

MODULES WHOSE MAXIMAL SUBMODULES β^* EQUIVALENT TO A SUPPLEMENT

ABSTRACT. Let R be a ring and M a left R -module. A module M is called ms^* -module if each maximal submodule is β^* equivalent to a supplement in M . In this work, we focus on ms^* -module and study various properties of this module.

1. INTRODUCTION

Throughout this paper R denotes associative ring with unity and all modules are unital left R -modules. For any module M , $\text{Rad}(M)$ and $\text{Soc}(M)$ denote the Jacobson radical and socle of M , respectively. We briefly recall some definitions and properties of (cofinitely) (weak) supplemented modules. Let M be an R -module and N, K be submodules of M . A submodule K of a module M is called *small* (or superfluous) in M , denoted by $K \ll M$, if for every submodule N of M , the equality $K + N = M$ implies $N = M$. K is called a *supplement* of N in M , if K is a minimal element in the set of submodules L of M with $N + L = M$. Equivalently, K is a supplement (weak supplement) of N in M if and only if $K + N = M$ and $K \cap N \ll K$ ($K \cap N \ll M$). A submodule K of M is called a *supplement submodule* if it is a supplement of any submodule of M . Obviously, direct summands are supplements. A module M is called *supplemented module* (*weakly supplemented*), if every submodule of M has a supplement (weak supplement) in M . A module M is $\oplus$ -*supplemented module* if every submodule of M has a supplement which is a direct summand in M . A submodule N of a module M is said to be *cofinite* if $\frac{M}{N}$ is finitely generated. A module M is called a *cofinitely supplemented module* if every cofinite submodule of M has a supplement in M . It is clear that every supplemented module is cofinitely supplemented. In [1] Alizade and Büyükaşık defined *cofinitely weak supplemented* (or briefly *cws-module*) module if every cofinite submodule has a weak supplement. Obviously, cofinitely supplemented module is

2010 *Mathematics Subject Classification.* 16D10, 16D99.

Key words and phrases. cofinitely supplemented, cofinitely weakly supplemented, weakly supplemented, Goldie*-supplemented module.

cofinitely weak supplemented. For additional properties and results of supplemented, cofinitely supplemented and weakly supplemented modules, we refer to [1], [6], [9], [12].

2. PROPERTIES OF β^* RELATION

In [4], G.F.Birkenmeier et.al. defined β^* relation. In this part, some properties of this relation is mentioned without proofs. The proofs of the following notions can be found in [4].

Definition 2.1. Any submodules X, Y of M are β^* equivalent, $X\beta^*Y$, if and only if $\frac{X+Y}{X} \ll \frac{M}{X}$ and $\frac{X+Y}{Y} \ll \frac{M}{Y}$.

By ([4], Lemma 2.2), β^* is an equivalence relation and the zero submodule is β^* equivalent to any small submodule.

Theorem 2.2. ([4], Theorem 2.3) Let X, Y be submodules of M . The following are equivalent:

- (1) $X\beta^*Y$.
- (2) $\frac{X+Y}{X} \ll \frac{M}{X}$ and $\frac{X+Y}{Y} \ll \frac{M}{Y}$.
- (3) For each submodule A of M such that $X+Y+A = M$, then $X+A = M$ and $Y+A = M$.
- (4) If $K+X = M$ for any submodule K of M , then $Y+K = M$ and if $Y+H = M$ for any submodule H of M , then $X+H = M$.

Theorem 2.3. ([4], Theorem 2.6) Let X, Y be submodules of M such that $X\beta^*Y$. Then

- (1) $X \ll M$ if and only if $Y \ll M$.
- (2) X has a (weak) supplement C in M if and only if C is a (weak) supplement for Y .

Lemma 2.4. ([4], Corollary 2.7) Let X, Y be submodules of M such that $X \subseteq Y$ and X has a weak supplement C in M . Then $X\beta^*Y$ if and only if $Y \cap C \ll M$.

Theorem 2.5. ([4], Proposition 2.10) Let X be a submodule of M and K a maximal submodule of M .

- (1) Let $C_1, C_2 \leq M$ such that $C_1 + C_2 = M$, $C_2 \neq M$ and $X\beta^*C_1$. Then $X \not\subseteq C_2$.
- (2) If $X\beta^*Y$ and $X \subseteq K$, then $Y \subseteq K$.
- (3) If $X\beta^*K$, then $X \subseteq K$. Hence if $X\beta^*Y$, then $X \subseteq \text{Rad}(M)$ if and only if $Y \subseteq \text{Rad}(M)$.

- (4) If $X\beta^*K$ and $X+W=M$ with $X\cap W\ll M$, then $K=X+(K\cap W)$ and $K\cap W\ll M$.

Proposition 2.6. *If a supplement submodule S in M is β^* equivalent to a cyclic submodule X of M , then S is also cyclic.*

Proof. Let X be a cyclic submodule and S supplement submodule of M such that $X\beta^*S$. Then there exists a submodule L of M such that $M=S+L$ and $S\cap L\ll S$. So L is weak supplement of S in M . By Theorem 2.3, L is also weak supplement of X in M . Thus we have $M=X+L$ and $X\cap L\ll M$. Since X is cyclic, we find

$$\frac{S}{S\cap L} \cong \frac{S+L}{L} = \frac{M}{L} = \frac{X+L}{L} \cong \frac{X}{X\cap L}$$

is cyclic. By the relation $S\cap L\ll S$, it follows from ([12], 11.6) that S is cyclic. $\square$

A module M is *distributive* if for submodules A, B and C of M , $A\cap(B+C) = (A\cap B) + (A\cap C)$.

Proposition 2.7. *Let M be distributive module. Let $M=D\oplus D'$ and A, B be submodules of D . Then $A\beta^*B$ in M where B is supplement submodule in M if and only if $A\beta^*B$ in D where B is supplement submodule in D .*

Proof. ($\Rightarrow$) Suppose $A\beta^*B$ in M such that B is supplement submodule in M . Then $M=B+C$ and $B\cap C\ll B$ for some submodule C of M . Let $D=A+B+X$ for some submodule X of D . We claim that $D=A+X$ and $D=B+X$. Since $M=D\oplus D'$, then $M=A+B+X+D'$. If $A\beta^*B$ in M , then $M=A+X+D'$ and $M=B+X+D'$. By modular law, $D=D\cap M=D\cap(A+X+D')=A+X+(D\cap D')=A+X$ and $D=D\cap M=D\cap(B+X+D')=B+X$. Let us show B supplement submodule in D . From the modularity, $D=D\cap M=D\cap(B+C)=B+(D\cap C)$ and $B\cap(D\cap C)=B\cap C\ll B$. It means that B is supplement of $D\cap C$ in D .

($\Leftarrow$) To prove the converse, suppose $A\beta^*B$ in D . Then $\frac{A+B}{A} \ll \frac{D}{A}$ and $\frac{A+B}{B} \ll \frac{D}{B}$. So $\frac{A+B}{A} \ll \frac{M}{A}$ and $\frac{A+B}{B} \ll \frac{M}{B}$ by ([12], 19.3). Thus $A\beta^*B$ in M . We shall show that B is supplement in M . Since B is supplement submodule in D , so $D=B+C$ and $B\cap C\ll B$ for some submodule C of D . Therefore $M=D\oplus D'=B+(C\oplus D')$. It follows from the distributive condition that $B\cap(C\oplus D')=(B\cap C)\oplus(B\cap D')=B\cap C\ll B$. Therefore B is supplement of $C\oplus D'$ in M . $\square$

3. MS^* -MODULES

In [4], authors defined Goldie*-supplemented modules using β^* equivalence relation. A module M is *Goldie*-supplemented* (or shortly $\mathcal{G}^*$ -supplemented)

if for each submodule X of M , there exists a supplement submodule S of M such that $X\beta^*S$. It is clear that hollow modules and semisimple modules are $\mathcal{G}^*$ -supplemented. They showed that for quasi-projective modules, supplemented and Goldie*-supplemented modules coincide.

We start this section by giving the definition of ms^* -module and obtain some results of this module. We prove that if M is finitely generated refinable module and each maximal submodule is β^* equivalent to a supplement submodule, then M is $\mathcal{G}^*$ -supplemented.

Definition 3.1. A module M is called *ms*-module* if, for each maximal submodule K of M , there exists a supplement submodule S of M such that $K\beta^*S$.

Example 3.2. Local modules and semisimple modules are ms^* -modules.

Theorem 3.3. ([1], Theorem 2.16) *For a module M , the following statements are equivalent.*

- (1) M is cofinitely weak supplemented (cws) module,
- (2) Every maximal submodule of M has a weak supplement,
- (3) $M/cws(M)$ has no maximal submodules.

We can use Theorem 3.3 to say ms^* -modules imply cofinitely supplemented modules.

Proposition 3.4. *Let M be ms^* -module. Then M is cofinitely weak supplemented module.*

Proof. Let K be a maximal submodule of M . It is enough to show that K has a weak supplement in M . By hypothesis, $K\beta^*S$ where S is supplement submodule in M . Then there exists a submodule L of M such that $M = S + L$ and $S \cap L \ll S$. So it follows from ([12], 19.3) that $S \cap L \ll M$. This means that L is weak supplement of S in M . By Theorem 2.3, L is also weak supplement of K in M . Hence M is cofinitely weak supplemented module. $\square$

The following is immediate consequence of Proposition 3.4.

Corollary 3.5. *If every maximal submodule of M is β^* equivalent to a supplement which is direct summand in M , then M is cofinitely supplemented module by ([2], Theorem 2.8).*

The converse of Proposition 3.4 does not hold in general. Cofinitely weak supplemented modules need the refinable condition to become ms^* -module.

A module M is said to be *refinable*, if, for any submodules U, V of M with $M = U + V$, there is a direct summand U' of M such that $U' \subseteq U$ and $M = U' + V$.

Proposition 3.6. *If M is refinable cofinitely weak supplemented, then M is ms^* -module.*

Proof. We will show that for each maximal submodule of M is β^* equivalent to any supplement submodule of M . For this, let K be a maximal submodule of M . It is clear that K is cofinite submodule in M . By assumption, K has a weak supplement A in M . Then $M = K + A$ and $K \cap A \ll M$. Since M is refinable, there exists a direct summand K' of M such that $K' \subseteq K$ and $M = K' + A$. We claim that $K\beta^*K'$. If $K \cap A \ll M$, $K' \cap A \ll M$, then we have that A is weak supplement of K' in M . By Lemma 2.4, $K\beta^*K'$ where K' is direct summand supplement in M . $\square$

Theorem 3.7. ([2], Theorem 2.8) *Let R be any ring. The following statements are equivalent for an R -module M :*

- (1) M is cofinitely supplemented,
- (2) Every maximal submodule of M has a supplement in M ,
- (3) The module $M/\text{Loc}(M)$ does not contain a maximal submodule,
- (4) The module $M/\text{Cof}(M)$ does not contain a maximal submodule.

Proposition 3.8. *Let M be refinable module. Then M is ms^* -module if and only if M is cofinitely supplemented.*

Proof. $(\Rightarrow)$ Let K be a maximal submodule of M . By previous theorem, it is enough to show that K has a supplement in M . Since M is ms^* -module, we can say that there exists a supplement submodule S of M such that $K\beta^*S$. Then $M = S + L$ and $S \cap L \ll S$ for some submodule L of M . If M is refinable, there exists a direct summand S' of M such that $S' \subseteq S$ and $M = S' + L$. So $M = S' \oplus S''$ for some submodule S'' of M . In this case, S'' is supplement of S' in M . If $S' \subseteq S$ and $S \cap L \ll S$, then $S' \cap L \ll S$. By ([12], 19.3), $S' \cap L \ll M$. This means that L is weak supplement of S' in M . If $S \cap L \ll S$, then $S \cap L \ll M$ from ([12], 19.3). By Lemma 2.4, $S\beta^*S'$. Since β^* is equivalence relation, from transitivity, if $K\beta^*S$ and $S\beta^*S'$, we get $K\beta^*S'$. It follows from Theorem 2.3 that K has a supplement S'' in M .

$(\Leftarrow)$ Since every cofinitely supplemented module is cofinitely weak supplemented, M is ms^* -module by Proposition 3.6. $\square$

Proposition 3.9. *Let M be finitely generated ms^* -module. Then M is weakly supplemented module.*

Proof. Let N be a submodule of M . Then $\frac{M}{N}$ is finitely generated because M is finitely generated, so N is cofinite submodule in M . By Proposition 3.4, N has a weak supplement in M . $\square$

Corollary 3.10. *Let M be finitely generated ms^* -module. Then M is semilocal.*

Proof. By ([9], Proposition 2.2) and previous Proposition, M is semilocal. $\square$

Büyükaşık and Pusat Yılmaz [5] introduced the concept of ms -modules and md -modules. A module M is said to be ms -module if every maximal submodule of M is supplement submodule in M . A module M is called md -module if every maximal submodule of M is a direct summand in M . Clearly, md -module is ms^* -module. Let K be a maximal submodule of M . If M is md -module, then K is direct summand in M . Since $K\beta^*K$, M is ms^* -module where K is supplement submodule in M .

Proposition 3.11. *Let M be ms^* -module with $Rad(M) = 0$. Then M is md -module.*

Proof. Let K be a maximal submodule of M . Now we shall prove that K is direct summand in M . Since M is ms^* -module, there exists a supplement submodule S of M such that $K\beta^*S$. So we have $M = S + L$ and $S \cap L \ll S$ for some submodule L of M . Therefore $S \cap L \subseteq Rad(M)$. If $Rad(M) = 0$, then the supplement submodule S is direct summand in M , that is, $M = S \oplus L$. It is clear that L is supplement of S in M . By Theorem 2.3, L is also supplement of K in M , that is, $M = K + L$ and $K \cap L \ll L$. In a similar manner we see that $M = K \oplus L$. $\square$

Theorem 3.12. *Let M be finitely generated refinable ms^* -module. Then M is $\mathcal{G}^*$ -supplemented.*

Proof. Let N be a submodule of M . By Corollary 3.10, M is semilocal, that is, $\frac{M}{Rad(M)}$ is semisimple. So $\frac{M}{Rad(M)}$ is $\mathcal{G}^*$ -supplemented. Then $\frac{N + Rad(M)}{Rad(M)}\beta^*\frac{X}{Rad(M)}$ such that $\frac{X}{Rad(M)}$ is supplement submodule in $\frac{M}{Rad(M)}$. Therefore by ([4], Proposition 2.9), $N\beta^*X$ in M . Let $\frac{X}{Rad(M)}$ be a supplement of $\frac{Y + Rad(M)}{Rad(M)}$ in $\frac{M}{Rad(M)}$ for some submodule Y of M . In this case, $\frac{M}{Rad(M)} = \frac{X}{Rad(M)} + \frac{Y + Rad(M)}{Rad(M)}$ and $\frac{X}{Rad(M)} \cap \frac{(Y + Rad(M))}{Rad(M)} = \frac{(X \cap Y) + Rad(M)}{Rad(M)} \ll \frac{X}{Rad(M)}$. Thus $M = X + Y$ and by ([6], 2.2), $X \cap Y \ll M$. Since M is refinable, there exists a direct summand U of M such that $U \subseteq X$ and $M = U + Y$. If $U \cap Y \subseteq X \cap Y \ll M$, then $U \cap Y \ll M$. We can say that Y is weak supplement of U in M . By Lemma 2.4, $X\beta^*U$. Since the relation β^* is equivalence relation, $N\beta^*X$ and $X\beta^*U$ imply $N\beta^*U$ such that U is supplement submodule in M . Hence M is $\mathcal{G}^*$ -supplemented. $\square$

Example 3.13. Let M be a radical module, that is, $Rad(M) = M$. Then by ([5], 2.3 Proposition), M is md-module. Hence M is ms^* -module.

Example 3.14. The $\mathbb{Z}$ -module $\mathbb{Q}$ is ms^* -module.

Clearly, every $\mathcal{G}^*$ -supplemented module is ms^* -module. Observe that ms^* -module need not to be $\mathcal{G}^*$ -supplemented.

Example 3.15. Let K be the quotient field of a discrete valuation domain R which is not complete. Let $M = K \oplus K$. Since $Rad(K) = K$, $Rad(M) = M$, that is, M is radical module. By Example 3.13, M is ms^* -module but not $\mathcal{G}^*$ -supplemented by ([4], Example 3.9 (iii))

Proposition 3.16. M is ms^* -module if and only if for maximal submodule K of M , $K = S + H$ where S is supplement submodule in M and $H \ll M$.

Proof. ($\Rightarrow$) Let K be a maximal submodule of M . By assumption, $K\beta^*S$ where S is supplement submodule in M . Then $M = S + L$ and $S \cap L \ll S$ for some submodule L of M . So L is weak supplement of S in M . By Theorem 2.3, L is also weak supplement of K in M . In this case we have $M = K + L$ and $K \cap L \ll M$. If $M = S + L$, by modularity, $K = S + (K \cap L)$.

($\Leftarrow$) To prove M is ms^* -module it is enough to show that for each maximal submodule K of M , there exists a supplement submodule S of M such that $K\beta^*S$. Let K be a maximal submodule of M . By assumption, $K = S + H$ where S is supplement submodule in M and $H \ll M$. Since β^* is equivalence relation, $K\beta^*K$, that is, $K\beta^*(S + H)$. By ([4], Corollary 2.12), we say that $K\beta^*S$. $\square$

Proposition 3.17. Let M be ms^* -module with $Rad(M) = 0$. Then any coclosed submodule N of M with $Soc(M) \subseteq N$ is ms^* -module.

Proof. Suppose M is ms^* -module with $Rad(M) = 0$. Then M is md-module by Proposition 3.11. From ([5], 2.4 Proposition), N is md-module. Therefore N is ms^* -module. $\square$

Proposition 3.18. Let M be ms^* -module. For small submodule N of M , $\frac{M}{N}$ is ms^* -module.

Proof. Let $\frac{K}{N}$ be a maximal submodule of $\frac{M}{N}$. We note that K is maximal submodule in M . By assumption, $K\beta^*S$ where S is supplement submodule in M . Let $\sigma : M \rightarrow \frac{M}{N}$ be a canonical epimorphism. By ([4], Proposition 2.9), $\frac{K}{N}\beta^*(\frac{S+N}{N})$. From ([3], Lemma 4), $\frac{S+N}{N}$ is supplement in $\frac{M}{N}$. In other words, $\frac{M}{N}$ is ms^* -module. $\square$

Corollary 3.19. *Let M be ms^* -module with $Rad(M) \ll M$. Then $\frac{M}{Rad(M)}$ is ms^* -module.*

For the converse of Proposition 3.18, refinable condition is needed.

Proposition 3.20. *Let $\frac{M}{N}$ be refinable ms^* -module for small submodule N of M . Then M is ms^* -module.*

Proof. Let K be a maximal submodule of M containing N . Then $\frac{K}{N}$ is maximal submodule in $\frac{M}{N}$. If $\frac{M}{N}$ is ms^* -module, so $\frac{K}{N} \beta^* \frac{S}{N}$ where $\frac{S}{N}$ is supplement submodule in $\frac{M}{N}$. Then there exists submodule $\frac{A+N}{N}$ in $\frac{M}{N}$ such that $\frac{M}{N} = \frac{A+N}{N} + \frac{S}{N}$ and $\frac{S}{N} \cap \frac{A+N}{N} = \frac{(S \cap A) + N}{N} \ll \frac{S}{N}$. This yields $M = A + S$ and $S \cap A \ll M$, i.e., S is weak supplement of A in M . In the refinable situation, there exists a direct summand S' of M such that $S' \subseteq S$ and $M = S' + A$. Moreover, in this case, S' is a supplement submodule in M . Our claim is to show that $K \beta^* S'$. We know that $S' \cap A \subseteq S \cap A \ll M$ implies $S' \cap A \ll M$, that is, S' has a weak supplement A in M . By Lemma 2.4, $S \beta^* S'$. Let $\sigma : M \rightarrow \frac{M}{N}$ be a canonical epimorphism. By ([4], Proposition 2.9(ii)), $\sigma^{-1}(\frac{K}{N}) \beta^* \sigma^{-1}(\frac{S}{N})$. Thus $K \beta^* S$. Since $S \beta^* S'$, we get $K \beta^* S'$ with the property of transitivity of β^* . $\square$

Proposition 3.21. *Let M be projective and refinable module. Then M is ms^* -module if and only if every simple factor of M has a projective cover.*

Proof. ($\Rightarrow$) Suppose that M is ms^* -module. Let K be a maximal submodule of M . Then $K \beta^* S$ where S is supplement submodule in M . Then $M = S + L$ and $S \cap L \ll S$ for some submodule L of M . Since M is refinable, there exists a direct summand S' of M such that $S' \subseteq S$ and $M = S' + L$. So $S' \cap L \subseteq S \cap L \ll S$, that is, $S' \cap L \ll S$. It implies that $S' \cap L \ll M$ by ([12], 19.3). We have that L is weak supplement of S' in M . By Lemma 2.4, $S \beta^* S'$. By transitivity of β^* relation, we mention that $K \beta^* S'$. Since S' is direct summand in M , that is, $M = S' \oplus S''$ for some submodule S'' of M , by Theorem 2.3, S'' is supplement of K in M because S'' is supplement of S' in M . Thus $\frac{M}{K}$ has a projective cover from ([12], 42.1).

($\Leftarrow$) Let K be a maximal submodule of M . With the assumption we can say that $\frac{M}{K}$ has a projective cover. By ([12], 42.1), there exists a direct summand V of $\frac{M}{K}$ such that $M = K + V$ and $K \cap V \ll V$. Since M is refinable, there

exists a direct summand K' of M such that $K' \subseteq K$ and $M = K' + V$. Then as similar to above $K' \cap V \ll M$, that is, V is weak supplement of K' in M . Since $K \cap V \ll M$, so $K\beta^*K'$ by Lemma 2.4 where K' is supplement submodule in M . $\square$

Theorem 3.22. *Let R be a semiperfect ring. Then every projective refinable module is ms^* -module.*

Proof. Let M be projective and refinable R -module. Then by ([12], 42.6), every simple factor module of M has a projective cover. Hence M is ms^* -module from previous proposition. $\square$

Proposition 3.23. *Let M be ms^* -module with $Rad(M) \ll M$. Then every maximal submodule of $\frac{M}{Rad(M)}$ is direct summand. The converse holds if M is refinable.*

Proof. Let M be ms^* -module with $Rad(M) \ll M$. Then by Proposition 3.4, M is cofinitely weakly supplemented. Then by ([1], Theorem 2.21), every maximal submodule of $\frac{M}{Rad(M)}$ is a direct summand. For the converse, suppose that M is refinable. Let K be a maximal submodule of M . Then $\frac{K}{Rad(M)}$ is maximal submodule of $\frac{M}{Rad(M)}$. By assumption, $\frac{M}{Rad(M)} = \frac{K}{Rad(M)} \oplus \frac{A + Rad(M)}{Rad(M)}$ for some submodule $\frac{A + Rad(M)}{Rad(M)}$ of $\frac{M}{Rad(M)}$ where A is submodule of M . So we have $M = A + K$ and $A \cap K \subseteq Rad(M)$. Since $Rad(M) \ll M$, $A \cap K \ll M$, this means that K is a weak supplement of A in M . By ([1], Theorem 2.21), M is cofinitely weakly supplemented. If M is refinable, from Proposition 3.6 we say that M is ms^* -module. $\square$

A module M is said to be *coatomic* if for every submodule N of M , $Rad(\frac{M}{N}) = \frac{M}{N}$ implies $\frac{M}{N} = 0$, equivalently, every proper submodule of M is contained in a maximal submodule of M . For instance, finitely generated and semisimple modules are coatomic. It is clear that every factor module of coatomic module is coatomic.

Proposition 3.24. *Let M be finitely generated ms^* -module over a Dedekind domain and let K be a maximal submodule of M . Then the following hold:*

- (1) *Every weak supplement of K is coatomic.*
- (2) *Every maximal submodule of M is coatomic.*

Proof. 1) Let L be a weak supplement of K in M . Then $M = K + L$ and $K \cap L \ll M$. Since R is dedekind domain, by ([8], Lemma 2.1), $K \cap L$

is coatomic. Let $\sigma : \frac{M}{K} \rightarrow \frac{L}{K \cap L}$ be an isomorphism. If K is maximal submodule of M , then $\frac{M}{K}$ is simple, that is, $\frac{M}{K}$ is coatomic. Then $\frac{L}{K \cap L}$ is coatomic. Let

$$0 \rightarrow K \cap L \rightarrow L \rightarrow \frac{L}{K \cap L} \rightarrow 0$$

be an short exact sequence of modules. By ([7], Lemma 3), L is coatomic.

2) Let K be a maximal submodule of M . Since M is ms^* -module, there exists a supplement submodule S in M such that $K\beta^*S$. So $\frac{K}{S} \ll \frac{M}{S}$ by definition of β^* relation. It follows from ([8], Lemma 2.1) that $\frac{K}{S}$ is coatomic. Since S is supplement submodule of M and M is finitely generated, S is finitely generated too. It implies that S is coatomic. Let

$$0 \rightarrow S \rightarrow K \rightarrow \frac{K}{S} \rightarrow 0$$

be an short exact sequence of modules. If S and $\frac{K}{S}$ are coatomic modules, then K is coatomic from ([7], Lemma 3). $\square$

Proposition 3.25. *Let M be projective and refinable module. If M is ms^* -module, then every direct summand of M is ms^* -module.*

Proof. Let N be a direct summand of M . Then $M = N \oplus L$ for some submodule L of M . Since M is projective, so N is projective. By assumption and ([6], 11.31), M has the finite exchange property. Then N has the finite exchange property from ([6], 11.9). Again by ([6], 11.31), we have N is refinable. Let K be a maximal submodule of N . To use Proposition 3.21 we want to show that $\frac{N}{K}$ has a projective cover. Let $\pi : M \rightarrow N$ be a projection and $f : N \rightarrow \frac{N}{K}$ be an epimorphism. Thus $g : M \rightarrow \frac{N}{K}$ is an epimorphism such that $g = f\pi$. In this case, $Ker g = K \oplus L$. By the isomorphism theorem, $\frac{M}{K \oplus L} \cong \frac{N}{K}$ is simple. By Proposition 3.21, $\frac{M}{K \oplus L}$ has a projective cover. So $h : P \rightarrow \frac{M}{K \oplus L}$ is a projective cover of $\frac{M}{K \oplus L}$ such that $Ker h \ll P$. Let $\alpha : \frac{M}{K \oplus L} \rightarrow \frac{N}{K}$ be an isomorphism. Then $\alpha h : P \rightarrow \frac{N}{K}$ is a projective cover of $\frac{N}{K}$ with $Ker \alpha h \ll P$. By Proposition 3.21, N is ms^* -module. $\square$

Theorem 3.26. *Let M be a finitely generated projective and refinable module. Then the following are equivalent.*

- (1) M is ms^* -module,

- (2) M is cofinitely supplemented,
- (3) M is supplemented,
- (4) M is semiperfect.

Proof. (3) $\Rightarrow$ (2) Clear.

(2) $\Leftrightarrow$ (1) By Proposition 3.8.

(1) $\Rightarrow$ (4) Let M be ms^* -module. By Proposition 3.21, every simple factor of M has a projective cover. By ([12], 42.5), M is semiperfect.

(4) $\Rightarrow$ (3) By ([12], 42.3). $\square$

Proposition 3.27. *Let R be a noetherian ring. If every cyclic submodule of M is a direct summand of M , then M is ms^* -module.*

Proof. Let R be a noetherian ring. Then M is locally noetherian. Then every finitely generated submodule of M is noetherian. By assumption, M is semisimple (kontrol et). From Example 3.2, M is ms^* -module. $\square$

REFERENCES

- [1] Alizade, R., Büyükaşık, E., Cofinitely Weak Supplemented Modules, *Communication in Algebra*, Vol 31, No 11, (2003), 5377-5390.
- [2] Alizade, R., Bilhan, G., Smith, P., Modules whose maximal submodules have supplements, *Comm. Algebra*, 29 (6), (2001), 2389-2405.
- [3] Acar, U., Harmancı, A., Principally Supplemented Modules, *Albanian Journal of Math.*, Vol.4, 3, (2010), 79-88.
- [4] Birkenmeier, G.F., Mutlu, F.T., Nebiyev, C., Sökmez, N., Tercan, A., Goldie*-Supplemented Modules, *Glasgow Math. Journal*, 52(A), (2010), 41-52.
- [5] Büyükaşık, E., Pusat Yılmaz, D., Modules whose maximal submodules are supplement, *Hacettepe Journal of Mathematics and Statistic*, Vol 39(4), (2010), 477-487.
- [6] Clark, J., Lomp, C., Vanaja, N., Wisbauer, R., *Lifting Modules: Frontiers in Mathematics*, Boston, USA, Birkhäuser-Verlag, (2006).
- [7] Güngöroğlu, G., Coatomic Modules, *Far East Journal Math. Sci.*, Special Volume 2, (1998), 153-162.
- [8] Güngöroğlu, G. and Harmancı, A., Coatomic Modules over Dedekind Domains, *Hacet. Bull. Nat. Sci. Eng. Ser. B*, 28, (1999), 25-29.
- [9] Lomp, C., On Semilocal Modules and Rings, *Communication in Algebra*, 27(4), (1999), 1921-1935.
- [10] Türkmen, B.N and Türkmen, E., On A Generalization Of Weakly Supplemented Modules, *An.t.Univ.Ovidius Constanta*, (2014), 0-13, DOI:10.1515/aicu-2015-0012.
- [11] Türkmen, E., Rings whose modules are $\oplus$ -cofinitely supplemented, *Analele Stiintifice Ale Universitatii*, DOI:10.2478/aicu-2013-0025.
- [12] Wisbauer, R., *Foundations of Modules and Rings*, Gordon and Breach, (1991).
- [13] Zöschinger, H., Komplementierte moduln über dedekindringen, *Journal of Algebra* , 29, (1974a), 42-56.