

A New Spectral-Collocation Method Using Legendre Multi-wavelets for Solving of Nonlinear Fractional Differential Equations

ABSTRACT

In this paper, a novel spectral collocation method using Legendre multi-wavelets as the basis functions is presented to obtain the numerical solution of nonlinear fractional differential equations. The fractional derivative is described in the Caputo sense. The two-scale relations of Legendre multi-wavelets and the properties of block pulse functions have been used in the evaluation of the fractional integral operational matrix and expansion coefficients of the nonlinear terms for the Legendre multi-wavelets. Due to the aforementioned properties, the original differential equation is converted into a nonlinear system of algebraic equations which can be solved by existing tools. The numerical results are compared with exact solutions and existing numerical solutions found in the literature and demonstrate the validity and applicability of the proposed method.

Keywords: Legendre multi-wavelets; fractional integral operational matrix; two-scale relations; fractional differential equation

2010 Mathematics Subject Classification: 35R11, 65M06, 65M12

1. INTRODUCTION (ARIAL, BOLD, 11 FONT, LEFT ALIGNED, CAPS)

Fractional differential equation is a very effectively tool for the description of memory and hereditary properties in various materials and processes [1-4], such as gas diffusion and heat conduction in fractal porous media. Although a lot of attentions have been paid on fractional calculus and a large literature for solving fractional differential equation exists, there remain several challenges, and therefore, it is necessary to improve the current methods or develop new ones. Among the issues are as follows: (i) how to accurately represent fractional integral or differential operator, (ii) how to effectively deal with high nonlinearity and (iii) how to significantly reduce the computational cost.

In the past two decades there has been a considerable interest to the wavelets methods in the numerical solution of fractional differential equations. Wavelets possess orthogonality, compactly supported, and ability to accurately represent a variety of functions and operators at different levels of resolution. Moreover, wavelets establish a connection with fast numerical algorithms [5]. Wavelets methods, such as Haar wavelets [6-8], Chebyshev wavelets [9,10], CAS [11,12] and Legendre wavelets [13-15] have been successfully developed to solve the fractional order differential equations and proved to be effective and powerful in simulation of fractional phenomena. A detailed description about wavelets

methods in fractional differential equations has been presented in reference [16]. The main attraction of wavelets methods is that it can exploit multi-level parallelism by employing the multi-scale analysis and hierarchy structure of Legendre wavelets

As an extension of wavelets, multi-wavelets are a discontinuous, orthogonal, compactly supported, multiscale set of functions with vanishing moments and can approximate a large class functions in L_2 space by using multiresolution analysis (MRA) in terms of translates of linear combinations of the scaling function vector. Sparse representation of differential and integral operators due to vanishing moments of the wavelets is another property of multi-wavelets [17-23]. What's more, the numerical methods using multi-wavelets as basis are highly stable and the related computation is economic [24]. Legendre multi-wavelets [17] were firstly developed in 1990 by Alpert to provide sparse representations of integral operator for the solution of integral equations and have been successfully used to solve partial differential equations [20-22]. It has been proved that Legendre multi-wavelets can reduce the error to a level below that of wavelets [23-24]. Comparing to Haar wavelets, Legendre multi-wavelets can converge more rapidly and produce piecewise polynomial solutions of any order. What's more, the properties of piecewise polynomials can be used to decrease the saving and computational complexity.

To the best of the authors' knowledge, Legendre multi-wavelets have not been used previously for solving fractional differential equation. Motivated and inspired by the ongoing research, we develop a new spectral collocation method using Legendre multi-wavelets (SCLMW) to the solution of fractional differential equation. The remainder of the paper is organized as follows. In section 2, we describe some preliminaries about fractional calculus theory and also illustrate how to construct Legendre multi-wavelets. Fractional integral matrix and nonlinear terms approximation of Legendre multi-wavelets are derived in section 3. The proposed method is detailed in section 4. Five examples are given in Section 5 to demonstrate the validity and applicability of the proposed method. Finally the concluding remarks are given in Section 6.

2. PRELIMINARIES AND NOTATIONS

2.1 FRACTIONAL CALCULUS THEORY

For the reader's convenience, some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development are listed. More details about fractional calculus theory see reference [31,32].

Definition 2.1 A real function $h(x)$, $x > 0$, is said to be in the space C_μ , $\mu \in R$, if there exists a real number $p > \mu$, such that $h(x) = x^p h_1(x)$, where $h_1(x) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)} \in C_\mu$, $n \in N$.

Definition 2.2 Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$ is defined as,

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad x > 0, \quad (1)$$

$$J^0 f(x) = f(x),$$

85 $\Gamma(\alpha)$ is the well known Gamma function. Some properties of the operator J^α can be found,
 86 for example, in [31,32]. We recall only the following, for $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$,
 87 we have,

$$\begin{aligned} J^\alpha J^\beta f(x) &= J^{\alpha+\beta} f(x), \\ J^\alpha J^\beta f(x) &= J^\beta J^\alpha f(x), \\ J^\alpha x^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \end{aligned} \quad (2)$$

89 **Definition 2.3** The fractional derivative of $f(x)$ in the Caputo sense is defined as [33]

$$(D_x^\alpha f)(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha-m+1}} d\xi, & (\alpha > 0, m-1 < \alpha < m) \\ \frac{\partial^m f(x)}{\partial x^m}, & \alpha = m \end{cases}, \quad (3)$$

91 where $f: R \rightarrow R$, $x \rightarrow f(x)$ denotes a continuous (but not necessarily differentiable) function.
 92 Some useful formulas and results of Caputo derivative, which we will need here, are as
 93 follows

$$D_x^\alpha c = 0, \quad \alpha > 0, \quad c = \text{constant}, \quad (4)$$

$$D_x^\alpha [cf(x)] = cD_x^\alpha f(x), \quad \alpha > 0, \quad c = \text{constant}, \quad (5)$$

$$D_x^\alpha x^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \quad \beta > \alpha > 0, \quad (6)$$

$$D_x^\alpha [f(x)g(x)] = [D_x^\alpha f(x)]g(x) + f(x)[D_x^\alpha g(x)], \quad (7)$$

$$D_x^\alpha [f(x(t))] = f'_x(x)x^{(\alpha)}(t), \quad (8)$$

99 **Lemma 1.** Let $n-1 < \alpha \leq n$, $n \in N$, $x > 0$, $h \in C_\mu$, $\mu \geq -1$. Then

$$(J^\alpha D^\alpha)h(x) = h(x) - \sum_{k=0}^{n-1} h^{(k)}(0^+) \frac{x^k}{k!}. \quad (9)$$

101 2.2 LEGENDRE MULTIWAVELETS

102 In this section, Legendre multi-wavelets are introduced and some useful properties are also
 103 described. A detailed description about Legendre multi-wavelets can be seen in reference
 104 [17-19].
 105

107 2.2.1 Legendre multi-wavelets

108 Legendre scaling functions are defined on the interval $[0, 1)$ as follows:
 109

$$\phi_{J,k}^m(x) = \begin{cases} \sqrt{2m+1} 2^{J/2} L_m(2(2^J x - k) - 1), & \frac{k}{2^J} \leq x \leq \frac{k+1}{2^J} \\ 0, & \text{otherwise} \end{cases} \quad (10)$$

111 where $m = 0, 1, 2, \dots, r-1$, $k = 0, 1, \dots, 2^J - 1$ and $J = 0, 1, 2, \dots$. Here, $L_m(x)$ are the
 112 well-known Legendre polynomials of order m .

113 Legendre scaling function vector and Legendre multi-wavelets vector in subspaces

114 $\mathbf{V}_J = \mathbf{V}_{J-1} \oplus \mathbf{W}_{J-1}$ are defined as

$$\Phi_J(x) = [\phi_{J,0}^0(x), \dots, \phi_{J,0}^{r-1}(x), |, \dots, \phi_{J,(2^j-1)}^0(x), \dots, \phi_{J,(2^j-1)}^{r-1}(x)] \quad (11)$$

and

$$\Psi_J(x) = [\phi_{0,0}^0(x), \dots, \phi_{0,0}^{r-1}(x), |, \psi_{0,0}^0(x), \dots, \psi_{0,0}^{r-1}(x), |, \dots, |, \psi_{J-1,0}^0(x), \dots, \psi_{J-1,0}^{r-1}(x), |, \dots, \psi_{J-1,2^{j-1}-1}^0(x), \dots, \psi_{J-1,2^{j-1}-1}^{r-1}(x)]', \quad (12)$$

respectively.

Legendre scaling function vector $\Phi_J(t)$ satisfies a two-scale equation of the form

$$\Phi_J(x) = \sum_{k=0}^1 \mathbf{H}^{(k)} \Phi_J(2x - k), \quad (13)$$

where $\mathbf{H}^{(k)}$ is a finite set of matrices,

The two-scale relation of Legendre multi-wavelets vector is given in the following form

$$\Psi_J(x) = \sum_{k=0}^1 \mathbf{G}^{(k)} \Phi_J(2x - k), \quad (14)$$

where $\Psi_J(x)$ is the vector of corresponding wavelet functions. So the Legendre multiwavelets can be constructed by Legendre scaling functions by two-scale equation Eq. (14).

A function $f(x)$ in $\mathbf{V}_n^k$ can be approximated by using the Legendre scaling functions as

$$f(x) \approx P_J f(x) = \sum_{k=0}^{2^j-1} \sum_{m=0}^{r-1} c_{J,k}^m \phi_{J,k}^m(x) = \mathbf{C}^T \Phi_J(x), \quad (15)$$

and the decomposition of $f(x)$ has an equivalent Legendre multiwavelets given by

$$f(x) \approx P_J f(x) = \sum_{m=0}^{r-1} \left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\} = \mathbf{D}^T \Psi_J(x), \quad (16)$$

where

$$c_{j,k}^m = \langle f(x), \phi_{j,k}^m \rangle, \quad d_{j,k}^m = \langle f(x), \psi_{j,k}^m \rangle, \quad (17)$$

in which $\langle \cdot, \cdot \rangle$ denotes the stander inner product of the Hilbert space, and $\mathbf{C}$ and $\mathbf{D}$ are $N \times 1$ vectors with $N = r2^J$ given by

$$\mathbf{C}^T = [c_{J,0}^0, \dots, c_{J,0}^{r-1} | \dots | c_{J,(2^j-1)}^0, \dots, c_{J,(2^j-1)}^{r-1}], \quad (18)$$

and

$$\mathbf{D}^T = [d_{0,0}^0, \dots, d_{0,0}^{r-1}, |, d_{0,0}^0, \dots, d_{0,0}^{r-1} |, \dots, |, d_{J-1,0}^0, \dots, d_{J-1,0}^{r-1} |, \dots, |, d_{J-1,2^{j-1}-1}^0, \dots, d_{J-1,2^{j-1}-1}^{r-1}]. \quad (19)$$

The operator P_J is named as the orthogonal projection of function f onto space V_J .

For example the cubic Legendre scaling functions consist of four functions are given in the following:

$$\begin{cases} \phi^0(x) = 1, & 0 \leq x < 1 \\ \phi^1(x) = \sqrt{3}(2x-1), & 0 \leq x < 1 \\ \phi^2(x) = \sqrt{5}(6x^2-6x+1), & 0 \leq x < 1 \\ \phi^3(x) = \sqrt{7}(20x^3-30x^2+12x-1), & 0 \leq x < 1 \end{cases}, \quad (20)$$

The corresponding Legendre multi-wavelets $\psi^0(x)$, $\psi^1(x)$, $\psi^2(x)$ and $\psi^3(x)$ have the form as

$$\psi^0(x) = \begin{cases} -\sqrt{\frac{15}{17}}(224x^3 - 216x^2 + 56x - 3), & 0 \leq x < \frac{1}{2}, \\ \sqrt{\frac{15}{17}}(224x^3 - 456x^2 + 296x - 61), & \frac{1}{2} \leq x < 1 \end{cases}, \quad (21)$$

$$\psi^1(x) = \begin{cases} \sqrt{\frac{1}{21}}(1680x^3 - 1320x^2 + 270x - 11), & 0 \leq x < \frac{1}{2}, \\ \sqrt{\frac{1}{21}}(1680x^3 - 3720x^2 + 2670x - 619), & \frac{1}{2} \leq x < 1 \end{cases}, \quad (22)$$

$$\psi^2(x) = \begin{cases} -\sqrt{\frac{35}{17}}(256x^3 - 174x^2 + 30x - 1), & 0 \leq x < \frac{1}{2}, \\ \sqrt{\frac{35}{17}}(256x^3 - 594x^2 + 450x - 111), & \frac{1}{2} \leq x < 1 \end{cases}, \quad (23)$$

$$\psi^3(x) = \begin{cases} \sqrt{\frac{5}{21}}(420x^3 - 246x^2 + 36x - 1), & 0 \leq x < \frac{1}{2}, \\ \sqrt{\frac{5}{21}}(420x^3 - 1014x^2 + 804x - 209), & \frac{1}{2} \leq x < 1 \end{cases}, \quad (24)$$

which are determined using Eqs. (13) and (14). The matrix $\mathbf{H}^{(0)}$ and $\mathbf{G}^{(0)}$ for $r = 4$ are defined as [24]

$$\mathbf{H}^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{\sqrt{15}}{4} & \frac{1}{4} & 0 \\ \frac{\sqrt{7}}{8} & \frac{\sqrt{21}}{8} & -\frac{\sqrt{35}}{8} & \frac{1}{8} \end{pmatrix} \text{ and } \mathbf{G}^{(0)} = \begin{pmatrix} 0 & \frac{2}{\sqrt{85}} & 2\sqrt{\frac{3}{17}} & -\sqrt{\frac{21}{85}} \\ -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{7}} & -\frac{1}{2}\sqrt{\frac{5}{21}} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{1}{4}\sqrt{\frac{21}{85}} & -\frac{3}{4}\sqrt{\frac{7}{17}} & -\frac{8}{\sqrt{85}} \\ \frac{5}{8}\sqrt{\frac{5}{21}} & \frac{5}{8}\sqrt{\frac{5}{7}} & \frac{23}{8\sqrt{21}} & \frac{\sqrt{15}}{8} \end{pmatrix}. \quad (25)$$

The elements $h_{i,j}^{(1)}$ of matrix $\mathbf{H}^{(1)}$ and $g_{i,j}^{(1)}$ of matrix $\mathbf{G}^{(1)}$ can be calculated by using the following formulae

$$h_{i,j}^{(1)} = (-1)^{i+j} h_{i,j}^{(0)} \quad (26)$$

and

$$g_{i,j}^{(1)} = (-1)^{i+j+K} g_{i,j}^{(0)}, \quad (27)$$

respectively.

2.2.2 Transition matrix of Legendre multiwavelets

Eq. (13) can be rewritten as

$$\Phi_k = \mathbf{H}_k \Phi_{k+1}, \quad (28)$$

where $\mathbf{H}_k$, $k = 1, 2, \dots, J$ is a $(r2^{k-1}, r2^k)$ matrix.

Similarly, Eq. (14) can be rewritten as

$$\Psi_k = \mathbf{G}_k \Phi_{k+1}, \quad (29)$$

where $\mathbf{G}_k$, $k = 1, 2, \dots, J$ is a $(r2^{k-1}, r2^k)$ matrix.

166

167 From Eqs. (28) and (29), one can have

$$168 \quad \Psi_J = \mathbf{G}\Phi_{J+1}, \quad (30)$$

169 where $\mathbf{G}$ is $(N \times N)$ transit matrix and can be calculated as follows [23]

$$170 \quad \mathbf{G} = \begin{bmatrix} \mathbf{H}_1 \times \mathbf{H}_2 \times \cdots \times \mathbf{H}_J \\ \mathbf{G}_1 \times \mathbf{H}_2 \times \cdots \times \mathbf{H}_J \\ \vdots \\ \mathbf{G}_{J-2} \times \mathbf{H}_{J-1} \times \cdots \times \mathbf{H}_J \\ \mathbf{G}_{J-1} \times \mathbf{H}_J \\ \mathbf{G}_J \end{bmatrix}_{N \times N}. \quad (31)$$

171 Relations (30) and (31) can be used to develop the fast algorithms for transition between
172 different scales of the multiresolution analysis. It's worth mentioning that the transition
173 operator between coefficients and values is sparse and possess hierarchy structure.

174

175 2.3 BLOCK-PULSE FUNCTIONS

176

177 The N -set of block-pulse functions is defined on $[0, I)$ as follows

$$178 \quad b_i(x) = \begin{cases} 1, & \frac{(i-1)I}{N} \leq x < \frac{iI}{N}, \\ 0, & \text{otherwise,} \end{cases} \quad (32)$$

179 where $i = 1, 2, \dots, N$. The functions $b_i(x)$ are disjoint and orthogonal. That is, for $x \in [0, I)$

180 and $i, j = 1, 2, \dots, N$

$$181 \quad b_i(x)b_j(x) = \begin{cases} b_i(x), & i = j \\ 0, & i \neq j \end{cases}, \quad (33)$$

182 and

$$183 \quad \int_0^I b_i(x)b_j(x)dx = \begin{cases} \frac{I}{N}, & i = j \\ 0, & i \neq j \end{cases}, \quad (34)$$

184 The orthogonality property of block-pulse functions is obtained from the disjointness property
185 for $i = 1, 2, \dots, N$. It is also known that for arbitrary absolutely integrable function $f(x)$ on
186 $[0, I)$ can be expanded in block-pulse functions:

$$187 \quad f(x) = \boldsymbol{\xi}^T \mathbf{B}_N(x), \quad (35)$$

188 in which

$$189 \quad \boldsymbol{\xi}^T = [f_1, f_2, \dots, f_N] \quad (36)$$

190 and

$$191 \quad \mathbf{B}(x) = [b_1(x), b_2(x), \dots, b_N(x)], \quad (37)$$

192 where $f_i (i = 1, 2, \dots, N)$ are the coefficients of the block-pulse function given by

$$193 \quad f_i = \frac{N}{I} \int_0^I f(x)b_i(x)dx = \frac{N}{I} \int_{((i-1)I/N)^+}^{(iI/N)^-} f(x)b_i(x)dx. \quad (38)$$

194 So the disjointness property of BPF's follows

$$195 \quad \mathbf{B}_N(x)\mathbf{B}_N^T(x)\boldsymbol{\xi} = \text{diag}(\boldsymbol{\xi})\mathbf{B}_N(x). \quad (39)$$

196

197 3. NONLINEAR TERM APPROXIMATION AND FRACTIONAL INTEGRAL OF 198 LEGENDRE MULTIWAVELETS

199 3.1 NONLINEAR TERM APPROXIMATION

202 The Legendre scale functions can be expanded into N -set of block-pulse functions as

$$203 \quad \Phi_J(x) = \Phi_{N \times N} \mathbf{B}_N(x). \quad (40)$$

204 where $N = 2^J r$.

205 Taking the collocation points as following

$$206 \quad x_i = \frac{i-1/2}{N}, \quad i = 1, 2, \dots, N. \quad (41)$$

207 The N -square Legendre matrix $\Phi_{N \times N}$ is defined as:

$$208 \quad \Phi_{N \times N} \triangleq [\Phi(x_1) \quad \Phi(x_2) \quad \dots \quad \Phi(x_N)] \quad (42)$$

209 and

$$210 \quad \Psi_J(x) = \mathbf{G} \Phi_{J+1}(x) = \mathbf{G} \Phi_{N \times N} \mathbf{B}_N(x). \quad (43)$$

211 The operational matrix of product of Legendre multi-wavelets vector $\Psi_J(x)$ can be obtained
212 by using the properties of BPFs. Let $f_1(x)$ and $f_2(x)$ are two absolutely integrable functions,
213 which can be expanded in Legendre multi-wavelets as $f_1(x) = \mathbf{F}_1^T \Psi_J(x)$ and $f_2(x) = \mathbf{F}_2^T \Psi_J(x)$,
214 respectively.

215 From (40) and (43), we have

$$216 \quad \begin{aligned} f_1(x) &= \mathbf{F}_1^T \Psi_J(x) = \mathbf{F}_1^T \mathbf{G} \Phi_{N \times N} \mathbf{B}(x) \\ f_2(x) &= \mathbf{F}_2^T \Psi_J(x) = \mathbf{F}_2^T \mathbf{G} \Phi_{N \times N} \mathbf{B}(x) \end{aligned} \quad (44)$$

217 By employing Lemma 1 in [28] and Eq. (44), we get

$$218 \quad \begin{aligned} f_1(x) f_2(x) &= (\mathbf{F}_1^T \mathbf{G} \Phi_{N \times N} \otimes \mathbf{F}_2^T \mathbf{G} \Phi_{N \times N}) \mathbf{B}(x) \\ &= (\mathbf{F}_1^T \mathbf{G} \Phi_{N \times N} \otimes \mathbf{F}_2^T \mathbf{G} \Phi_{N \times N}) \text{inv}(\mathbf{G} \Phi_{N \times N}) \mathbf{G} \Phi_{N \times N} \mathbf{B}(x), \\ &= (\mathbf{F}_1^T \mathbf{G} \Phi_{N \times N} \otimes \mathbf{F}_2^T \mathbf{G} \Phi_{N \times N}) \text{inv}(\mathbf{G} \Phi_{N \times N}) \Psi_J(x) \end{aligned} \quad (45)$$

219 where $\mathbf{A} \otimes \mathbf{B} = (a_{ij} b_{ij})_{N \times N}$ is the Hadamara products of matrix $\mathbf{A} = (a_{ij})_{N \times N}$ and $\mathbf{B} = (b_{ij})_{N \times N}$.

220 Here and after we denote $\Psi_{N \times N} = \mathbf{G} \Phi_{N \times N}$.

221 Figure 1 shows the sparse structure of matrices $\mathbf{T}$ and $\Psi_{N \times N}$. It can be found that the
222 transition matrices $\mathbf{T}$ and $\Psi_{N \times N}$ are sparse and possess hierarchy structure which can be
223 used to save memory cost.

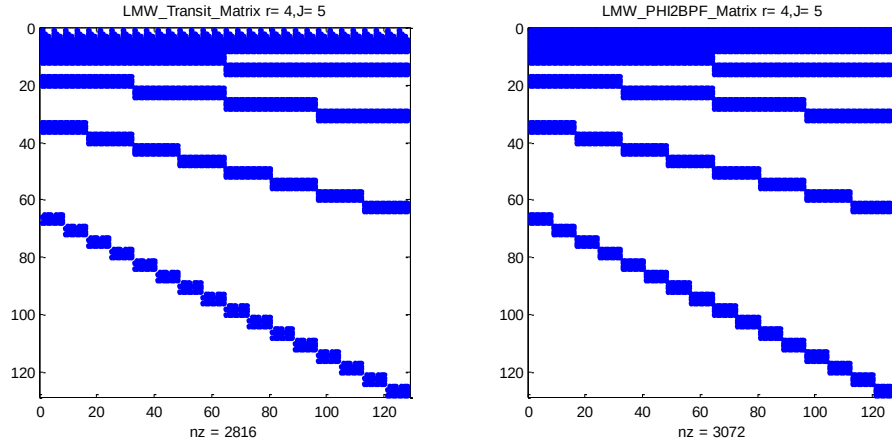


Fig.1. Sparse structure of matrices of $\mathbf{T}$ and $\Psi_{N \times N}$

3.2 THE FRACTIONAL INTEGRAL OPERATOR OF LEGENDRE MULTIWAVELETS

The Legendre multi-wavelets can be expanded into N -set of block-pulse functions as

$$\Psi(x) = \Psi_{N \times N} \mathbf{B}(x). \quad (46)$$

The fractional integral of block-pulse function vector can be written as

$$(I^\alpha \mathbf{B})(x) \approx \mathbf{P}_B^\alpha \mathbf{B}(x), \quad (47)$$

where $\mathbf{P}_B^\alpha$ is given in [27] and is defined as

$$\mathbf{P}_B^\alpha = \frac{1}{N^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{N-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{N-2} \\ 0 & 0 & 1 & \cdots & \xi_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad (48)$$

in which

$$\xi_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1}. \quad (49)$$

The fractional integral of vectors $\Psi_J(x)$ and $\Phi_J(x)$ can be expressed as

$$(I^\alpha \Phi_J)(x) = \mathbf{P}_\phi^\alpha \Phi_J(x), \quad (I^\alpha \Psi_J)(x) = \mathbf{P}_\psi^\alpha \Psi_J(x), \quad (50)$$

where $\mathbf{P}_\phi^\alpha$ and $\mathbf{P}_\psi^\alpha$ are operational matrices of fractional integral for Legendre scaling functions and Legendre multi-wavelets, respectively. The matrix $\mathbf{P}_\phi^\alpha$ can be obtained by the following process.

By using Eq. (40)

$$(I^\alpha \Phi_J)(x) \approx (I^\alpha \Phi_{N \times N} \mathbf{B})(x) = \Phi_{N \times N} (I^\alpha \mathbf{B})(x) = \Phi_{N \times N} \mathbf{P}_B^\alpha \mathbf{B}(x) = \Phi_{N \times N} \mathbf{P}_B^\alpha \Phi_{N \times N}^{-1} \Phi_J(x), \quad (51)$$

so

$$\mathbf{P}_\phi^\alpha \approx \Phi_{N \times N} \mathbf{P}_B^\alpha \Phi_{N \times N}^{-1}. \quad (52)$$

From Eqs. (30) and (50), we have

$$(I^\alpha \Psi_J)(x) = \mathbf{G} (I^\alpha \Phi_{J+1})(x) = \mathbf{G} \mathbf{P}_\phi^\alpha \Phi_{J+1}(x) = \mathbf{G} \mathbf{P}_\phi^\alpha \mathbf{G}^{-1} \Phi_J(x). \quad (53)$$

Finally, we obtain

$$\mathbf{P}_\psi^\alpha = \mathbf{G} \mathbf{P}_\phi^\alpha \mathbf{G}^{-1}. \quad (54)$$

4. NUMERICAL METHOD AND CONVERGENCE ANALYSIS

4.1 DESCRIPTION OF NUMERICAL METHOD

Consider the following initial value problem:

$$D^\alpha u(x) + N[u(x)] + L[u(x)] = g(x), \quad \alpha > 0, \quad (55)$$

$$u^{(k)}(0) = c_k, \quad k = 0, 1, 2, \dots, m-1, \quad m-1 < \alpha \leq m, \quad (56)$$

where L is a linear operator, N is a nonlinear operator, and D^α is the Caputo fractional derivative of order α .

Applying the fractional integral operator I^α to both sides of (55) and using Lemma 1, yields

$$u(x) + I^\alpha (N[u(x)] + L[u(x)] - R(x)) = 0, \quad \alpha > 0, \quad (57)$$

in which $R(x) = \sum_{k=0}^{n-1} u^{(k)}(0^+) \frac{x^k}{k!} + g(x)$.

To solve the problem, $u(x)$ and $R(x)$ are approximated by Legendre multiwavelets as

$$u(x) = \mathbf{D}^T \Psi_J(x), \quad R(x) = \mathbf{R}^T \Psi_J(x). \quad (58)$$

Now for the nonlinear part $N[u(x)]$, by nonlinear term approximation described in Section 3.1, we have

$$N[u(x)] = \mathbf{N}^T \Psi_J(x). \quad (59)$$

For the linear part, we have

$$L[u(x)] = \mathbf{L}^T \Psi_J(x) = a \mathbf{D}^T \Psi_J(x). \quad (60)$$

where $\mathbf{L} = a\mathbf{D}$ has a linear relation with $\mathbf{D}$ and a is constant.

Substituting Eq. (58)-(60) into Eq. (55), we can get a system of algebraic equations as

$$\mathbf{D}^T + (\mathbf{N}^T + \mathbf{L}^T - \mathbf{R}^T) \mathbf{P}_\psi^\alpha = 0, \quad \alpha > 0, \quad (61)$$

According to the Wu's [25] technology for determining the initial iteration value, the first guess can be determined as

$$u_0(x) = \sum_{k=0}^{n-1} u^{(k)}(0^+) \frac{x^k}{k!} + J^\alpha (g_{\text{frac}}(x)), \quad (62)$$

where $g_{\text{frac}}(x)$ is the fractional component of $g(x)$ with respect to x . The coefficient matrix $\mathbf{D}^T$ can be computed by using the MATLAB function `fsolve()` or the method described in [26]. The power of SCLMW depends on the occurrence of the exact solution in Eq. (62) and the accurate representation of fractional integral. If the exact solution exists, SCLMW can converge very fast to the exact solution. What's more, for the linear fractional equation, the solution can be obtained by following equation

$$\mathbf{D}^T = \mathbf{R}^T \mathbf{P}_\psi^\alpha (\mathbf{I} + a \mathbf{P}_\psi^\alpha)^{-1}. \quad (63)$$

where $\mathbf{I}$ is an identical matrix.

4.2 CONVERGENCE ANALYSIS

Theorem 4.1. Suppose that the function $f(x): [0,1] \rightarrow \mathbf{R}$ and $f(x) \in C^{r-1}[0,1]$. Then $P_J f(x)$ approximates f with mean error bounded as follows [19]:

$$\|f(x) - P_J f(x)\| \leq M(r, J). \quad (64)$$

in which

$$M(r, J) = \frac{1}{(r-1)!} \frac{1}{2^{(J+2)(r-1)-1}} \sup_{x \in [0,1]} |f^{(r-1)}(x)|. \quad (65)$$

Theorem 4.2. Suppose that the function $f(x) : [0,1] \rightarrow \mathbf{R}$ and $f(x) \in C^{r-1}[0,1]$. Then $P_J f(x)$ converges to f :

Proof. Let $P_J f(x) = y_J^r = \sum_{m=0}^{r-1} \left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\}$ be a sequence of partial sums, we have

$$\begin{aligned} \langle f(x), y_J^r \rangle &= \left\langle f(x), \sum_{m=0}^{r-1} \left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\} \right\rangle \\ &= \sum_{m=0}^{r-1} \left\{ \bar{c}_{0,0}^m \langle f(x), \phi_{0,0}^m(x) \rangle + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \bar{d}_{j,k}^m \langle f(x), \psi_{j,k}^m(x) \rangle \right\}, \\ &= \sum_{m=0}^{r-1} \left\{ |c_{0,0}^m|^2 + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} |d_{j,k}^m|^2 \right\} \end{aligned} \quad (66)$$

Without loss of generality, assuming that $r_2 > r_1$ and $J_2 > J_1$, then we can get

$$\begin{aligned} \|y_{J_1}^{r_1} - y_{J_2}^{r_2}\| &= \left\| \sum_{m=0}^{r_1-1} \left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=0}^{J_1-1} \sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\} - \sum_{m=0}^{r_2-1} \left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=0}^{J_2-1} \sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\} \right\| \\ &= \left\| \sum_{m=r_1}^{r_2-1} \left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=J_1}^{J_2-1} \sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\} \right\|^2 \\ &= \left\langle \sum_{m=r_1}^{r_2-1} \left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=J_1}^{J_2-1} \sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\}, \sum_{m=r_1}^{r_2-1} \left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=J_1}^{J_2-1} \sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\} \right\rangle \\ &= \sum_{m=r_1}^{r_2-1} \left\{ |c_{0,0}^m|^2 + \sum_{j=J_1}^{J_2-1} \sum_{k=0}^{2^j-1} |d_{j,k}^m|^2 \right\} \end{aligned} \quad (67)$$

So, as $N = r2^J \rightarrow \infty$, from Bessel's inequality and Theorem 4.1, we have

$$\sum_{m=r_1}^{r_2-1} \left\{ |c_{0,0}^m|^2 + \sum_{j=J_1}^{J_2-1} \sum_{k=0}^{2^j-1} |d_{j,k}^m|^2 \right\} = \|y_{J_1}^{r_1} - y_{J_2}^{r_2}\| \leq \|y(x) - y_{J_1}^{r_1}\| + \|y(x) - y_{J_2}^{r_2}\| \leq M(r_1, J_1) + M(r_2, J_2). \quad (68)$$

It can be found that $\sum_{m=0}^{r-1} \left\{ |c_{0,0}^m|^2 + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} |d_{j,k}^m|^2 \right\}$ is convergent, thus $\{y_J^r\}$ is a Cauchy sequence.

Further

$$\begin{aligned} \left\langle f(x) - \lim_{N \rightarrow \infty} y_J^r, \phi_{j,k}^m(x) \right\rangle &= \left\langle f(x), \phi_{j,k}^m(x) \right\rangle - \left\langle \lim_{N \rightarrow \infty} y_J^r, \phi_{j,k}^m(x) \right\rangle \\ &= c_{j,k}^m - \left(\lim_{N \rightarrow \infty} \sum_{k=0}^{2^j-1} \sum_{m=0}^{r-1} c_{j,k}^m \phi_{j,k}^m(x), \phi_{j,k}^m(x) \right) \\ &= c_{j,k}^m - c_{j,k}^m = 0 \end{aligned} \quad (69)$$

It implies that sequence $\{y_J^r\}$ converges to $f(x)$.

Theorem 4.3. Assume that $f(x) \in C^1[0,1]$, and $\exists M > 0, \forall x \in (0,1), |f'(x)| \leq M$, as $N \rightarrow \infty$, $\xi^T \mathbf{B}_N(x)$ is point-wise convergent to $f(x)$ with error bounded as follows [36,37]

$$\|f(x) - \xi^T \mathbf{B}_N(x)\| \leq \frac{M}{N}. \quad (70)$$

Theorem 4.1, 4.2 and 4.3 give the guarantee of convergence of SCLWM. It is clear that the approximation will be more accurate if N is increased.

5. APPLICATIONS AND RESULTS

In this section, we first give two linear examples with exact solution to analysis the accuracy of SCLMW with difference values of r and J , and then use three examples to demonstrate the validity and applicability of SCLMW in the solution of nonlinear fractional differential equations. It should be note that linear examples are solved by using Eq. (63) while the nonlinear cases using Matlab function `fsolve()`.

Table 1 L_2 error for different values of r and J .

r	$J=2$	$J=3$	$J=4$	$J=5$	$J=6$	$J=7$	$J=8$
2	0.0041	0.0015	5.2449e-4	1.8684e-4	6.6467e-5	2.3621e-5	8.3868e-6
3	4.0370e-4	1.4915e-4	5.4740e-5	1.9973e-5	7.2508e-6	2.6209e-6	9.4393e-7
4	2.6530e-4	9.8094e-5	3.5978e-5	1.3111e-5	4.7531e-6	1.7158e-6	6.1719e-7

Example 1. Consider the composite fractional oscillation equation [13]

$$D^{0.25}u(x) + u(x) - x^2 - \frac{2}{\Gamma(2.75)}x^{1.75} = 0, \quad (71)$$

with the initial condition $u(0) = 0$ and the exact solution is $u(x) = x^2$.

Table 1 show the L_2 error of example 1 for different values of r and J . It is easy to deduce that the accuracy of SCLMW is improving with the increase of r and J and converge rapidly which depend on the accuracy of the integral of fraction-order. It is worth mentioning that obtained results agree well with exact solutions even for small values of r and J .

Example 2. Consider the following fractional linear equation [13, 42-44]

$$D^\alpha u(x) + u(x) = 0, \quad 0 < x < 1, \quad 0 < \alpha \leq 2, \quad (72)$$

$$u(0) = 1, \quad u'(0) = 0, \quad (73)$$

The exact solution of this problem is $u(x) = \sum_{k=0}^{\infty} \frac{(-x^\alpha)^k}{\Gamma(\alpha k + 1)}$.

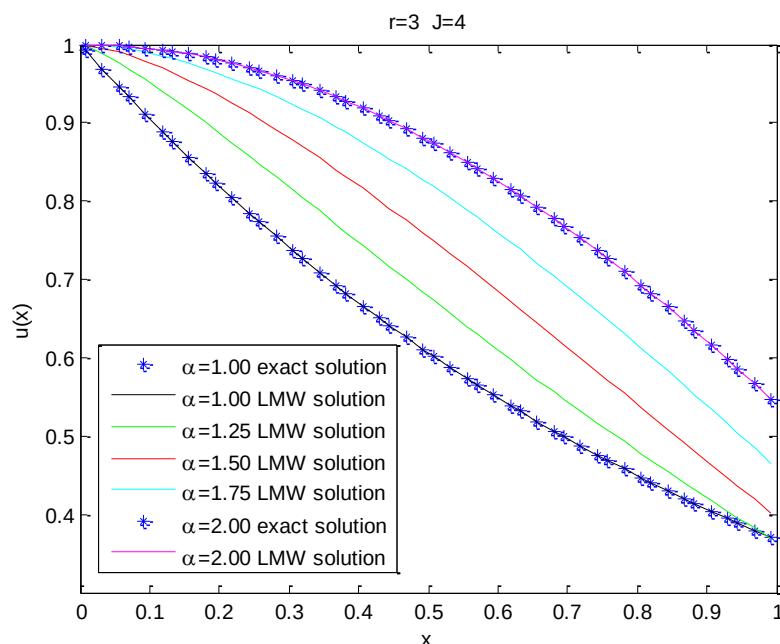


Fig.2. Numerical solutions of Example 2 for difference α

Table 2. L_2 error for example 2.

r	α	$J=4$	$J=5$	$J=6$	$J=7$	$J=8$
3	1	4.8361e-05	1.7182e-05	6.0904e-06	2.1561e-06	7.6280e-07
	2	2.0809e-05	7.3880e-06	2.6177e-06	9.2652e-07	3.2776e-07
4	1	3.1739e-05	1.1219e-05	3.9663e-06	1.4023e-06	4.9578e-07
	2	1.3633e-05	4.8198e-06	1.7040e-06	6.0246e-07	2.1300e-07

Table 3. Numerical results of example 2 for $\alpha = 0.50$ and 0.75 with $r=3$ and $J=5$.

α	0.50			0.75		
x	JOM[34]	SFJTM[35]	SCLMW	JOM[34]	SFJTM[35]	SCLMW
0.1	0.737355	0.723578	0.723585	0.830145	0.828251	0.828258
0.2	0.633468	0.643788	0.643789	0.731281	0.732585	0.732588
0.3	0.593029	0.592018	0.592019	0.660086	0.660337	0.660339
0.4	0.563169	0.553606	0.553606	0.603789	0.602121	0.602122
0.5	0.512164	0.523157	0.523156	0.552192	0.553603	0.553603
0.6	0.499543	0.498025	0.498024	0.512051	0.512285	0.512285
0.7	0.485038	0.476703	0.476703	0.477957	0.476555	0.476555
0.8	0.447227	0.458246	0.458246	0.44395	0.445292	0.445293
0.9	0.452686	0.442021	0.442021	0.418733	0.417682	0.417682
1.0	0.439646	0.427584	0.427583	0.393598	0.393108	0.393108

We solved the problem by applying the method described in Section 4. Figure 2 exhibits the numerical results for different values of α with $r=3$ and $J=4$. The L_2 errors for $\alpha=1.0$ and $\alpha=2.0$ are shown in Table 2. From Figure 2, one can see that SCLMW can achieve a good approximation with the exact solution for $\alpha=1.0$ and $\alpha=2.0$. In Table 3, we compare our numerical results with those obtained by the shifted fractional-order Jacobi orthogonal functions (SFJTM) [35] and by the Jacobi operational matrix method (JOM) [34]. From Table

3, it is clear that SCLMW can achieve a good approximation of solution nearly the same as SFJTM and better than JOM.

351
352
353

Table 4 L_2 error of Example 3 for different values of r and J

r	$J=1$	$J=2$	$J=3$	$J=4$	$J=5$	$J=6$	$J=7$
2	0.0134	0.0048	0.0017	6.2198e-4	2.2195e-4	7.9073e-05	2.8133e-05
3	0.0014	5.4055e-4	2.0460e-4	7.5963e-5	2.7894e-5	1.0168e-05	5.5354e-15
4	9.7490e-4	3.6690e-4	1.3642e-4	5.0248e-5	1.8373e-5	4.4024e-15	6.3071e-15

354
355

Example 3. Consider the composite fractional oscillation equation [14]

356
$$D^{0.25}u(x) + u^2(x) = \frac{2}{\Gamma(2.75)}x^{1.75} + x^4, \quad (74)$$

357 with the initial condition $u(0) = 0$ and the exact solution is $u(x) = x^2$.

358

359 Table 4 show the L_2 error of example 3 for different values of r and J . It is also support the
360 conclusion that the accuracy of SCLMW is improving with the increase of r and J and
361 converge rapidly. It can be found that SCLMW can converge to the exact solution with large
362 values of r and J . This is because that the exact solution exists in first guess defined in Eq.
363 (62) and the error of fractional integral operator is smaller than the tolerance in Matlab
364 function `fsolve()`.
365

366

Example 4. Consider the following nonlinear Riccati equation [10, 30]:

367
$$D^\alpha u(x) + u^2(x) = 1, \quad 0 < \alpha \leq 1, \quad 0 < x < 1, \quad (75)$$

368 subject to the initial state $u(0) = 0$. The exact solution of this problem, when $\alpha=1$,
369 is $u(x) = (e^{2x} - 1)/(e^{2x} + 1)$.

370 Figure 3 exhibits the approximate solutions of Example 4 for difference α with $r=3$ and $J=6$. It
371 can be found that SCLMW's solution is in very good agreement with the exact solution for
372 $\alpha=1.00$. Therefore, numerical results for $\alpha=0.25, 0.50$ and 0.75 are also credible. Table 5
373 compares SCLMW with variational iteration method using fractional-order Legendre
374 Functions [28]. It can be found that our result is as accurate as those in Ref [28] in the case
375 of $\alpha=1.00$ and nearly the same for $\alpha=0.25, 0.50$ and 0.75 . This demonstrates the importance
376 of presented numerical scheme in solving nonlinear fractional differential equations.
377

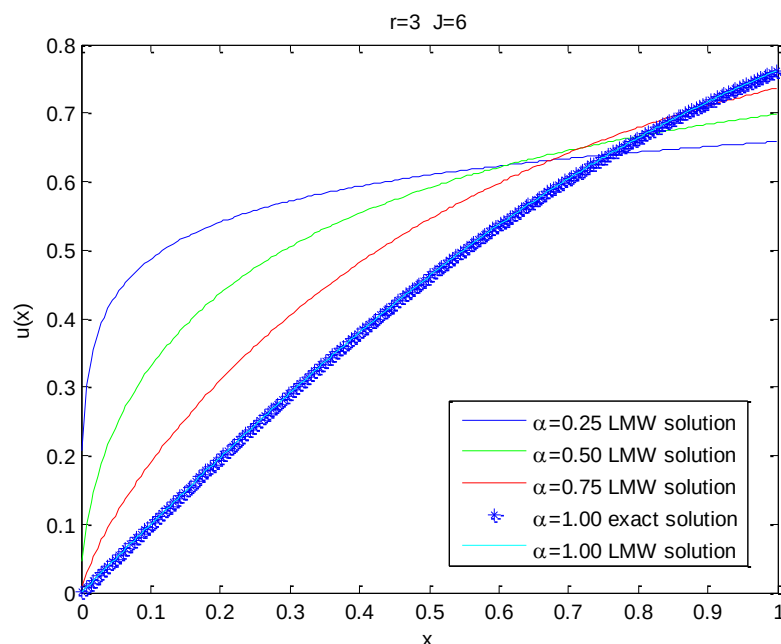


Figure 3: Numerical solutions of Example 4 for difference α .

Table 5. Numerical results of Example 4 for $r=3$ and $J=6$ with comparison to Ref. [28].

x	$\alpha = 0.50$		$\alpha = 0.75$		$\alpha = 1.00$		Exact
	Ref. [28]	SCLMW	Ref. [28]	SCLMW	Ref. [28]	SCLMW	
0.1	0.330108	0.330107	0.190101	0.190100	0.099668	0.099668	0.099668
0.2	0.436839	0.436839	0.309976	0.309975	0.197375	0.197375	0.197375
0.3	0.504889	0.504890	0.404615	0.404614	0.291313	0.291312	0.291313
0.4	0.553782	0.553782	0.481632	0.481631	0.379949	0.379949	0.379949
0.5	0.591195	0.591194	0.545090	0.545089	0.462117	0.462117	0.462117
0.6	0.621014	0.621014	0.597783	0.597783	0.537050	0.537049	0.53705
0.7	0.645485	0.645486	0.641820	0.641820	0.604368	0.604367	0.604368
0.8	0.666020	0.666019	0.678850	0.678849	0.664037	0.664037	0.664037
0.9	0.683554	0.683552	0.710175	0.710175	0.716298	0.716298	0.716298

Example 5. Consider fractional Riccati equation [30]

$$D^\alpha u(x) - 2u(x) + u^2(x) = 1, \quad 0 < \alpha \leq 1, \quad (76)$$

subject to the initial condition $u(0) = 0$, for $\alpha = 1$ the exact solution is

$$u(x) = 1 + \sqrt{2} \tanh \left[\sqrt{2}x + \frac{1}{2} \log \left(\frac{(\sqrt{2}-1)}{(\sqrt{2}+1)} \right) \right].$$

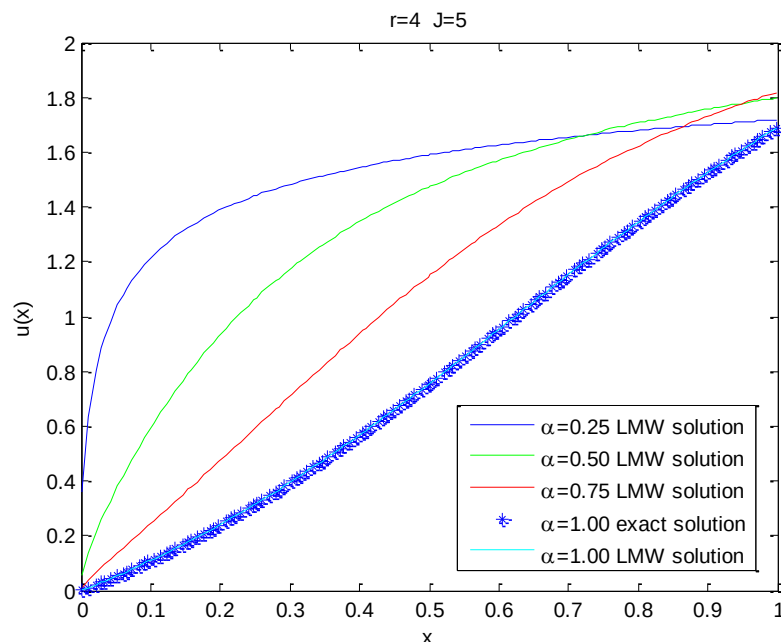


Fig. 4. Numerical solutions of Example 5 for difference α .

Table 6. Numerical results of Example 5 for $r=4$ and $J=5$ with comparison to Ref. [28].

x	$\alpha=0.50$		$\alpha=0.75$		$\alpha=1.0$		exact
	Ref[28]	SCLMW	Ref[28]	SCLMW	Ref[28]	SCLMW	
0.1	0.592833	0.592785	0.245446	0.245443	0.110308	0.110300	0.110295
0.2	0.933104	0.933200	0.475051	0.475108	0.241990	0.241983	0.241976
0.3	1.174069	1.173996	0.710050	0.710032	0.395119	0.395111	0.395104
0.4	1.346694	1.346661	0.938523	0.938543	0.567830	0.567819	0.567812
0.5	1.473790	1.473890	1.149016	1.149066	0.756032	0.756020	0.756014
0.6	1.570577	1.570573	1.334339	1.334336	0.953583	0.953571	0.953566
0.7	1.646302	1.646199	1.491949	1.491927	1.152968	1.152953	1.152946
0.8	1.706644	1.706880	1.622950	1.622992	1.346381	1.346367	1.346363
0.9	1.756349	1.756643	1.730575	1.730612	1.526927	1.526914	1.526911

Figure 4 gives approximate solution of fractional Riccati differential equation in Example 5. From Fig. 4 it can be found that the numerical solution is in very good agreement with the exact solution when $\alpha = 1.0$. Therefore, it should be held that the solution for $\alpha = 0.25, 0.50$ and 0.75 is also credible. Table 6 compares the results of SCLMW with those obtained by variational iteration method using fractional-order Legendre Functions. It can be found that our result is more accurate than that in Ref [28] for $\alpha = 1.00$. So it can be held that SCLMW's results are better for $\alpha = 0.25, 0.50$ and 0.75 . This demonstrates the importance of presented numerical scheme in solving nonlinear fractional differential equations.

6. CONCLUSION

In this work Legendre multi-wavelets operational matrix of fractional order integration is derived and used to solve fractional differential equations. By using the two-scales difference relations of Legendre multi-wavelets and properties of BPFs, the operational matrices of fractional integration and nonlinear term expansion coefficients vector are derived.

409 Consequently, the solution of fractional differential equation is converted to the solution of a
 410 sparse nonlinear system of algebraic equations. The achieved results are compared with
 411 exact solutions and with the solutions obtained by some other numerical methods. The
 412 numerical results show that SCLMW is accuracy and applicable even for small values of r
 413 and J . what's more, SCLMW can get the exact solution when exact solution exists in first
 414 guess defined in Eq. (62) and the error of fractional integral operator is smaller than the
 415 tolerance in Matlab function `fsolve()`. It is worth mentioning that the proposed algorithms can
 416 be extended to solve a number of fractional partial differential equations.
 417

423 REFERENCES

- 424 1 R. P. Agarwal, B. de Andrade, and C. Cuevas. Weighted pseudo-almost periodic solutions
 425 of a class of semilinear fractional differential equations. *Nonlinear Analysis: Real World*
 426 *Applications*, vol. 11, no. 5, pp. 3532-3554, 2010.
- 427 2 R. P. Agarwal, V. Lakshmikantham, and J. J. Nieto. On the concept of solution for
 428 fractional differential equations with uncertainty. *Nonlinear Analysis: Theory, Methods &*
 429 *Applications*, vol. 72, no. 6, pp. 2859-2862, 2010.
- 430 3 R. Garrappa and M. Popolizio. On the use of matrix functions for fractional partial
 431 differential equations. *Mathematics and Computers in Simulation*, vol. 81, no. 5, pp. 1045-
 432 1056, 2011.
- 433 4 R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing, River
 434 Edge, NJ, USA, 2000.
- 435 5 Beylkin G, Coifman R, Rokhlin V. Fast wavelet transforms and numerical algorithms.
 436 *Communication on Pure and Applied Mathematics* 1991; 44(2):141-183.
- 437 6 Li Yuanlu, Zhao Weiwei, Haar wavelet operational matrix of fractional order integration and
 438 its applications in solving the fractional order differential equations. *Appl Math Comput*
 439 2010;216:2276-2285.
- 440 7 S. Saha Ray, On haar wavelet operational matrix of general order and its application for
 441 the numerical solution of fractional Bagley Torvik equation, *Applied Mathematics and*
 442 *Computation*, vol. 218, no. 9, pp. 5239-5248, 2012.
- 443 8 S. S. Ray and A. Patra, Haar wavelet operational methods for the numerical solutions of
 444 fractional order nonlinear oscillatory vanderPol system, *Applied Mathematics and*
 445 *Computation*, vol. 220, pp. 659-667, 2013.
- 446 9 Li Yuanlu, Solving a nonlinear fractional differential equation using Chebyshev wavelets,
 447 *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 9, pp. 2284-
 448 2292, 2010. 15:2284-2292.
- 449 10 Y. Wang and Q. Fan, The second kind Chebyshev wavelet method for solving fractional
 450 differential equations, *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8592-
 451 8601, 2012.
- 452 11 Saeedi H, Mohseni Moghadam M, Mollahasani N, Chuev GN, A CAS wavelet method for
 453 solving nonlinear Fredholm integro-differential equations of fractional order, *Communications*
 454 *in Nonlinear Science and Numerical Simulation*, 2011;16:1154-1163.
- 455 12 Saeedi H, Mohseni Moghadam M. Numerical solution of nonlinear Volterra integro-
 456 differential equations of arbitrary order by CAS wavelets. *Appl Math Comput* 2011;16:1216-
 457 1126
- 458 13 H. Jafari, S. A. Yousefi, M. A. Firoozjaee, S. Momani, and C. M. Khalique, Application of
 459 Legendre wavelets for solving fractional differential equations, *Computers & Mathematics*
 460 *with Applications*, vol. 62, no. 3, pp. 1038-1045, 2011.

- 14 M. Rehman and R. A. Khan, The legendre wavelet method for solving fractional differential equations, Communications in Nonlinear Science and Numerical Simulation, vol. 16, no. 11, pp. 4163-4173, 2011.
- 15 M. A. Mohamed and M. S. Torky, Approximate solution of fractional nonlinear partial differential equations by legendre multiwavelet Galerkin method, Journal of Applied Mathematics, vol. 2014, Article ID 192519, 12 pages, 2014.
- 16 A. K. Gupta and S. Saha Ray, Wavelet Methods for Solving Fractional Order Differential Equations, Mathematical Problems in Engineering, Volume 2014, Article ID 140453, 11 pages <http://dx.doi.org/10.1155/2014/140453>
- 17 B.K.Alpert. Sparse Representation of Smooth Linear Operator. PhD Thesis, Yale University,1990
- 18 B. Alpert, G. Beylkin, R. R. Coifman, V. Rokhlin, Wavelet-like bases for the fast solution of second-kind integral equations, SIAM J. Sci. Statist. Comput. 14, 1 (1993), 159-184.
- 19 B.Alpert, A class of bases in L2 for the sparse representation of integral operators, SIAM J. Appl. Math. Anal 24(1):246-262, 1993.
- 20 G.Beylkin, J.M.Keiser and L.Volzovoi, A new class of stable time discretization schemes for the solution of nonlinear PDEs. J. Comp Physics 147:132-137, 1998.
- 21 G.Beylkin and J.M.Keiser. On the adaptive numerical solution of nonlinear partial differential equations in the wavelet bases .J.Comp.Physics.,132:233-259,1997.PAM report 262,1995.
- 22 B.Alpert, G.Beylkin, D.Gines and L.Vozovoi, Adaptive solution of partial differential equations in multiwavelet bases, J.Comput.Phys., 182(2002).
- 23 M. Lakestani, B. N. Saray, M. Dehghan, Numerical solution for the weakly singular Fredholm integro-differential equations using Legendre multiwavelets, J. Comput. Appl. Math., 235, (2011), 3291-3303.
- 24 Swaraj Paul, M.M. Panja, B.N. Mandal ,Multiscale approximation of the solution of weakly singular second kind Fredholm integral equation in Legendre multiwavelet basis, Journal of Computational and Applied Mathematics 300 (2016) 275-289
- 25 G. C. Wu, Challenge in the variational iteration method- a new approach to identification of the Lagrange multipliers, Journal of King Saud University, vol. 25, pp. 175-178, 2013.
- 26 F. Yin, J. Song, and F. Lu, A coupled method of Laplace transform and Legendre wavelets for nonlinear Klein-Gordon equations, Mathematical Methods in the Applied Sciences, 2013.
- 27 Adem Kilicman, Zeyad Abdel Aziz Al Zhour. Kronecker operational matrices for fractional calculus and some applications. Applied Mathematics and Computation 187 (2007) 250-265
- 28 Fukang Yin, Junqiang Song, Hongze Leng, and Fengshun Lu. Couple of the Variational Iteration Method and Fractional-Order Legendre Functions Method for Fractional Differential Equations. The Scientific World Journal, Volume 2014, Article ID 928765, 9 pages, <http://dx.doi.org/10.1155/2014/928765>
- 29 A. Saadatmandi and M. Dehghan. A new operational matrix for solving fractional-order differential equations. Computers and Mathematics with Applications, vol. 59, no. 3, pp. 1326-1336, 2010.
- 30 Z. Odibat and S. Momani. Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order. Chaos, Solitons & Fractals, vol. 36, no. 1, pp.167–174, 2008.
- 31 I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.
- 32 I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation, Fractional Calculus and Applied Analysis, vol. 5, pp. 367-386, 2002.
- [33] M. Caputo, Linear models of dissipation whose Q is almost frequency independent-part II, Geophysical Journal of the Royal Astronomical Society, vol. 13, pp. 529-539, 1967.

512 34 E.H. Doha, A.H. Bhrawy, S.S. Ezz-Eldien, A new Jacobi operational matrix: An
513 application for solving fractional differential equations, *Applied Mathematical Modelling*, 36
514 (2012) 4931-4943.
515 35 A.H. Bhrawy, M.A. Zaky, Shifted fractional-order Jacobi orthogonal functions: Application
516 to a system of fractional differential equations, *Applied Mathematical Modelling*, 40 (2016)
517 832-845.
518 36 Monireh Nosrati Sahlan, Hamid Reza Marasi, Farzaneh Ghahramani, Block-pulse
519 functions approach to numerical solution of Abel's integral equation, *Cogent Mathematics*
520 (2015), 2:1047111.
521 37 Wen-Liang Chen & Chun-Liang Lee, On the convergence of the block-pulse series
522 solution of a linear time-invariant system, *International Journal of Systems Science*, (1982),
523 13:5, 491-498, DOI: 1080/00207728208926363
524
525