

Pseudo Lifting Modules

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ABSTRACT

Let R be a ring with identity, and M is a unitary R -module. In this paper we introduce and study the concepts of pseudo lifting modules and give basic properties, examples and characterizations of these concepts.

Keywords: Direct Summand, Hollow Module, Lifting Module, Pseudo Lifting Module, Radical, R -Module, Semi-simple, Submodule, Z -Module.

1. INTRODUCTION

The basic properties and definitions are given in [2], [3], [6], [9] and [11].

Definition 1.1: Let R be a ring and M be an R -module. A submodule N of M is small, denoted by: $N \ll M$, if for any submodule X of M , $X + N = M$ implies that $X = M$.

Definition 1.2: Let M be an R -Module and let $N_1, N_2, \dots, N_K$ be submodules of M . The direct summand $N_1 + N_2 + \dots + N_K$ denoted by $\sum_{i=1}^k N_i$ is defined by: $\sum_{i=1}^k N_i = \{x: x = a_1 + a_2 + \dots + a_k; a_i \in N_i\}$, [8].

Definition 1.3 An R -module is called lifting if for every submodule N of M there exists a submodule K of N such that $M = K \oplus K'$ and $N \cap K' \ll K'$, [5].

Lifting modules have been studied recently by several authors, such as, K. Oshiro [10]. We call a submodule N of an R -module M a pseudo stable, if $f(N) \leq N$ for each R -monomorphism $f: N \rightarrow M$ [1]

In this paper, we introduce and study the concept pseudo lifting modules. An R -module M is called pseudo lifting module if for every submodule N of M there exists a pseudo stable submodule K of N such that $M = K \oplus K'$ and $N \cap K' \ll K'$.

Definition 1.4: Let M be a module is called a pseudo lifting module if for every submodule N of M there exists a pseudo stable submodule K of N such that $M = K \oplus K'$ and $N \cap K' \ll K'$.

Note: A ring R is called pseudo lifting if R is pseudo lifting R -module.

Every pseudo lifting module is lifting module, but the converse is not true. This is shown in the following example:

The Z -module $M = Z_{p^\infty} \oplus Z_{p^\infty}$ is lifting Z -module because M is projective and it is well-known that every projective module is lifting [5, Theorem 4-1-16], while M is not pseudo lifting.

The following lemma gives some properties of small submodules which are needed later

2. Applications

Lemma 2.1 : [10] Let N, K and L are submodules of M .

- 1- If $N \ll K$ and $K \ll L$, then $N \ll L$.
- 2- If $N \ll M$, $N \leq K$ and K is a direct summand of M then $N \ll K$.
- 3- If $N \ll M$ and $f : M \rightarrow M'$ is a submodule homomorphism, then $f(N) \ll f(M')$

The following theorem gives another characterization of pseudo lifting modules.

Theorem 2.2: Let M be an R -module. The following statements are equivalent:

1. M is pseudo lifting module
2. Every pseudo stable submodule N of M can be written as $N = A \oplus S$, where A is a direct summand of M and $S \ll M$.
3. For every pseudo stable submodule N of M , there exists a direct summand K of M such that $K \ll N$ and $N/K \ll M/K$.

Proof : (1 $\Rightarrow$ 2)

Let M be a pseudo lifting module. There exists a pseudo stable submodule K of N such that $M = K \oplus K'$ and $N \cap K' \ll K$. So $N \cap K' \ll M$, by lemma 2.1. Now, $N = N \cap M$

$$= N \cap (K \oplus K') = K \oplus N \cap K', \text{ by the modular law.}$$

We take $A = K$ and $S = N \cap K'$. Therefore $N = A \oplus S$ with A is a pseudo stable direct summand of M and $S \ll M$.

(2 $\Rightarrow$ 3): Let N be a submodule of M . By (2) $N = A \oplus S$ where A is a pseudo stable direct summand of M and $S \ll M$. The proof is concluded if we can show that $N/A \ll M/N$.

To see that, suppose that $N/A + L/A = M/A$. Then $(A \oplus S)/A + L/A = M/A$.

Therefore, $M = A + S + L = S + L$, since $A \leq L = L$, and $S \ll M$. Hence, $N/A \ll M/A$.

(3 $\Rightarrow$ 1): Let N be a submodule of M , there exists a pseudo stable submodule K of N such that $M = K \oplus K'$ and $N/K \ll M/K$. We want to show that $N \cap K' \ll K'$. Suppose that $N \cap K' + B = K'$, where $B \leq K'$. Therefore, $M = K + K' = K + N \cap K' + B$. Thus

$$M/K = (K + N \cap K' + B)/K = (N \cap K + K)/K + (K + B)/K.$$
 Since

$$(N \cap K' + K)/K \leq N/K, \text{ then } (N \cap K' + K)/K \ll M/K.$$

Therefore, $(K + B)/K = M/K$. Thus, $M = K + B$. Since, $B \leq K'$ and $K \cap K' = 0$, then $B \cap K = 0$ and hence, $M = K \oplus B$. This implies $B = K'$. Hence, $N \cap K' \ll K'$.

The following theorem gives another characterization of pseudo lifting modules.

Theorem 2.3: An R -module M is a pseudo lifting if and only if for each submodule A of M there is a direct decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$ where M_1 is a pseudo stable submodule of M and $A + M_2$ is small in M .

Proof: ($\Rightarrow$) Suppose that M is pseudo lifting module. Let A be a submodule of M . Thus there exists a pseudo stable direct summand K of M such that A is small in K . That is

$M = K \oplus K_1$ for some sub module K_1 of M . Also, since A is small in K and K_1 is small in K_1 , then $A \oplus K_1$ is small in $M = K \oplus K_1$. Hence $A \oplus K_1$ is small submodule of M .

($\Leftarrow$) Let A be a submodule of M , then there is a direct decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$. Where M_1 is pseudo stable sub module of M and $A + M_2$ is small in M . To prove that M is pseudo lifting module we claim that A is small in M_1 . Let K be a non-zero sub module of M_1 hence K is a submodule of M , since $A + M_2$ in M . Then $(A + M_2) \cap K \neq (0)$. Let $k = a + m_2$ be a non-zero element of K , where $a \in A$ and $m_2 \in M_2$, thus $m_2 = k - a \in M_1 \cap M_2 = (0)$. Therefore, $k = a \in K \cap A$, then $K \cap A \neq (0)$.

Hence A is small in M . Thus M is a pseudo lifting module.

The following proposition gives some properties of small submodules.

Proposition 2.4 : [12] Let K, L and N be a submodules of an R -module M , then

- 1- If $K \ll M$ and $f : M \rightarrow M'$ is a homomorphism, then $f(K) \ll M'$.
- 2- If $K \subseteq L \subseteq N \subseteq M$ and $L \ll N$, then $K \ll M$.
- 3- If $K \ll M$ and $L \ll M$, then $K + L \ll M$.
- 4- If L is a direct summand of M and $K \subseteq L$ with $K \ll M$, then $K \ll L$.

Proof : Let N be any submodule of M , since M is pseudo lifting module, so by definition (1) there exists a submodule K of N such that $M = K \oplus K'$ and $N \cap K' \ll K'$, but $K' \subseteq M$, hence $N \cap K' \ll M$

Recall that a module M is semi-simple, if every submodule is a direct summand of M , [7].

It is clear that (0) is the only small submodule in a semi-simple module.

Proposition 2.5: Every semi-simple module is pseudo lifting module.

Proof: Let M be a semi-simple module and N a pseudo stable submodule of M . Since M is semi-simple, N is a direct summand of M , i.e. $M = N \oplus T$, $N \cap T = 0 \ll M$. Hence, M is pseudo lifting.

Recall that the Jacobson Radical of a module M "notion $J(M)$ " is the intersection of all maximal submodules of M .

Note that the sum of all submodule of a module M is equal to $J(M)$. [4].

Proposition 2.6: Let M be a module with $J(M) = 0$, if M is Artinian, then M is semi-simple and hence, a pseudo lifting module.

Proof: Let N be a pseudo stable submodule of M . Set $S = \{L : M = N + L, N \ll M\}$. It is clear that $S \neq \emptyset$, since $M \in S$. Now M is Artinian implies that S has a minimal element say K . Let $M = U + (N \cap K)$, where $U \ll M$, $M = N + K = N + N \cap K = N + U \cap K$ by the minimality of K , we have $U \cap K = K$, i.e., $K \leq U$. Hence, $U = M$. This implies that $N \cap K \ll M$. Thus, $N \cap K \leq J(M) = 0$. i.e. $N \cap K = 0$. Therefore, $M = N \oplus K$ and thus M is semi-simple.

Definition 2.7 A module M is said to be hollow, if every proper submodule is small in M .

Proposition 2.8: Every hollow modules is pseudo lifting module.

Proof : Let N be a pseudo stable submodule of M , then $N = M \oplus 0$ and by Theorem 2.2, M is pseudo lifting .

Proposition 2.9, shows that any direct summand of pseudo lifting module is pseudo lifting.

Proposition 2.9: Any direct summand of a pseudo lifting module is pseudo lifting.

Proof: Let M be a pseudo lifting module. Suppose that $M = M_1 \oplus M_2$.

We want to show that M_1 is pseudo lifting, let N be a submodule of M_1 , so $N \leq M$.

By Theorem (3), $N = A \oplus S$, where A is a pseudo stable direct summand of M and $S \ll M$

By Lemma (2) $S \ll M_1$. Now $M = A \oplus T$, where $T \leq M$, since A is pseudo stable direct summand of M . The proof is concluded if we can show that A is a pseudo stable direct summand of M_1 . Now $M_1 = M_1 \cap M = M_1 \cap (A \oplus T) = A \oplus (M_1 \cap T)$, by the modular law.

Thus, A is a pseudo stable direct summand of N . By Theorem (3), M_1 is pseudo lifting.

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