{s}{2}-\frac{3}{2}} w\left(\frac{1}{x}\right) + \frac{1}{2} \left(x^{\frac{s}{2}-\frac{3}{2}} - x^{\frac{s}{2}-1} \right) \right] dx \\ &= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}} w\left(\frac{1}{x}\right) dx + \frac{1}{2} \int_0^1 \left[x^{\frac{s}{2}-\frac{3}{2}} - x^{\frac{s}{2}-1} \right] dx \\ &= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}} w\left(\frac{1}{x}\right) dx + \frac{1}{2} \left[\frac{1}{\frac{s}{2}-\frac{1}{2}} x^{\frac{s}{2}-\frac{1}{2}} - \frac{1}{\frac{s}{2}} x^{\frac{s}{2}} \right] dx \end{aligned}$$

$$= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}} w\left(\frac{1}{x}\right) + \frac{1}{s(s-1)}$$

Now, if we change $x = \frac{1}{u}$ and $dx = -\frac{1}{u^2} du$

Hence

$$= \int_{\infty}^1 \left(\frac{1}{u}\right)^{\frac{s}{2}-\frac{3}{2}} w(u) \left(-\frac{du}{u^2}\right) + \frac{1}{s(s-1)}$$

Now

$$= \int_1^{\infty} \left(\frac{1}{x}\right)^{\frac{s}{2}-\frac{3}{2}} w(x) \left(-\frac{dx}{u^2}\right) + \frac{1}{s(s-1)} \quad |_0^1 \rightarrow |_{\infty}^1$$

$$\int_0^1 x^{\frac{s}{2}-1} w(x) dx = \int_1^{\infty} x^{-\frac{s}{2}-\frac{1}{2}} dx + \frac{1}{s(s-1)}$$

$$\int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx = \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_0^1 x^{\frac{s}{2}-1} w(x) dx$$

$$\int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx = \int_1^{\infty} x^{\frac{s}{2}-1} w(x) dx + \int_1^{\infty} x^{-\frac{s}{2}-\frac{1}{2}} dx + \frac{1}{s(s-1)}$$

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1} w(x) dx$$

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^{\infty} \left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right) \frac{w(x)}{x} dx - \frac{1}{s(s-1)}$$

Finally, we obtain

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

6.4 Fourth Method

This method depend on self-reciprocal function. If $\sigma > 1$ could be written

$$\zeta(s)\Gamma(s) = \int_0^1 \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{s-1} dx + \frac{1}{s-1} + \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \quad (6.19)$$

For σ greater than zero and analytic continuation (4.6.1) is hold. Moreover, if $0 < \sigma < 1$ then

$$\frac{1}{s-1} = - \int_1^\infty \frac{x^{s-1}}{x} dx \quad (6.20)$$

we apply(6.20) in (6.19) , Hence we obtain that

$$\zeta(s)\Gamma(s) = \int_0^1 \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{s-1} dx \quad for \ 0 < \sigma < 1 \quad (6.21)$$

The function $f(x) = \frac{1}{e^{x\sqrt{(2\pi)}} - 1} - \frac{1}{x\sqrt{(2\pi)}}$ is self-reciprocal for transforms

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(y) \sin xy \ dy$$

Therefore, letting $x = \epsilon\sqrt{(2\pi)}$ in (6.21)

$$\zeta(s)\Gamma(s) = (2\pi)^{\frac{1}{2}s} \int_0^\infty f(\epsilon) \epsilon^{s-1} d\epsilon \int_0^\infty f(y) \sin \epsilon y \ dy$$

let we invert the integration, then

$$2^{\frac{1}{2}s+\frac{1}{2}} \pi^{\frac{1}{2}s-\frac{1}{2}} \int_0^\infty f(y) \ dy \int_0^\infty \epsilon^{s-1} \sin \epsilon y \ d\epsilon$$

Put $\epsilon = u$

$$\begin{aligned} & 2^{\frac{1}{2}s+\frac{1}{2}} \pi^{\frac{1}{2}s-\frac{1}{2}} \int_0^\infty f(y) \ dy \int_0^\infty u^{s-1} \sin u \ du \\ &= 2^{\frac{1}{2}s+\frac{1}{2}} \pi^{\frac{1}{2}s-\frac{1}{2}} (2\pi)^{\frac{1}{2}s-\frac{1}{2}} \Gamma(1-s) \zeta(1-s) \frac{\pi}{2 \cos \frac{1}{2}\pi s \Gamma(1-s)} \end{aligned}$$

Now,we justify the investigation ,we know that the integral $\int_0^\infty f(y) \sin \epsilon y \ dy$ is uniformly convergent when $0 < \delta \leq \epsilon \leq \Delta$ and it is enough to prove this function equal to zero.

$$\lim_{\delta \rightarrow 0} \lim_{\Delta \rightarrow \infty} \int_0^{\infty} f(y) dy \left(\int_0^{\delta} + \int_{\Delta}^{\infty} \right) \epsilon^{s-1} \sin \epsilon y d\epsilon = 0$$

$$\int_0^{\delta} \epsilon^{s-1} \sin \epsilon y d\epsilon = \int_0^{\delta} O(\epsilon^{s-1} \epsilon y) d\epsilon = O(\delta^{\sigma+1} y)$$

Furthermore, we have

$$= y^{-s} \int_0^{\delta y} u^{s-1} \sin u du = O(y^{-\sigma})$$

Since,

$$f(y) = O(1)$$

as y tend to zero and $O(y^{-1})$ as y tend to infinity, we get that

$$\int_0^{\infty} f(y) dy \int_0^{\delta} \epsilon^{s-1} \sin \epsilon y d\epsilon = \int_0^1 O(\delta^{\sigma+1} y) dy + \int_1^{\frac{1}{\delta}} O(\delta + 1) dy + \int_{\frac{1}{\delta}}^{\infty} O(y^{-\delta-1}) dy$$

$$= O(\epsilon^{\delta}) \rightarrow 0$$

7 Generalization of $\zeta(s)$

7.1 The Dirichlet L-Function

[16] There are many various ways in which the zeta function has been generalized. We will introduce one of these generalizations. In this section, we will give the definition of the Dirichlet L-Function, Analytic continuation of $L(s, \chi)$ and the functional equation of $L(s, \chi)$. Most of information taken from the references [7] and [16].

The function $L(s, \chi)$ was introduced by Dirichlet in 1837, it defined as

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

This function is a holomorphic function for the real part of a complex variable s greater than 1. Moreover, the L-function can be written as an Euler's product for $\zeta(s)$

$$L(s, \chi) = \prod \left(1 - \frac{\chi(p)}{p^s} \right)^{-1} \quad (7.1)$$

For $Re(s) > 1$

The proof of (7.1) is precisely the same proof of theorem (5.1)

Assume $\chi(n) = \chi_1(n)$ where $\chi_1(n)$ is principal character modulo ℓ and if

$$\chi_1(n) = \begin{cases} 1 & \text{if } (\ell, n) = 1 \\ 0 & \text{if } (\ell, n) > 1 \end{cases} \quad (7.2)$$

$$(7.3)$$

For the real part $Re(s) > 1$, the $L(s, \chi)$ connected with a character χ modulo ℓ

$$L(s, \chi_1) = \prod_p \left(1 - \frac{\chi_1(p)}{p^s} \right)^{-1}$$

Applying (7.2) in (7.1) we have that

$$\begin{aligned} &= \prod_{(p, \ell)=1} \left(1 - \frac{1}{p^s} \right)^{-1} \\ &= \prod_{p|\ell} \left(1 - \frac{1}{p^s} \right) \prod_p \left(1 - \frac{1}{p^s} \right)^{-1} \end{aligned}$$

By this relation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s} \right)^{-1}$$

We obtain that

$$L(s, \chi_1) = \zeta(s) \prod_{p|\ell} \left(1 - \frac{1}{p^s}\right)$$

The different between $L(s, \chi_1)$ and $\zeta(s)$ merely by a simple factor.

7.2 Analytic continuation of $L(s, \chi_1)$ for $Re(s) > 0$

By following next lemma , the analytic continuation is obtained

Lemma 7.1. *Assume*

$$S(x) = \sum_{n \leq x} \chi(n)$$

such that $\chi(n)$ is non principal character modulo ℓ . Then, for $Re(s)$ greater than zero we have that

$$L(s, \chi) = s \int_1^{\infty} S(x) x^{-s-1} dx$$

Proof. Suppose $Re(s) > 1$ and $N \geq 1$

we use Abel summation (partial summation)

$$\sum_{n \leq N} \chi(n) n^{-s} = - \int_1^N c(x) (x^{-s})' + c(N) N^{-s}$$

taking derivative of (x^{-s})

$$\begin{aligned} &= - \int_1^N c(x) ((-s)x^{-s-1}) + c(N) N^{-s} \\ &= s \int_1^N c(x) x^{-s-1} + c(N) N^{-s} \\ &= 1 + s \int_1^N c(x) x^{-s-1} + c(N) N^{-s} \end{aligned}$$

Such that $c(x) = S(x) - 1$ and $x \geq |c(x)|$, let N goes to infinity we get that

$$\begin{aligned} L(s, \chi) &= 1 + s \int_1^{\infty} c(x) x^{-s-1} dx \\ &= 1 + s \int_1^{\infty} (S(x) - 1) x^{-s-1} dx \end{aligned}$$

$$= s \int_1^{\infty} S(x) x^{-s-1} dx$$

The lemma is proved. □

Corollary 7.1.1.

$$L(s, \chi) = s \int_1^{\infty} S(x) x^{-s-1} dx$$

gives an analytic continuation of L-function $L(s, \chi)$ for $\text{Re}(s) > 0$

7.3 Functional equation for $L(s, \chi)$

Theorem 7.2. *Assume*

$$\delta = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases} \quad (7.4)$$

$$(7.5)$$

such that χ is primitive character modulo k

The function (ξ, χ) defined as

$$\xi(s, \chi) = (\pi k^{-1})^{-(s+\delta/2)} \Gamma\left(\frac{s+\delta}{2}\right) L(s, \chi)$$

Then, we have the identity

$$\xi(1-s, \bar{\chi}) = \frac{i^\delta \sqrt{k}}{g(\chi)} \xi(s, \chi)$$

before prove this theorem, we present an important Lemma for this theorem

Lemma 7.3. ([7] Lemma 4.2)

Suppose that χ is primitive character modulo k , if χ is even character define $\theta(\tau, \chi)$ by setting

$$\theta(\tau, \chi) = \sum_{n=-\infty}^{\infty} \chi(n) \exp(-\pi \tau n^2 / k) \quad , \tau > 0 \quad (7.6)$$

if χ is an odd character defined

$$\theta_1(\tau, \chi) = \sum_{n=-\infty}^{\infty} n \chi(n) \exp(-\pi \tau n^2 / k) \quad , \tau > 0 \quad (7.7)$$

These functions $\theta(\tau, \chi)$ and $\theta_1(\tau, \chi)$ satisfy the following functional equations

$$g(\bar{\chi}) \theta(\tau, \chi) = \sqrt{k\tau^{-1}} \theta(\tau^{-1}, \bar{\chi}) \quad (7.8)$$

$$g(\overline{\chi}) \theta_1(\tau, \chi) = \sqrt{k\tau^{-3}} \theta(\tau^{-1}, \overline{\chi}) \quad (7.9)$$

where $g(\chi)$ is Gauss sum

$$g(\chi) = \sum_{a=1}^k \exp(2\pi i a/k)$$

Proof. the theorem (7.1)

Firstly, let χ is an even character as in (7.4), then

$$\Pi^{-s/2} k^{s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi) = \int_0^\infty \tau^{s/2-1} \left(\sum_{n=1}^\infty \chi(n) \exp(-\pi \tau n^2/k) d\tau \right) \quad (7.10)$$

Using lemma (7.6) we have that

$$\sum_{n=1}^\infty \chi(n) \exp(-\pi \tau n^2/k) = \frac{1}{2} \theta(\tau, \chi) \quad (7.11)$$

Applying (7.11) in (7.10), we obtain that

$$\Pi^{-s/2} k^{s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi) = \frac{1}{2} \int_0^\infty \tau^{s/2-1} \theta(\tau, \chi) d\tau$$

Dividing the integral into two parts, changing the variable from τ to τ^{-1} , consequently, we have

$$\begin{aligned} \Pi^{-s/2} k^{s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi) &= \frac{1}{2} \int_1^\infty \tau^{s/2-1} \theta(\tau, \chi) d\tau + \frac{1}{2} \int_1^\infty \tau^{-s/2-1} \theta(\tau^{-1}, \chi) d\tau \\ &= \frac{1}{2} \int_1^\infty \tau^{s/2-1} \theta(\tau, \chi) d\tau + \frac{1}{2} \frac{\sqrt{k}}{g(\overline{\chi})} \int_1^\infty \tau^{-s/2-1} \theta(\tau, \overline{\chi}) d\tau \end{aligned} \quad (7.12)$$

Therefore, (7.12) gives an analytic continuation of L-function $L(s, \chi)$ onto the whole s-plane. $L(s, \chi)$ is regular because $\Gamma\left(\frac{s}{2}\right) \neq 0$

Using the fact

$$g(\chi)g(\overline{\chi}) = g(\chi)\overline{g(\chi)} = k$$

Since, $\chi(-1) = 1$ we show that if we changed the variables s by $1-s$ and χ by $\overline{\chi}$. The Right hand side of (7.12) multiplied by $\frac{\sqrt{k}}{g(\chi)}$. this provide us the theorem in(7.4) when $\delta = 0$.

Now, let χ is odd character $\chi(-1) = -1$, by (7.7) in lemma (7.3) we have that

$$\Pi^{-s+1/2} k^{s+1/2} \Gamma\left(\frac{s+1}{2}\right) n^{-s} = \int_0^\infty n \exp(-\pi \tau n^2/k) \tau^{\frac{s}{2}-\frac{1}{2}} d\tau$$

Multiplying both of sides by $\sum_{n=1}^{\infty} \chi(n)$ as a result, for $\text{Re}(s)$ greater than 1

$$\pi^{-s+1/2} k^{s+1/2} \Gamma\left(\frac{s+1}{2}\right) \sum_{n=1}^{\infty} \chi(n)/n^{-s} = \int_0^{\infty} \tau^{\frac{s}{2}-\frac{1}{2}} \sum_{n=1}^{\infty} \chi(n) n \exp(-\pi \tau n^2/k) d\tau$$

$$\pi^{-s+1/2} k^{s+1/2} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi) = \frac{1}{2} \int_0^{\infty} \theta_1(\tau, \chi) \tau^{\frac{s}{2}-\frac{1}{2}} d\tau$$

Dividing the integral into two parts

$$\int_1^{\infty} \theta_1(\tau, \chi) \tau^{\frac{s}{2}-\frac{1}{2}} d\tau + \frac{1}{2} \frac{i\sqrt{k}}{g(\bar{\chi})} \int_1^{\infty} \theta_1(\tau, \chi) \tau^{-s/2} d\tau \quad (7.13)$$

Therefore, (7.13) provide us an analytic continuation of $L(s, \chi)$ onto the whole s -plane. Then, the $L(s, \chi)$ is regular. Moreover, if s is changed by $1-s$ and χ by $\bar{\chi}$, the Right hand side of (7.13) is multiplied by $\frac{i\sqrt{k}}{g(\bar{\chi})}$.

Using the fact

$$g(\chi)g(\bar{\chi}) = -k$$

Hence, this give us the theorem in (7.2) □

8 The Zeros of $\zeta(s)$

According the theory of the Riemann zeta function, the critical strip is the region $0 \leq \sigma \leq 1$ of the complex plane. The critical line is the line $\sigma = 1/2$. The trivial zeros of the Riemann zeta function is the real zero of $\zeta(s)$ at $-2, -4, -6, \dots, -2n..$ the negative even numbers. The non trivial zeros are the complex of $\zeta(s)$. This section will gives the theorems of trivial and non trivial zeros. However, before that it will be state a few important basic theorems which help us to reaching the results. Note that by the references [13] and [14] we found these results. One can see (chapter 7 in [14]) for more details.

Theorem 8.1. ([14] Theorem 1.3)

The function $\Gamma(s)$ initially defined for $\text{Re}(s) > 0$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ whose only singularities are simple poles at the negative integers.

Lemma 8.2. *the function $1/\Gamma(s)$ is an entire function of s in $\mathbb{C}$ with simple zeros, such that s are negative even integer and this function has zeros nowhere else.*

Theorem 8.3. *(Trivial zeros)*

The only zeros of the zeta function outside of the critical strip are the negative even integers.

Proof. From the third method of the functional equation we see that

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \Pi^{-\frac{1}{2}s-1}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

and the fact that $\zeta(s)$ has no zero when $s > 1$, also with the fact $\Gamma(s)$ has no zero. Hence, $\zeta(1-s)$ can only be zero when $\Gamma(1-s/2)$ has a simple pole at $s = -2, -4, -6, \dots, -2n$... We find that when $1 - s/2$ is negative even integer. Therefore, when $1 - s$ is negative even integer. It follows that for $s < 0$ the $\zeta(s)$ can be only zero if $s = -2, -4, -6, \dots, -2n$..

□

Theorem 8.4. *(Non Trivial zeros)*

The zeta function has merely zeros at $s = -2, -4, -6, \dots - 2n, \dots$ and the complex numbers λ_n that lie in the critical strip ($0 < \text{Re}(s) < 1$). Moreover, on the critical strip the zeros are situated symmetrically respecting the lines $\text{Im}(s)$ equal to zero to $\text{Re}(s)$ equal to $1/2$.

Proof. Firstly, we will Define the Möbius function $\mu(s)$ as

$$\mu(s) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^\ell & \text{if } n = p_1 \dots p_\ell \text{ and } p_1 \dots p_\ell \text{ are distinct primes} \\ 0 & \text{Otherwise} \end{cases}$$

The relation between this function and Euler Product is

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \mu(n)n^{-s}$$

Hence

$$\left|\frac{1}{\zeta(s)}\right| = \left|\sum_{n=1}^{\infty} \mu(n)n^{-s}\right| < \left|\sum_{n=1}^{\infty} n^{-\sigma}\right| < \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^{-\sigma}} dx + 1$$

$$\begin{aligned} &= \int_1^{\infty} \frac{1}{x^{\sigma}} dx + 1 = \left. \frac{x^{\sigma+1}}{-\sigma+1} \right|_1^{\infty} \\ &= \sigma/\sigma - 1 \end{aligned}$$

From the inequality , we prove the zeta function has no zero when $Re(s)$ greater than 1. Also, for $Re(s)$ less than zero and by the Riemann's function equation, we obtained $\zeta(s)$ has no zero. However, excepting the zero at $s = -2, -4, -6, \dots -2n \dots$

□

Theorem 8.5. *The $\zeta(s)$ has no zero when $Re(s)$ equal 1.*

The proof of this theorem need some work with some theorems. We will state some important lemmas. Furthermore, we present an essential theorems about zeros and poles.

Lemma 8.6. *If the real part $Re(s)$ greater than 1, we have that*

$$\log \zeta(s) = \sum_{p, \ell} \frac{p^{-\ell s}}{\ell} = \sum_{n=1}^{\infty} \frac{C_n}{n^s}$$

Lemma 8.7. *If φ in $\mathbb{R}$, we have that*

$$0 \leq 3 + 4 \cos \varphi + \cos 2\varphi$$

Lemma 8.8. *If the real part $Re(s)$ greater than 1 and let $t \in \mathbb{R}$ then*

$$0 \leq \log |\zeta^3(\sigma) \zeta^4(\sigma + it) \zeta(\sigma + 2it)|$$

Theorem 8.9. *([14] Theorem 1.1)*

Suppose f be holomorphic in a connected open set Ω such that it has a zero at $s_0 \in \Omega$ and does not vanish identically in Ω . Then there exists a neighborhood $U \in \Omega$ of s_0 a non-vanishing holomorphic function g on U , and a unique positive integer n such that

$$f(s) = (z - z_0)^n g(s)$$

for all $s \in U$

Theorem 8.10. *([14] Theorem 1.1)*

If f has a pole at $s_0 \in \Omega$ then in a neighborhood of that point there exist a non-vanishing holomorphic function h and a unique positive integer n such that

$$f(s) = (z - z_0)^{-n} h(s)$$

Proof. We will prove it by contradiction

let we have a point $s_0 = (1 + it)$ such that $\zeta(1 + it) = 0$ for $t \neq 0$, we know that the zeta function is holomorphic at the point $(1 + it)$, it must disappear at least to order 1 at $(1 + it)$, by theorem 1 there exist a constant C , it is greater than 0

Therefore,

$$\frac{|\zeta(\sigma + it)|^4}{C(\sigma - 1)^4} \leq 1 \quad \text{as } \sigma \text{ goes to } 1$$

Likewise, by theorem (8.9) the zeta function has a simple pole at $s = 1$ and there exist a constant C' , it is greater than 0 So

$$\frac{|\zeta(\sigma)|^3}{C'(\sigma - 1)^3} \leq 1 \quad \text{as } \sigma \text{ goes to } 1$$

At the end, since the zeta function is holomorphic at $\sigma + 2it$, $|\zeta(\sigma + 2it)|$ is remains bounded as σ tend to 1

Hence,

$$\lim_{\sigma \rightarrow 1} |\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| = 0$$

However, this is contradiction with Lemma(8.8). Since the logarithm of $\mathbb{R}$ between zero and one is non positive. Thus, the zeta function has no zero on the real line $Re(s) = 1$.

□

8.1 Weierstrass product for $\zeta(s)$ and $L(s, \chi)$

Lemma 8.11. $\xi(s)$ and $\xi(s, \chi)$ are entire functions of order 1.

In order to understand and prove this lemma we need to review some theorems and corollary.

Theorem 8.12. If $\xi(s)$ is an entire function, then it satisfies that

$$\xi(s) = \xi(1 - s) \tag{8.1}$$

Theorem 8.13. Let $G(s)$ is an entire function such that $G(s)$ does not equal zero. The set of zeros of $G(s)$ is denoted by P , and the multiplicity of zero of G at λ is denoted by $v_\lambda(G)$. Assume that

$$\log \max_{|s|=R} |G(s)| \ll_\epsilon (R + 1)^{1+\epsilon}$$

for $\forall \epsilon > 0$ and $R > 0$. Then

(i)

$$f(s) = \prod_{\lambda \in P, \lambda \geq 0} \left(\left(1 - \frac{s}{\lambda}\right) e^{s/\lambda} \right)^{v_\lambda(G)}$$

This product is uniformly convergent on every compact subset of the complex plane , therefore it determines an entire function $f(s)$.

(ii)

$$G(s) = s^{v_\lambda(G)} \exp(As + b) f(s)$$

such that A and B are belong to $\mathbb{C}$

(iii)

$$\sum_{\lambda \in P, \lambda \geq 0} v_\lambda(G) / |\lambda|^2$$

This series is convergent.

(iv)

$$\sum_{\lambda \in P, \lambda \geq 0} v_\lambda(G) / |\lambda|$$

If this series is convergent. Then $\exists c > 0$, c is absolutely constant. one has

$$\log \max_{|s|=R} |G(s)| \leq c(R + 1)$$

Corollary 8.13.1.

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2} N^{-s} + s \int_N^\infty \frac{\lambda(u) du}{u^{s+1}}$$

Such that N belong to $\mathbb{N}$ and $\lambda = \frac{1}{2} - [u]$.

Theorem 8.14. (Stirling formula)

(1)

$$\log \Gamma(s) = \left(\frac{s-1}{2} \right) \log s - s + \log \sqrt{2\pi} + O(|s|^{-1})$$

such that δ greater than 0 and args between $\delta - \pi$ and $\pi - \delta$. The constant O -big depend on δ .

(2)

$$\log \Gamma(s) = \left(\frac{s-1}{2} \right) \log s - s + \log \sqrt{2\pi} + \int_0^\infty \frac{\lambda(u)}{u+s} du$$

such that $\lambda = \frac{1}{2} - [u]$.

Proof. According to the theorem (8.12) we have that $\xi(s) = \xi(1-s)$, it will suffice to prove $|\xi(s)|$ for $\operatorname{Re}(s) \geq \frac{1}{2}$

By the corollary (8.3.1), when $\operatorname{Re}(s) \geq \frac{1}{2}$ and $N = 1$ and since $|\zeta(s)| = O(|s|)$ and $|s(s-1)| = O(|s|^2)$ we have that

$$\zeta(s)s(s-1) = O(|s|^3)$$

Moreover, By the theorem (8.3) obviously that

$$|\Gamma(s)| \leq \exp(C|s| \log|s|) \quad , \quad |\zeta(s)| < (C|s|)$$

and

$$|\pi^{-\frac{s}{2}}| \leq \exp(C|s|)$$

It concludes that

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad (8.2)$$

The order of this function at most 1. However, when s goes to $+\infty$ i.e. $s > 1$ one has

$$\log \Gamma(s) \sim s \log s \quad \text{i.e.} \quad \log \xi(s) \sim s \log s$$

Therefore, the $\xi(s)$ is a function of order 1. $\xi(s, \chi)$ has similar steps to prove it and if we use the theorem (7.1), we obtain that $\xi(s, \chi)$ is a function of order 1 as well. $\square$

* By using the lemma and the notice that $s \rightarrow +\infty$

$$|\xi(s)| \geq e^{cs} \quad , \quad |\xi(s, \chi)| \geq e^{cs}$$

such that the constant $c > 0$ and it arbitrary large however, fixed.

Applying the theorem (8.3) to follow that each of the functions $\xi(s)$ and $\xi(s, \chi)$ could be represented as

$$\xi(s) = e^{A+Bs} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right) e^{s/\lambda_n} \quad (8.3)$$

where A and B are constant and the series $\sum_{n=1}^{\infty} |\lambda_n|^{-1-\epsilon}$ convergent for $\forall \epsilon > 0$ and the series $\sum_{n=1}^{\infty} |\lambda_n|^{-1}$ divergent. The product of (8.3) has infinity many factors, therefore $\xi(s)$ and $\xi(s, \chi)$ have infinity several zeros, t. We can see as a simple argument, when the λ_n lie in $0 \leq \sigma \leq 1$ where $s = \sigma + it$. In other words, if we take the function $\xi(s)$.

According to (8.1),(8.2), it is enough to see that $\xi(s) \neq 0$ when $\sigma > 1$. We can show that if we prove $\zeta(s) \neq 0$ when $\sigma > 1$.

By the theorem (4.1) and when $\sigma > 1$ such that $s = \sigma + it$, we have that

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right)^{-1}$$

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

$$\begin{aligned} \frac{1}{|\zeta(s)|} &= \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right| < \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} < 1 + \int_1^{\infty} \frac{du}{u^{\sigma}} \\ &= 1 + \frac{u^{-\sigma+1}}{-\sigma+1} \Big|_1^{\infty} = 1 + \frac{1}{\sigma-1} = \frac{\sigma}{\sigma-1} \end{aligned}$$

Since

$$\frac{1}{|\zeta(s)|} = \frac{\sigma}{\sigma-1}$$

Hence

$$|\zeta(s)| > \frac{\sigma-1}{\sigma}$$

The function $\xi(s, \chi)$ has the same proof. By (8.1), it easy to show that $\xi(0)$ does not equal 0 and $\xi(1)$ does not equal 0 as well. Since, $\Gamma(s/2) \neq 0$. Therefore, all of zeros of the function (8.3) are zeros of the zeta function $\zeta(s)$, for $\xi(s, \chi)$ the same hold. Note that the last subsection taken from the reference [5] and [7].

8.2 Theorems relating to the zeros of $\zeta(s)$

This section will gives the consequence of the functional equation. It studies with the logarithmic derivative $\zeta'(s)/\zeta(s)$ to know the poles and zeros of the Riemann zeta function. Moreover, it focuses on the zeros for $\sigma > 1$ where $s = \sigma + it$ there are no zero. By (6.20) in third method, the zeros of the Riemann zeta function in $\sigma > 0$ are simple zeros at $-2, -4, -6, \dots, 2n, \dots$. Most of results had been taken from the references [7] and [8].

8.2.1 Results of the functional equation for the Riemann zeta function $\zeta(s)$

[7] Since,

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad (8.4)$$

and

$$\xi(s) = \xi(1-s) \quad (8.5)$$

such that $s = \sigma + it$ also, as before can be written as

$$\xi(s) = e^{A+Bs} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right)^{s/\lambda_n} \quad (8.6)$$

Therefore, we obtain that

$$\xi(s) = \frac{1}{2}s(s-1) \prod_{n=1}^{\infty} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \xi(s) = e^{A+Bs} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right)^{s/\lambda_n}$$

By analytic continuation of $\zeta(s)$, the $\zeta(s)$ can represent as

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + s \int_N^{\infty} \frac{\lambda(u)du}{u^{s+1}}$$

For $\sigma > 0$ and N equal 1 we find that

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s \int_1^{\infty} \frac{\lambda(u)du}{u^{s+1}}$$

Multiplying the right hand side and the left hand side by $(s-1)$, then take limit when s tend to 1, we get

$$\lim_{s \rightarrow 1} (s-1)\zeta(s) = 1$$

and

$$\lim_{s \rightarrow 1} \xi(s) = \frac{1}{2} \left(\prod\right)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2}$$

From the equation (8.5) we have that

$$\xi(0) = \xi(1) = \frac{1}{2}$$

Since, the inverse of $\Gamma(s)$ has expansion

$$(\Gamma(s))^{-1} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{\ell}\right) e^{-s/\ell}$$

Consequently, $\lim_{s \rightarrow 0} s\Gamma(s) = 1$ $e^A = \xi(0) = -\zeta(0) = \frac{1}{2}$ then we can say that $\zeta(0) = -\frac{1}{2}$ Hence, we obtain

$$\zeta(0) = -\frac{1}{2}$$

and

$$\xi(1) = \frac{1}{2}$$

Now, for $\sigma > 1$, we get

$$\begin{aligned} (1 - 2^{1-s})\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^s} \\ &= 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots \end{aligned}$$

For $s > 0$, the last terms of series are converges. Then,

$$(1 - 2^{1-s})\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots$$

This demonstrated that $\zeta(s) \neq 0$ for all real positive s . Therefore, the λ_n in (8.6) is complex. It means that, the zeros of $\zeta(s)$ is also the zero of $\xi(s)$, are does not belong to $\mathbb{R}$.

Besides of the λ_n , the zeta function ζ has another zeros by

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

Since $s = \sigma + it$, if $Re(s) = \sigma$ less than zero then, $Re(1-s)$ greater than one.

$\pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$ does not vanish, However, $\Gamma\left(\frac{s}{2}\right)$ has poles at $\frac{s}{2} = -n$ such that $n = 1, 2, 3, 4, \dots$. The zeta function $\zeta(s)$ vanishes when $s = -2n$ such that $n = 1, 2, 3, 4, \dots$

Next that by

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} (s^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}} w(x)) dx$$

such that $w(x) = \sum e^{-\pi n^2 x}$ and

$$\xi(s) = \xi(1-s)$$

It follow that

$$\overline{\xi(s)} = \xi(\bar{s})$$

Theorem 8.15. *The Riemann zeta function satisfies that*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \left(\frac{1}{s+2n} - \frac{1}{2n} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{s-\lambda_n} - \frac{1}{\lambda_n} \right) - \frac{1}{s-1} + c \quad (8.7)$$

such that c is constant and λ_n runs through all of the complex zeros of the zeta function.

Corollary 8.15.1. *The number of λ_n where λ_n is the zeros of $\zeta(s)$ such that $T \leq |Im\lambda_n| \leq T+1$ less than $c \log T$. By the $O(\log T)$, the λ_n in the region $0 \leq Re(s) \leq 1$ and $T \leq Im(s) \leq T+1$ is bounded.*

Corollary 8.15.2. *For T greater than or equal 2 then we have that*

$$O(\log T) = \sum_{|T-\gamma_n|>1} \frac{1}{|T-\gamma_n|}$$

Theorem 8.16. *assume T greater than or equal 2 and let $\lambda_n = \beta_n + i\gamma_n$ such that $n = 1, 2, 3, 4, \dots$ it's complex zeros of the zeta function hence,*

$$\sum_{n=1}^{\infty} \frac{1}{1+(T-\gamma_n)^2} \leq c \log T$$

Theorem 8.17. *Let $-1 \leq \sigma \leq 2$ where $s = \sigma + it$ then,*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{1}{s-\lambda_n} - \frac{1}{s-1} + O(\log|t| + 2)$$

Proof. $m = |t| + 2$ and for $-1 \leq \sigma \leq 2$ where $s = \sigma + it$ we have that

$$\left| \sum_{n=1}^{\infty} \left(\frac{1}{2n+s} - \frac{1}{2n} \right) \right| \leq \sum_{n \leq m} \left(\frac{1}{n} + \frac{1}{n} \right) + \sum_{n \leq m} \frac{|s|}{n^2} = O(\log m)$$

By the theorem (8.7) we have that

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \left(\frac{1}{s-\lambda_n} + \frac{1}{\lambda_n} - \frac{1}{s-1} \right) + O \log(m)$$

Subtracting the same equality when $s = 2 + it$

$$\sum_{n=1}^{\infty} \left(\frac{1}{s-\lambda_n} - \frac{1}{2+it-\lambda_n} \right) + O \log(m)$$

Then we have If $|\gamma_n - t| > 1$ hence, we obtain that

$$\left| \frac{1}{\sigma + it - \lambda_n} - \frac{1}{2 + it - \lambda_n} \right| \leq \frac{2}{(\gamma_n - t)^2} \leq \frac{3}{(\gamma_n - t)^2}$$

□

8.2.2 Results of the functional equation for $L(s, \chi)$

The basic consequence focusing on the zeros of L-Function are similar to the zeros of the zeta function.

If χ even character we have that

$$\sqrt{k}(\pi k^{-1})^{-s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi) = (\pi k^{-1})^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) L(1-s, \bar{\chi})$$

By (7.1) we have

$$L(s, \chi) = \prod \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \text{ for } \sigma > 1$$

$$|L(s, \chi)| = \left| \prod \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \right| = \left| \left(\sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} \right)^{-1} \right|$$

Then

$$\begin{aligned} \frac{1}{|L(s, \chi)|} &= \left| \prod \left(1 - \frac{\chi(p)}{p^s}\right) \right| = \left| \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} \right| < \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \\ &< 1 + \int_1^{\infty} \frac{1}{u^\sigma} du = 1 + (u^{-\sigma+1} / -\sigma + 1) = \sigma / \sigma - 1 \end{aligned}$$

Thus

$$|L(s, \chi)| = \sigma - 1/\sigma$$

when $Re(s)$ greater than 0, therefore that $Re(1-s)$ greater than or equal 1. We find that the merely zeros of L-Function are pole of $\Gamma(s/2)$ at negative even integers.

If χ an odd character then

$$i\sqrt{k} (\pi k^{-1})^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi) = g(\chi) (\pi k^{-1})^{-(2-s)/2} \Gamma\left(\frac{2-s}{2}\right) L(1-s, \bar{\chi})$$

As a result, when $\sigma < 0$ such that $s = \sigma + it$ the zeros of $L(s, \chi)$ are poles of $\Gamma\left(\frac{s+1}{2}\right)$ at negative odd integers. These zeros called trivial zeros. By the last section we know that $L(s, \chi)$ has infinity many non trivial zeros.

8.2.3 The Theorem of de La Vallée-Poussin of $\zeta(s)$

Theorem 8.18. *The Riemann zeta function does not have any zeros in the region of the s -plane and c is constant which is greater than zero, then we have that*

$$\sigma \geq 1 - \frac{c}{\log(|t| + 2)}$$

Proof. Firstly, assume $\lambda = \beta + i\gamma$ is zero of the zeta function, hence $\gamma_1 < |\gamma|$ such that $\gamma_1 > 0$ Since,

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$$

such that $\text{Re}(s)$ greater than one Then, we have

$$-Re \left(\frac{\zeta'(s)}{\zeta(s)} \right) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \cos(t \log n)$$

Since, $\varphi \in \mathbb{R}$ then

$$3 + 4 \cos \varphi + \cos 2\varphi = 2(1 + \cos \varphi)^2 \geq 0$$

Thus

$$3 \left(\frac{\zeta'(\sigma)}{\zeta(\sigma)} \right) - 4 \left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} \right) + \left(-Re \frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + 2it)} \right) \geq 0 \quad (8.8)$$

for every terms on the left hand side in(8.8) we obtain an upper bound and by corollary (8.15.1) we get that for $s = \sigma$ and $1 < \sigma \leq 2$

$$A_1 + \frac{1}{\sigma - 1} > -\frac{\zeta'(\sigma)}{\zeta(\sigma)}$$

such that A_1 is constant and it's greater than zero. Using Theorem (8.15)

$$-Re \left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} \right) < -\sum_{n=1}^{\infty} Re \left(\frac{1}{s - \lambda_k} + \frac{1}{\lambda_k} \right) + A_2 \log(|t| + 2)$$

such that A_2 is constant and it's greater than zero. Since, $\lambda_k = \beta_k + it, \beta_k > 0$

$$Re \frac{1}{s - \lambda_k} = Re \frac{1}{\sigma - \beta_{k+i(t-\gamma_k)}} = \frac{\sigma - \beta_k}{(\sigma - \beta_k)^2 + (t - \gamma_k)^2} > 0$$

$$Re \frac{1}{\lambda_k} = \frac{\beta_n}{\beta_k^2 + \gamma_k^2} \geq 0$$

Then

$$-Re \left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} \right) < -\frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2} + A_2 \log(|t| + 2)$$

We substitute $-\sigma - \beta/(\sigma - \beta)^2 + (t - \gamma)^2$ by a weaker one. In addition, assume $t = 2t$ then

$$-Re \left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} \right) < +A_2 \log(2|t| + 2)$$

Now, the terms on the left hand side in (8.8)are bounded. Hence, we obtain

$$\frac{3}{\sigma - 1} - \frac{4(\sigma - \beta)}{(\sigma - \beta)^2 + (t - \gamma)^2} + A \log(|t| + 2) \geq 0$$

such that A is greater than one and it is constant, for $|t| > \gamma_1$ t and σ when $1 < \sigma \leq 2$ we have $t = \sigma$

$$\sigma = 1 + \frac{1}{2A \log(\gamma) + 2}$$

We have that

$$\frac{3}{\sigma - 1} + A \log(|\gamma| + 2) \geq \frac{4}{\sigma - \beta}$$

Finally, we get

$$\beta \leq 1 - \frac{1}{14A \log(|\gamma| + 2)}$$

□

8.2.4 The Theorem of de La Vallée-Poussin of $L(s, \chi)$

Theorem: ([7], Theorem 7.2.2) Let χ be a complex character modulo k , and let $s = \sigma + it$. Then $L(s, \chi)$ has no zeros in the region

$$\operatorname{Re}(s) = \sigma \geq 1 - \frac{c}{\log k(|t| + 2)}$$

If on the other hand, χ is a real character modulo k , $s = \sigma + it$, then $L(s, \chi)$ has no zeros in the region

$$\operatorname{Re}(s) = \sigma \geq 1 - \frac{c}{\log k(|t| + 2)}, \quad |t| > 0$$

9 Prime number theorem

The last part of the aim in this dissertation is the application of the Riemann zeta function. In this section, we will state the theorem of the prime numbers, will not prove it here. However, one can find the proof in [7]. Then, we will briefly give the link between The Distribution of prime numbers and the Riemann zeta function.

The prime number theorem can be defined as

$$\psi(x) \sim x \tag{9.1}$$

as x tends to infinity, such that the function $\psi(x)$ is Chebyshev's function. It satisfies that

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

we will present an important theorem of formula (9.1)

Theorem 9.1. *Let $x \geq 1$, there exist c is absolutely constant and it is greater than zero such that*

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = x + O(x \exp(-c\sqrt{\log x}))$$

9.1 The Connection between The Distribution of Prime Numbers and The Riemann Zeta Function $\zeta(s)$

Let the $\pi(x)$ is the number of prime. For $\sigma > 1$ such that $s = \sigma + it$ we have that

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

According the partial sum ,we get

$$\log \zeta(s) = - \sum \log \left(1 - \frac{1}{p^s}\right) \quad (9.2)$$

$$- \int_1^\infty \log \left(1 - \frac{1}{x^s}\right) d\pi(x) = \int_1^\infty \pi(x) d \log \left(1 - \frac{1}{x^s}\right) - \log \left(1 - \frac{1}{x^s}\right) \pi(x) \Big|_1^\infty$$

Hence, we find that

$$= s \int_1^\infty \frac{\pi(x) dx}{x(x^2 - 1)}$$

the formula (9.2) is most significant method to study the number of prime numbers $\pi(x)$.

But, the Chebyshev function $\psi(x)$ is more appropriate than $\pi(x)$ from analytic point of view.

Since,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} \quad (9.3)$$

such that

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \\ 0 & \text{otherwise} \end{cases}$$

Then,we can write

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

To obtain (9.3), we find that

$$-\frac{\zeta'(s)}{\zeta(s)} = (-\log \zeta(s))' = \left(-\log \prod_p \left(1 - \frac{1}{p^s}\right) \right)'$$

$$\begin{aligned}\sum_p \log \left(1 - \frac{1}{p^s}\right)' &= \sum_p \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots\right) \\ &= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\end{aligned}$$

All of series is uniformly convergent for $\sigma > 1 + \delta$ such that δ is positive number.

Now, we have that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \int_1^{\infty} \frac{1}{x^s} d\psi(x)$$

Taking the integral , then

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{\psi(x)}{x^s} \Big|_1^{\infty} - \int_1^{\infty} \psi(x) d\left(\frac{1}{x^2}\right)$$

From the definition of $\psi(x)$, we know that

$$\psi(x) = O(x \log x)$$

as x tend to infinity and $\psi(0) = 1$ Hence,

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^{\infty} \frac{\psi(x)}{x^{s+1}} dx$$

The slimier role is played by $L(s, \chi)$ we have that

$$\psi(x, \chi) = \sum_{n \leq x} \Lambda(n) \chi(n)$$

By definition of L-Function we have that

$$-\frac{L'(s, \chi)}{L(s, \chi)} = (-\log L(s, \chi))'$$

slimier steps of (9.10) we obtain that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n) \chi(n)}{n^s}$$

Therefore

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_1^{\infty} \frac{1}{x^2} d\psi(x, \chi) = s \int_1^{\infty} \frac{\psi(x, \chi)}{x^{s+1}}$$

10 Conclusion

We have demonstrated the most important of the zeta function as its analytic continuation, prove the functional equation and introduce the application. Reviewing what we have already presented. This dissertation into the background of the zeta function and we knew the reason of this function is named after Riemann. Additionally, we have proved some of interesting properties of the zeta function. For example,

(i) The function $\zeta(s) - [1/(s-1)]$ extend to the right half plane for $\sigma > 0$ and this function is analytic when $\sigma > 0$.

(ii) let $\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ such that $\xi(s)$ is analytic continuation on the entire plane also $\xi(s)$ satisfies $\xi(s) = \xi(1-s)$.

Then, we have proved the functional equation by fourth methods and it is very an important in this dissertation. The first method we have proved this theorem depend on the summation formula. *The function $\zeta(s)$ is regular for all value of s except $s=1$, where there is a simple pole with residue 1, it satisfies the functional equation*

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{1}{2} s \pi \Gamma(1-s) \zeta(1-s)$$

The second methods, we have proved The fundamental formula by using the theorem of residue and contour integration, one can see more detail in [13] The third method, we have proved this equation by the gamma function

$$\pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s) = \pi^{-\frac{1}{2}s-1} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

The last method depend on self-reciprocal function and we have proved by using Fourier transform functions. Then, we have introduced the Dirichlet L-Function. It is notice that it has similar the $\zeta(s)$ results. Moreover, we have shown the zeros of $\zeta(s)$ particularly, trivial zeros, non trivial zeros and the $\zeta(s)$ has no zero when $Re(s) > 1$. We have given a briefly discussion of the connection between the prime numbers and the zeta function.

Following this dissertation the Riemann zeta function and its analytic continuation has many other parts remain to be studied. One of this parts is to look the famous Riemann Hypothesis. Another parts is to look at Convexity and the Riemann zeta function.

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