

# Square-Normal Operator

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## 1 Abstract

In this paper we define a new class of operators in Hilbert space called square-normal operator and we give an example to show that the square-normal operator is not normal operator. We also consider the conditions on any operator to be a square-normal operator. Then we give a condition in order to get a normal operator from a square-normal operator.

Keywords: normal operators, numerical range of operators, self-adjoint operators, square-normal operators.

## 2 Introduction

In this paper  $A, A_1, A_2, N, M$  and  $E$  represent continuous linear operators on Hilbert space  $\mathcal{H}$ . If  $A$  is an operator on  $\mathcal{H}$ , then we denote  $\mathcal{W}(A)$  for the numerical range of  $A$ ,  $A^*$  denotes the adjoint of  $A$ ,  $\sigma(A)$  denotes the spectrum of  $A$ ,  $A$  is said to be normal if  $AA^* = A^*A$ , it is called self-adjoint if  $A = A^*$  and unitary if  $AA^* = A^*A = I$  where  $I$  is the identity operator. The real part of  $A$  is denoted by  $ReA = \frac{A+A^*}{2}$  and imaginary part of  $A$  is denoted by  $ImA = \frac{A-A^*}{2i}$ .

For a bounded linear operator  $A$  on  $\mathcal{H}$ , the numerical range  $\mathcal{W}(A)$  is the image of the unit sphere of  $\mathcal{H}$  under the quadratic form  $x \rightarrow \langle Ax, x \rangle$  associated with the operator. More precisely,

$$\mathcal{W}(A) := \{\langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

The following special notations will be used throughout this paper:  $B = A^2(A^*)^2$  and  $C = (A^*)^2A^2$ , where  $B$  and  $C$  are nonnegative definite.

## 3 Square-Normal Operator and Some Properties

We now define the following operator:

**Definition 1.** An operator  $A$  is said to be square-normal operator if  $A^2(A^*)^2 = (A^*)^2A^2$

**Proposition 1.** If  $A$  is a normal operator then  $A$  is a square-normal operator.

*Proof.*  $A$  is normal operator then

$$A^2(A^*)^2 = AAA^*A^* = AA^*AA^* = A^*AA^*A = A^*A^*AA = (A^*)^2A^2.$$

So  $A$  is a square-normal operator. □

The converse is not true. We give an example of a square-normal operator which is not normal:

**Example 1.**  $A = \begin{pmatrix} i & 0 \\ i & -i \end{pmatrix}$ ,  $A^* = \begin{pmatrix} -i & -i \\ 0 & i \end{pmatrix}$ .

Since  $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $(A^*)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ,

$A^2(A^*)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $(A^*)^2A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . So  $A$  is a square-normal operator.

But  $AA^* = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$  and  $A^*A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ . So  $A$  is not normal.

**Proposition 2.**  $A$  is a square-normal operator if and only if  $A^2$  is normal.

*Proof.* Let  $A$  be a square-normal operator, so

$$\begin{aligned} A^2(A^*)^2 &= (A^*)^2A^2 \\ \iff A^2(A^2)^* &= (A^2)^*A^2 \\ \iff A^2 &\text{ is normal} \end{aligned}$$

□

**Theorem 1.** (Fuglede-Putnam [1, 7]). If  $A, N$  and  $M$  are bounded operators such that  $M$  and  $N$  are normal, then

$$AN = MA \implies AN^* = M^*A,$$

and if  $N$  and  $M$  are unbounded, then " $=$ " is replaced by " $\subset$ " in the last displayed equation.

Now we state Embry's Theorem and some of its corollaries which we will use to prove our results. This Theorem and corollaries are stated in [5]:

**Theorem 2.** *If  $N$  and  $M$  are commuting normal operators and  $AN = MA$ , where  $0$  is not in the numerical range of  $A$ , then  $N = M$ .*

**Corollary 1.** *If  $A$  is an operator such that either  $\sigma(A) \cap \sigma(-A) = \emptyset$  or  $0 \notin \mathcal{W}(A)$  and  $AE = -EA$ , where either  $A$  or  $E$  is normal, then  $E = 0$ .*

**Corollary 2.** *If  $A^2$  is normal and  $0 \notin \mathcal{W}(A)$ , then  $A$  is normal.*

## 4 The Main Results

**Theorem 3.** *The following are equivalent*

- i.  $A$  is a square-normal.*
- ii. each of  $B$  and  $C$  commute with  $\operatorname{Re}A^2$ .*
- iii.  $B$  commutes with  $\operatorname{Re}A^2$  and  $C$  commutes with  $\operatorname{Im}A^2$ .*

*Proof.* (i)  $\iff$  (ii)

$$\begin{aligned} B(\operatorname{Re}A^2) &= B\left(\frac{A^2 + (A^*)^2}{2}\right) = \frac{BA^2 + B(A^*)^2}{2} \\ &= \frac{A^2(A^*)^2A^2 + A^2(A^*)^2(A^*)^2}{2} \\ &= \frac{A^2A^2(A^*)^2 + (A^*)^2A^2(A^*)^2}{2} \\ &= \left(\frac{A^2 + (A^*)^2}{2}\right)A^2(A^*)^2 = (\operatorname{Re}A^2)B \end{aligned}$$

$$\begin{aligned} C(\operatorname{Re}A^2) &= (A^*)^2A^2\left(\frac{A^2 + (A^*)^2}{2}\right) \\ &= \frac{(A^*)^2A^2A^2 + (A^*)^2A^2(A^*)^2}{2} \\ &= \frac{A^2(A^*)^2A^2 + (A^*)^2(A^*)^2A^2}{2} \\ &= \left(\frac{A^2 + (A^*)^2}{2}\right)(A^*)^2A^2 \\ &= (\operatorname{Re}A^2)C \end{aligned}$$

Conversely, let  $H = \operatorname{Re} A^2$  and  $K = \operatorname{Im} A^2$ .

Since we have

$$BA^2 = A^2(A^*)^2A^2 = A^2C, \quad (1)$$

also we have

$$(A^*)^2B = (A^*)^2A^2(A^*)^2 = C(A^*)^2. \quad (2)$$

If

$$BH = HB \quad \text{and} \quad CH = HC$$

then

$$\begin{aligned} B \left( \frac{A^2 + (A^*)^2}{2} \right) &= \left( \frac{A^2 + (A^*)^2}{2} \right) B \\ BA^2 + B(A^*)^2 &= A^2B + (A^*)^2B. \end{aligned}$$

This is equivalent to

$$B(A^*)^2 - (A^*)^2B = A^2B - BA^2.$$

Hence by (1) and (2) we have:

$$(B - C)(A^*)^2 = A^2(B - C) \quad (3)$$

and similarly

$$(B - C)A^2 = (A^*)^2(B - C) \quad (4)$$

Multiplying (3) on the left by  $(A^*)^2$  and multiplying (4) on the right by  $(A^*)^2$ , we get:

$$\begin{aligned} (B - C)A^2(A^*)^2 &= (A^*)^2A^2(B - C) \\ (B - C)B &= C(B - C). \end{aligned} \quad (5)$$

Multiplying (3) on the right by  $A^2$  and multiplying (4) on the left by  $A^2$ , we get:

$$\begin{aligned} (B - C)(A^*)^2A^2 &= A^2(A^*)^2(B - C) \\ (B - C)C &= B(B - C). \end{aligned} \quad (6)$$

Subtract (6) from (5), will give

$$\begin{aligned} (B - C)(B - C) &= (C - B)(B - C) \\ (B - C)^2 &= -(B - C)^2 \end{aligned}$$

$(B - C)^2 = 0$  implies  $B - C = 0$  since  $B$  and  $C$  are self-adjoint operators, and so is  $B - C$ . So  $A$  is square-normal.

(ii)  $\implies$  (iii)

$BReA^2 = ReA^2B$  is clear.

From (ii)  $\implies$  (i) we have  $A$  is a square-normal operator so we have

$$A^2C = CA^2 \quad \& \quad (A^*)^2C = C(A^*)^2. \quad (7)$$

So by (7) we have

$$\begin{aligned} C(ImA^2) &= C \left( \frac{A^2 - (A^*)^2}{2i} \right) \\ &= \frac{CA^2 - (A^*)^2C}{2i} \\ &= \frac{A^2C - (A^*)^2C}{2i} \\ &= \left( \frac{A^2 - (A^*)^2}{2i} \right) C = (ImA^2)C \end{aligned}$$

(iii)  $\implies$  (i)

Since

$$\begin{aligned} KC &= CK \\ \left( \frac{A^2 - (A^*)^2}{2i} \right) C &= C \left( \frac{A^2 - (A^*)^2}{2i} \right) \\ (A^2 - (A^*)^2)C &= C(A^2 - (A^*)^2) \\ A^2C - (A^*)^2C &= CA^2 - C(A^*)^2 \\ BA^2 - CA^2 &= (A^*)^2C - (A^*)^2B \end{aligned}$$

so

$$(B - C)A^2 = -(A^*)^2(B - C). \quad (8)$$

Since because of  $HB = BH$  then (3) is satisfied. Indeed, if

$$BH = HB$$

then

$$\begin{aligned} B \left( \frac{A^2 + (A^*)^2}{2} \right) &= \left( \frac{A^2 + (A^*)^2}{2} \right) B \\ BA^2 + B(A^*)^2 &= A^2B + (A^*)^2B. \end{aligned}$$

This is equivalent to

$$B(A^*)^2 - (A^*)^2B = A^2B - BA^2.$$

Hence by (1) and (2), (3) is satisfied.

Multiplying (8) on the left by  $A^2$  and multiplying (3) on the right by  $A^2$ , we get:

$$(B - C)C = -B(B - C)$$

$$BC - C^2 = BC - B^2.$$

We get  $C^2 = B^2$  and so  $B = C$  (2, p. 262).

Hence  $A$  is a square-normal operator.

□

**Theorem 4.** *Let  $S$  and  $T$  be two square-normal operators. For any bounded linear operator  $A$  if*

$$AS^2 = T^2A$$

*then*

$$A(S^*)^2 = (T^*)^2A$$

*Proof.* Since  $S$  and  $T$  are square-normal operators, so by Proposition 2 we see that  $S^2$  and  $T^2$  are normal operators. So by Fuglede-Putnam Theorem 1 we see

$$A(S^*)^2 = (T^*)^2A$$

□

**Theorem 5.** *Let  $S$  and  $T$  be two square-normal operators which commute, and let  $A$  be any bounded operator for which  $0 \notin \mathcal{W}(A)$ . If  $AS^2 = T^2A$  then  $S^2 = T^2$ .*

*Proof.* Since  $S$  and  $T$  are square-normal operators, so by Proposition 2 we see that  $S^2$  and  $T^2$  are normal, so Embry's Theorem 2 is applicable, resulting in  $S^2 = T^2$ . □

**Theorem 6.** *If  $A$  is a square-normal operator and  $0 \notin \mathcal{W}(A)$  then  $A$  is normal.*

*Proof.* If  $A$  is square-normal operator then by Proposition 2  $A^2$  is normal. So by Corollary 2 we see that  $A$  is normal. □

**Theorem 7.** *If  $B$  and  $C$  commute and  $0 \notin \mathcal{W}(A^2)$  then  $A$  is square-normal operator.*

*Proof.* Notice that

$$A^2(A^*)^2A^2 = A^2(A^*)^2A^2$$

so

$$A^2C = BA^2$$

Since  $B$  and  $C$  are normal even self-adjoint which commute and  $0 \notin \mathcal{W}(A^2)$ . So Embry's Theorem 2 is applicable, resulting in  $B = C$ . So  $A$  is square-normal operator.  $\square$

**Theorem 8.** *Let  $A$  be any bounded operator. for any bounded operator  $E$  for which either  $0 \notin \mathcal{W}(E)$  or  $\sigma(E) \cap \sigma(-E) = \emptyset$  if*

$$EB + BE = 0 \tag{9}$$

and

$$EC + CE = 0 \tag{10}$$

then  $A$  is square-normal, where  $B = A^2(A^*)^2$  and  $C = (A^*)^2 A^2$ .

*Proof.* Subtracting (10) from (9), we see

$$EB - EC + BE - CE = 0$$

$$E(B - C) = -(B - C)E.$$

Since  $0 \notin \mathcal{W}(E)$  or  $\sigma(E) \cap \sigma(-E) = \emptyset$  and  $B - C$  is normal (even self-adjoint). So by Corollary 1 we see

$$B - C = 0 \implies B = C.$$

So  $A$  is a square-normal operator.  $\square$

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