

# GENERALIZED HYERS-ULAM STABILITY OF A RECIPROCAL-CUBIC FUNCTIONAL EQUATION

ABSTRACT. In this paper, we obtain the generalized Hyers-Ulam stability of a new cubic reciprocal functional equation of the form

$$f(2x+y) + f(2x-y) = \frac{4f(x)f(y) \left[ 4f(y) + 3f(x)^{\frac{2}{3}}f(y)^{\frac{1}{3}} \right]}{\left( 4f(y)^{\frac{2}{3}} - f(x)^{\frac{2}{3}} \right)^3}$$

with  $f(y) \neq \frac{1}{8}f(x)$ . We also extend the results related to Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product-sum of powers of norms for the same equation.

---

2000 *Mathematics Subject Classification.* : 39B22, 39B52, 39B72.

*Key words and phrases.* : Reciprocal functional equation, Reciprocal-cubic functional equation, Generalized Hyers-Ulam stability.

## 1. INTRODUCTION

The classical theory of stability of functional equations is instigated by the question of S.M. Ulam [1] in 1940. In the subsequent year, D.H. Hyers [2] was the foremost mathematician to establish the result connected with the stability of functional equations. The result obtained by D.H. Hyers is called as Hyers-Ulam stability of functional equation. Later in 1950, T. Aoki [3] made further simplification to the theorem of D.H. Hyers. In 1978, Th.M. Rassias [4] took a broad view in the Hyers' result by taking the upper bound as sum of powers of norms. The theorem proved by Th.M. Rassias persuaded a lot of mathematicians to work on the stability of various functional equations and the result obtained by Th.M. Rassias is recognized as Hyers-Ulam-Rassias stability of functional equation. J.M. Rassias [5] provided further generalization of the result of D.H. Hyers by using weaker conditions controlled by a product of different powers of norms. The result proved by J.M. Rassias is termed as Ulam-Gavruta-Rassias stability of functional equation. Further, in 1994, P. Gavruta [6] provided a generalization of Th.M. Rassias' theorem by replacing a general control function as upper bound. The stability result ascertained by P. Gavruta is celebrated as generalized Hyers-Ulam stability of functional equation. In 2008, K. Ravi et al. [7] investigated the stability of a quadratic functional equation using mixed product-sum of powers of norms. This stability result acquired by J.M. Rassias is known as J.M. Rassias stability involving mixed product-sum of powers of norms.

In the year 2010, K. Ravi and B.V. Senthil Kumar [8] proved Ulam-Gavruta-Rassias stability for a new reciprocal type functional equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)} \quad (1.1)$$

where  $f : X \rightarrow \mathbb{R}$  is a mapping with  $X$  as the space of non-zero real numbers. The reciprocal function  $g(x) = \frac{c}{x}$  is a solution of the functional equation (1.1).

The other results pertaining to stability of different reciprocal type functional equations can be found in ([9], [10], [11]).

A. Bodaghi and S.O. Kim [12] introduced and studied the Ulam-Gavruta-Rassias stability for the quadratic reciprocal functional equation

$$f(2x + y) + f(2x - y) = \frac{2f(x)f(y)[4f(y) + f(x)]}{(4f(y) - f(x))^2}. \quad (1.2)$$

The quadratic reciprocal function  $f(x) = \frac{c}{x^2}$  is a solution of the functional equation (1.2).

A. Bodaghi and Y. Ebrahimdoost [13] generalized the equation (1.2) as

$$f((a + 1)x + ay) + f((a + 1)x - ay) = \frac{2f(x)f(y)[(a + 1)^2f(y) + a^2f(x)]}{((a + 1)^2f(y) - a^2f(x))^2} \quad (1.3)$$

where  $a \in \mathbb{Z}$  with  $a \neq 0$  and established the generalized Hyers-Ulam-Rassias stability for the functional equation (1.3) in non-Archimedean fields.

K. Ravi et al. [14] investigated the generalized Hyers-Ulam-Rassias stability of a reciprocal-quadratic functional equation of the form

$$r(x + 2y) + r(2x + y) = \frac{r(x)r(y) \left[ 5r(x) + 5r(y) + 8\sqrt{r(x)r(y)} \right]}{\left[ 2r(x) + 2r(y) + 5\sqrt{r(x)r(y)} \right]^2} \quad (1.4)$$

in intuitionistic fuzzy normed spaces.

In this paper, we obtain the generalized Hyers-Ulam stability of a cubic reciprocal functional equation of the form

$$f(2x + y) + f(2x - y) = \frac{4f(x)f(y) \left[ 4f(y) + 3f(x)^{\frac{2}{3}}f(y)^{\frac{1}{3}} \right]}{\left( 4f(y)^{\frac{2}{3}} - f(x)^{\frac{2}{3}} \right)^3}. \quad (1.5)$$

in the skeleton of non-zero real numbers with  $f(y) \neq \frac{1}{8}f(x)$ . We extend the results concerning Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product-sum of

powers of norms for the equation (1.5). It is easy to see that the reciprocal-cubic function  $f(x) = \frac{1}{x^3}$  is a solution of the functional equation (1.5).

## 2. GENERALIZED HYERS-ULAM STABILITY OF EQUATION (1.5)

In this section, we investigate the generalized Hyers-Ulam stability of equation (1.5) in the schema of non-zero real numbers. We also establish the results pertaining to Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by product-sum of powers of norms.

Throughout this paper, we assume that  $X$  is the set of non-zero real numbers. For notational pragmatism, we define the difference operator  $D_f : X \times X \rightarrow \mathbb{R}$  such that

$$D_f(x, y) = f(2x + y) + f(2x - y) - \frac{4f(x)f(y) \left[ 4f(y) + 3f(x)^{\frac{2}{3}}f(y)^{\frac{1}{3}} \right]}{\left( 4f(y)^{\frac{2}{3}} - f(x)^{\frac{2}{3}} \right)^3}$$

for all  $x, y \in X$ .

In the following results, we will set  $\frac{0^2}{0} = 0$  and assume  $f(x) \neq 0$  and  $x + y \neq 0$  for all  $x, y \in \mathbb{R}$ .

For proving our main results, we necessitate the following definition.

**Definition 2.1.** A mapping  $f : X \rightarrow \mathbb{R}$  is called as reciprocal-cubic mapping if  $f$  satisfies equation 1.5.

**Theorem 2.2.** Let  $\varphi : X \times X \rightarrow \mathbb{R}$  be a function satisfying

$$\sum_{j=0}^{\infty} \frac{1}{27^j} \varphi \left( \frac{x}{3^{j+1}}, \frac{y}{3^{j+1}} \right) < +\infty \quad (2.1)$$

for all  $x, y \in X$ . If a function  $f : X \rightarrow \mathbb{R}$  satisfies functional inequality

$$|D_f(x, y)| \leq \varphi(x, y) \quad (2.2)$$

for all  $x, y \in X$ , then there exists a unique reciprocal-cubic mapping  $R_c : X \rightarrow \mathbb{R}$  which satisfies (1.5) and the inequality

$$|f(x) - R_c(x)| \leq \sum_{j=0}^{\infty} \frac{1}{27^j} \varphi\left(\frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}\right) \quad (2.3)$$

for all  $x \in X$ .

*Proof.* Plugging  $(x, y)$  by  $(x, x)$  in (2.2), we obtain

$$\left| f(3x) - \frac{1}{27}f(x) \right| \leq \varphi(x, x) \quad (2.4)$$

for all  $x \in X$ . Now, substituting  $x$  by  $\frac{x}{3}$  in (2.4), we find

$$\left| f(x) - \frac{1}{27}f\left(\frac{x}{3}\right) \right| \leq \varphi\left(\frac{x}{3}, \frac{x}{3}\right) \quad (2.5)$$

for all  $x \in X$ . Now, replacing  $x$  by  $\frac{x}{3}$  in (2.5), dividing by 27 and then adding the ensuing inequality with (2.5), one finds

$$\left| f(x) - \frac{1}{27^2}f\left(\frac{x}{3^2}\right) \right| \leq \sum_{j=0}^1 \frac{1}{27^j} \varphi\left(\frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}\right)$$

for all  $x \in X$ . Similarly, continuing further and applying induction arguments on a positive integer  $n$ , we arrive

$$\left| f(x) - \frac{1}{27^n}f\left(\frac{x}{3^n}\right) \right| \leq \sum_{j=0}^{n-1} \frac{1}{27^j} \varphi\left(\frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}\right) \quad (2.6)$$

for all  $x \in X$ . For any positive integer  $k$  and  $x \in X$ , we have

$$\begin{aligned} \left| \frac{1}{27^{k+1}}f\left(\frac{x}{3^{k+1}}\right) - \frac{1}{27^k}f\left(\frac{x}{3^k}\right) \right| &= \frac{1}{27^k} \left| f\left(\frac{x}{3^k}\right) - \frac{1}{27}f\left(\frac{x}{3^{k+1}}\right) \right| \\ &\leq \frac{1}{27^k} \varphi\left(\frac{x}{3^{k+1}}, \frac{x}{3^{k+1}}\right). \end{aligned}$$

Hence for any integers  $n, m$  with  $n > m > 0$ , we obtain by using the triangular inequality

$$\begin{aligned}
 & \left| \frac{1}{27^n} f\left(\frac{x}{3^n}\right) - \frac{1}{27^m} f\left(\frac{x}{3^m}\right) \right| \\
 &= \left| \frac{1}{27^n} f\left(\frac{x}{3^n}\right) - \frac{1}{27^{n-1}} f\left(\frac{x}{3^{n-1}}\right) + \frac{1}{27^{n-1}} f\left(\frac{x}{3^{n-1}}\right) - \dots \right. \\
 & \quad \left. + \frac{1}{27^{m+1}} f\left(\frac{x}{3^{m+1}}\right) - \frac{1}{27^m} f\left(\frac{x}{3^m}\right) \right| \\
 &\leq \frac{1}{27^{n-1}} \varphi\left(\frac{x}{3^n}, \frac{x}{3^n}\right) + \dots + \frac{1}{27^m} \varphi\left(\frac{x}{3^{m+1}}, \frac{x}{3^{m+1}}\right) \\
 &\leq \sum_{k=m}^{n-1} \frac{1}{27^k} \varphi\left(\frac{x}{3^{k+1}}, \frac{x}{3^{k+1}}\right)
 \end{aligned} \tag{2.7}$$

for all  $x \in X$ . Taking the limit as  $n \rightarrow +\infty$  in (2.7) and considering (2.1), it follows that the sequence  $\{\frac{1}{27^n} f(\frac{x}{3^n})\}$  is a Cauchy sequence for each  $x \in X$ . Since  $\mathbb{R}$  is complete, we can define  $R_c : X \rightarrow \mathbb{R}$  by  $R_c(x) = \lim_{n \rightarrow \infty} \frac{1}{27^n} f(\frac{x}{3^n})$ . To show that  $R_c$  satisfies (1.5), replacing  $(x, y)$  by  $(3^{-n}x, 3^{-n}y)$  in (2.2) and dividing by  $27^n$ , we obtain

$$|27^{-n} D_f(3^{-n}x, 3^{-n}y)| \leq 27^{-n} \varphi(3^{-n}x, 3^{-n}y) \tag{2.8}$$

for all  $x, y \in X$  and for all positive integer  $n$ . Using (2.1) and (2.5) in (2.8), we see that  $R_c$  satisfies (1.5) for all  $x, y \in X$ . Taking limit  $n \rightarrow \infty$  in (2.5), we arrive (2.3). Now, it remains to show that  $R_c$  is uniquely defined. Let  $R'_c : X \rightarrow \mathbb{R}$  be another reciprocal-cubic mapping which satisfies (1.5) and the inequality (2.3). Clearly,  $R'_c(3^{-n}x) = 27^n R'_c(x)$ ,  $R_c(3^{-n}x) = 27^n R_c(x)$  and using (2.3), we arrive

$$\begin{aligned}
 |R'_c(x) - R_c(x)| &= 27^{-n} |R'_c(3^{-n}x) - R_c(3^{-n}x)| \\
 &\leq 27^{-n} (|R'_c(3^{-n}x) - f(3^{-n}x)| + |f(3^{-n}x) - R_c(3^{-n}x)|) \\
 &\leq 2 \sum_{k=0}^{\infty} \frac{1}{27^{n+k}} \varphi\left(\frac{x}{3^{n+k+1}}, \frac{x}{3^{n+k+1}}\right) \\
 &\leq 2 \sum_{k=n}^{\infty} \frac{1}{27^k} \varphi\left(\frac{x}{3^{k+1}}, \frac{x}{3^{k+1}}\right)
 \end{aligned} \tag{2.9}$$

for all  $x \in X$ . Allowing  $n \rightarrow \infty$  in (2.9), we find that  $R_c$  is unique. This completes the proof of Theorem 2.2.

**Theorem 2.3.** *Let  $\varphi : X \times X \rightarrow \mathbb{R}$  be a function satisfying*

$$\sum_{i=0}^{\infty} 27^i \varphi(3^i x, 3^i y) < +\infty \quad (2.10)$$

*for all  $x, y \in X$ . If a function  $f : X \rightarrow \mathbb{R}$  satisfies the functional inequality (2.2) for all  $x, y \in X$ , then there exists a unique reciprocal-cubic mapping  $R_c : X \rightarrow \mathbb{R}$  which satisfies (1.5) and the inequality*

$$|f(x) - R_c(x)| \leq \sum_{j=0}^{\infty} 27^{j+1} \varphi(3^j x, 3^j x) \quad (2.11)$$

*for all  $x \in X$ .*

*Proof.* Switching  $y$  to  $x$  in (2.2) and multiplying by 27, we find

$$|27f(3x) - f(x)| \leq 27\varphi(x, x) \quad (2.12)$$

for all  $x \in X$ . Again letting  $x$  as  $3x$  in (2.12) and multiplying by 27 and then summing the resultant inequality with (2.12), we obtain

$$|27^2 f(3^2 x) - f(x)| \leq 27 \sum_{j=0}^1 27^j \varphi(3^j x, 3^j x)$$

for all  $x \in X$ . Analogous to the previous steps, ensuing further and using mathematical induction, one can find

$$|27^n f(3^n x) - f(x)| \leq \sum_{j=0}^{n-1} 27^{j+1} \varphi(3^j x, 3^j x)$$

for all  $x \in X$ . The left over part of the proof is obtained by akin arguments as in Theorem 2.2.

The following theorem proves Hyers-Ulam stability ( $\epsilon$ -stability) of equation (1.5).

**Theorem 2.4.** *Let  $f : X \rightarrow \mathbb{R}$  be a mapping for which there exists a constant  $\epsilon > 0$  (independent of  $x$  and  $y$ ) such that the functional inequality*

$$|D_f(x, y)| \leq \frac{26\epsilon}{27} \quad (2.13)$$

*holds for all  $x, y \in X$ . Then the limit*

$$R_c(x) = \lim_{n \rightarrow \infty} \frac{1}{27^n} f\left(\frac{x}{3^n}\right). \quad (2.14)$$

*for all  $x \in X$ ,  $n \in \mathbb{N}$  and  $R_c(x) : X \rightarrow \mathbb{R}$  is the unique mapping satisfying the reciprocal-cubic functional equation (1.5), such that*

$$|f(x) - R_c(x)| \leq \epsilon$$

*for every  $x \in X$ . Moreover, the functional identity*

$$R_c(x) = \frac{1}{27^n} f\left(\frac{x}{3^n}\right) \quad (2.15)$$

*holds for all  $x \in X$ ,  $n \in \mathbb{N}$ .*

*Proof.* Putting  $y = x$  in (2.13), we get

$$\left| f(3x) - \frac{1}{27} f(x) \right| \leq \frac{26\epsilon}{27} \quad (2.16)$$

for all  $x \in X$ . Now, substituting  $x$  by  $\frac{x}{3}$  in (2.16), we find

$$\left| f(x) - \frac{1}{27} f\left(\frac{x}{3}\right) \right| \leq \frac{26\epsilon}{27} \quad (2.17)$$

for all  $x \in X$ . Replacing  $x$  by  $\frac{x}{3}$  in (2.17) and then dividing both sides by  $3^3$ , we have

$$\left| \frac{1}{3^3} f\left(\frac{x}{3}\right) - \frac{1}{3^6} f\left(\frac{x}{3^2}\right) \right| \leq \frac{26\epsilon}{27^2} \quad (2.18)$$

for all  $x \in X$ . It follows from (2.17) and (2.18) that

$$\left| f(x) - \frac{1}{3^6} f\left(\frac{x}{3^2}\right) \right| \leq \frac{26\epsilon}{27} \left( 1 + \frac{1}{3^3} \right) \quad (2.19)$$

for all  $x \in X$ . The above process can be repeated to obtain

$$\left| f(x) - \frac{1}{3^{3n}} f\left(\frac{x}{3^n}\right) \right| \leq \frac{26\epsilon}{27} \left( 1 + \frac{1}{3^3} + \frac{1}{3^6} + \cdots + \frac{1}{3^{3(n-1)}} \right) \quad (2.20)$$

for all  $x \in X$  and all  $n \in \mathbb{N}$ . In order to prove the convergence of the sequence  $\{(1/3^{3n}) f(x/3^n)\}$ , we have if  $n > k > 0$ , then by (2.21)

$$\begin{aligned} \left| \frac{1}{3^{3n}} f\left(\frac{x}{3^n}\right) - \frac{1}{3^{3k}} f\left(\frac{x}{3^k}\right) \right| &= \frac{1}{3^{3k}} \left| \frac{1}{3^{3(n-k)}} f\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^k}\right) \right| \\ &= \frac{1}{3^{3k}} \left| \frac{1}{3^{3(n-k)}} f\left(\frac{y}{3^{n-k}}\right) - f(y) \right| \\ &\leq \frac{1}{3^{3k}} \frac{26\epsilon}{27} \left( 1 + \frac{1}{3^3} + \frac{1}{3^6} + \cdots + \frac{1}{3^{3(n-k-1)}} \right) \\ &\leq 3^{-3k} \epsilon \end{aligned} \quad (2.21)$$

for all  $x \in X$  in which  $y = \frac{x}{3^k}$ . The above result shows that the mentioned sequence is a Cauchy sequence and thus limit (2.14) exists for all  $x \in X$ . Taking that  $n$  tends to infinity in (2.21), we can see that inequality (2.15) holds for all  $x \in X$ . Replacing  $(x, y)$  by  $(\frac{x}{3^n}, \frac{y}{3^n})$ , respectively in (2.13) and dividing both sides by  $3^{3n}$ , we deduce that

$$\frac{1}{3^{3n}} |D_f(3^n x, 3^n y)| \leq \frac{26\epsilon}{3^{3(n+1)}} \quad (2.22)$$

holds for all  $x, y \in X$ . Taking that  $n$  tends to infinity in (2.22), we see that  $R_c(x)$  satisfies (2.13) for all  $x, y \in X$ . To prove that  $R_c(x)$  is a unique cubic reciprocal functional satisfying (2.13) subject to (2.16), let us consider a  $R'_c(x) : X \rightarrow \mathbb{R}$  to be another cubic reciprocal function which satisfies (2.13) and inequality (2.16). Clearly  $R_c(x)$  and  $R'_c(x)$  satisfy (2.16) and using (2.15), we get

$$\begin{aligned} |R_c(x) - R'_c(x)| &= \lim_{n \rightarrow \infty} \frac{1}{3^{3n}} \left| R_c\left(\frac{x}{3^n}\right) - R'_c\left(\frac{x}{3^n}\right) \right| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{3^{3n}} \left\{ \left| R_c\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right) \right| + \left| f\left(\frac{x}{3^n}\right) - R'_c\left(\frac{x}{3^n}\right) \right| \right\} \\ &= 0 \end{aligned}$$

for all  $x \in X$ , which proves that  $R_c$  is unique. Hence the proof is complete.

**Corollary 2.5.** *For any fixed  $c_1 \geq 0$  and  $p \neq -3$ , be fixed constants. If  $f : X \rightarrow \mathbb{R}$  satisfies*

$$|D_f(x, y)| \leq c_1 (|x|^p + |y|^p)$$

for all  $x, y \in X$ , then there exists a unique reciprocal-cubic mapping  $R_c : X \rightarrow \mathbb{R}$  satisfying (1.5) and

$$|f(x) - R_c(x)| \leq \begin{cases} \frac{54c_1}{3^{p+3}-1}|x|^p & \text{for } p > -3 \\ \frac{54c_1}{1-3^{p+3}}|x|^p & \text{for } p < -3 \end{cases}$$

for all  $x \in X$ .

*Proof.* If we choose  $\varphi(x, y) = c_1(|x|^p + |y|^p)$ , for all  $x, y \in X$ , then by Theorem 2.2, we arrive

$$|f(x) - R_c(x)| \leq \frac{54c_1}{3^{p+3}-1}|x|^p$$

for all  $x \in X$  and  $p > -3$  and using Theorem 2.3, we arrive

$$|f(x) - R_c(x)| \leq \frac{54c_1}{1-3^{p+3}}|x|^p$$

for all  $x \in X$  and  $p < -3$ .

**Corollary 2.6.** Let  $f : X \rightarrow \mathbb{R}$  be a mapping and there exist  $\alpha, \beta$  with  $\rho = \alpha + \beta$  such that  $\rho \neq -3$ . If there exists  $c_2 \geq 0$  such that

$$|D_f(x, y)| \leq c_2 |x|^\alpha |y|^\beta$$

for all  $x, y \in X$ , then there exists a unique reciprocal-cubic mapping  $R_c : X \rightarrow \mathbb{R}$  satisfying the functional equation (1.5) and

$$|f(x) - R_c(x)| \leq \begin{cases} \frac{27c_2}{3^{\rho+3}-1}|x|^\rho & \text{for } \rho > -3 \\ \frac{27c_2}{1-3^{\rho+3}}|x|^\rho & \text{for } \rho < -3 \end{cases}$$

for all  $x \in X$ .

*Proof.* Considering  $\varphi(x, y) = c_2 |x|^\alpha |y|^\beta$ , for all  $x, y \in X$ , then by Theorem 2.2, we arrive

$$|f(x) - R_c(x)| \leq \frac{27c_2}{3^{\rho+3}-1}|x|^\rho$$

for all  $x \in X$  and  $\rho > -3$  and using Theorem 2.3, we arrive

$$|f(x) - R_c(x)| \leq \frac{27c_2}{1-3^{\rho+3}}|x|^\rho$$

for all  $x \in X$  and  $\rho < -3$ . □

**Corollary 2.7.** *Let  $c_3 > 0$  and  $r \neq -3$  be real numbers, and  $f : X \rightarrow \mathbb{R}$  be a mapping satisfying the functional inequality*

$$|D_f(x, y)| \leq c_3 \left( |x|^{\frac{r}{2}} |y|^{\frac{r}{2}} + (|x|^r + |y|^r) \right)$$

*for all  $x, y \in X$ . Then there exists a unique reciprocal-cubic mapping  $R_c : X \rightarrow \mathbb{R}$  satisfying the functional equation (1.5) and*

$$|f(x) - R_c(x)| \leq \begin{cases} \frac{81c_3}{3^{r+3}-1} |x|^r & \text{for } r > -3 \\ \frac{81c_3}{1-3^{r+3}} |x|^r & \text{for } r < -3 \end{cases}$$

*for all  $x \in X$ .*

*Proof.* Choosing  $\varphi(x, y) = c_3 \left( |x|^{\frac{r}{2}} |y|^{\frac{r}{2}} + (|x|^r + |y|^r) \right)$ , for all  $x, y \in X$ , then by Theorem 2.2, we arrive

$$|f(x) - R_c(x)| \leq \frac{81c_3}{3^{r+3}-1} |x|^r$$

for all  $x \in X$  and  $r > -3$  and using Theorem 2.3, we arrive

$$|f(x) - R_c(x)| \leq \frac{81c_3}{1-3^{r+3}} |x|^r$$

for all  $x \in X$  and  $r < -3$ .

## REFERENCES

- [1] **S.M. Ulam**, *Problems in Modern Mathematics*, Chapter VI, Wiley-Interscience, New York, 1964.
- [2] **D.H. Hyers**, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. 27(1941), 222-224.
- [3] **T. Aoki**, *On the stability of the linear transformation in Banach spaces*, J. Math.Soc. Japan, 2(1950), 64-66.
- [4] **Th.M. Rassias**, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72(1978), 297-300.
- [5] **J.M. Rassias**, *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal. 46(1982), 126-130.
- [6] **P. Gavruta**, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. 184(1994), 431-436.

- [7] **K. Ravi, M. Arunkumar and J.M. Rassias**, *Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, Int. J. Math. Stat. 3(A08)(2008), 36-46.
- [8] **K. Ravi and B.V. Senthil Kumar**, *Ulam-Gavruta-Rassias stability of Rassias Reciprocal functional equation*, Global J. of Appl. Math. and Math. Sci. 3(1-2) (Jan-Dec 2010), 57-79.
- [9] **K. Ravi, J.M. Rassias and B.V. Senthil Kumar**, *Ulam stability of Generalized Reciprocal Functional Equation in Several Variables*, Int. J. App. Math. Stat. 19(D10) 2010, 1-19.
- [10] **K. Ravi, J.M. Rassias and B.V. Senthil Kumar**, *Ulam stability of Reciprocal Difference and Adjoint Functional Equations*, The Australian J. Math. Anal. Appl., 8(1), Art. 13 (2011), 1-18.
- [11] **K. Ravi, E. Thandapani and B.V. Senthil Kumar**, *Stability of reciprocal type functional equations*, PanAmerican Math. Journal, 21(1) (2011), 59-70.
- [12] **A. Bodaghi and S.O. Kim**, *Approximation on the Quadratic Reciprocal Functional Equation*, Journal of Function Spaces and Applications, Vol.2014, Article ID532463, 5 pages.
- [13] **A. Bodaghi and Y. Ebrahimdoost**, *On the stability of quadratic reciprocal functional equation in non-Archimedean fields*, Asian-European J. Math. (submitted).
- [14] **K. Ravi, J.M. Rassias, B.V. Senthil Kumar and A. Bodaghi**, *Intuitionistic Fuzzy Stability of a Reciprocal-Quadratic Functional Equation*, Int. J. Appl. Sci. Math., 1(1) (2014), 9-14.