

COFINITELY G-SUPPLEMENTED MODULES

Berna Koşar

Ondokuz Mayıs University, Department of Mathematics

Kurupelit-Atakum-Samsun/Türkiye-Turkey

bernak@omu.edu.tr

Abstract

In this work, cofinitely g-supplemented modules are defined and investigated some properties of these modules. It is shown that an arbitrary sum of cofinitely g-supplemented modules is cofinitely g-supplemented. In addition, amply cofinitely g-supplemented modules are also defined and given some equivalences.

Key words: g-Small Submodules, Supplemented Modules, g-Supplemented Modules, Amply g-Supplemented Modules.

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1 INTRODUCTION

Throughout this paper all rings have an identity and all modules are unital left modules.

Let R be a ring and M be an R -module. We denote a submodule N of M by $N \leq M$. If M/N is finitely generated for $N \leq M$, then N is called a *cofinite submodule* of M . Let M be an R -module and $T \leq M$. If $K = 0$ for every $K \leq M$ with $T \cap K = 0$, then T is called an *essential submodule* of M and it is denoted by $T \trianglelefteq M$. Let $L \leq M$. If for every $T \leq M$ with $M = L + T$ implies that $T = M$, then L is called a *small submodule of M* and denoted by $L \ll M$. K is called a *generalized small* (briefly, *g-small*) *submodule* of M if for every $T \trianglelefteq M$ with $M = K + T$ implies that $T = M$, this is written by $K \ll_g M$ (in [15], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). If T is both essential and maximal submodule of M , then T is called a *generalized maximal submodule* of M . The intersection of all generalized maximal submodules of M is called the *generalized radical* of M and it is denoted by $Rad_g M$ (in [15], it is denoted by $Rad_e M$). If M have no generalized maximal

submodules, then the generalized radical of M is defined by $Rad_g M = M$. Let U and V be submodules of M . If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M . If $M = U + V$ and $M = U + T$ with $T \leq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M . If every submodule of M has a supplement in M , then M is called a *supplemented module*. M is called a *g-supplemented module*, if every submodule of M has a g-supplement in M . Let $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a g-supplement V' with $V' \leq V$, we say U has *ample g-supplements* in M . If every submodule of M has ample g-supplements in M , then M is called an *amply g-supplemented module*.

There are some important properties of g-small submodules in [5], [12], [13] and [15].

Lemma 1.1 *Let M be an R -module and $K, N \leq M$. The following conditions are hold. (See [13], [15])*

- (1) *If $K \leq N$ and $N \ll_g M$, then $K \ll_g M$.*
- (2) *If $K \ll_g N$, then K is an g-small submodule in submodules of M which contain N .*
- (3) *If $f : M \rightarrow N$ be an R -module homomorphism and $K \ll_g M$, then $f(K) \ll_g N$.*
- (4) *If $K \ll_g L$ and $N \ll_g T$ for $L, T \leq M$, then $K + N \ll_g L + T$.*

Lemma 1.2 *Let M be an R -module. Then $Rad_g M = \sum_{L \ll_g M} L$. (See [5])*

Lemma 1.3 *Let M be an R -module, $X \leq U \leq M$ and V be a g-supplement of U . Then $(V + X)/X$ is a g-supplement of U/X in M/X . [See 5]*

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Definition 2.1 *Let M be an R -module. If every cofinite submodule of M has a g-supplement in M , then M is called a *cofinitely g-supplemented module*.*

Clearly we see that every g-supplemented module is cofinitely g-supplemented, but the converse is not true in general.

Lemma 2.2 *Assume M be a finitely generated R -module. If M is cofinitely g-supplemented, then M is g-supplemented.*

Proof. Clear, since every submodule of M is cofinite. ■

Lemma 2.3 *Let M be a cofinitely g-supplemented module. Then every factor module of M is cofinitely g-supplemented.*

Proof. Let M/X be any factor module of M and U/X be a cofinite submodule of M/X . Since $\frac{M}{U} \cong \frac{M/X}{U/X}$, U is a cofinite submodule of M . Since M is cofinitely g-supplemented, U has a g-supplement V in M . Then by Lemma 1.3, $(V + X)/X$ is a g-supplement of U/X in M/X . Hence M/X is cofinitely g-supplemented. ■

Corollary 2.4 *Any homomorphic image of a cofinitely g -supplemented module is cofinitely g -supplemented.*

Proof. Clear from Lemma 2.3. ■

Proposition 2.5 *Let M be a cofinitely g -supplemented module. Then every cofinite submodule of $M/\text{Rad}_g M$ is a direct summand.*

Proof. Let $U/\text{Rad}_g M$ be a cofinite submodule of $M/\text{Rad}_g M$. Then U is a cofinite submodule of M . Since M is cofinitely g -supplemented, U has a g -supplement V in M . Hence $M = U + V$ and $U \cap V \ll_g V$. Then $\frac{M}{\text{Rad}_g M} = \frac{U}{\text{Rad}_g M} + \frac{V + \text{Rad}_g M}{\text{Rad}_g M}$. Since $U \cap V \ll_g V$, by Lemma 1.1 and Lemma 1.2, $U \cap V \leq \text{Rad}_g M$. Hence $\frac{U}{\text{Rad}_g M} \cap \frac{V + \text{Rad}_g M}{\text{Rad}_g M} = \frac{U \cap V + \text{Rad}_g M}{\text{Rad}_g M} = 0$ and $U/\text{Rad}_g M$ is a direct summand of $M/\text{Rad}_g M$. ■

Lemma 2.6 *Let M be an R -module, $M_1 \leq M$, U be a cofinite submodule of M and M_1 be a cofinitely g -supplemented module. If $M_1 + U$ has a g -supplement in M , then so does U .*

Proof. Let X be a g -supplement of $M_1 + U$ in M . Then $M_1 + U + X = M$ and $(M_1 + U) \cap X \ll_g X$. Since U is a cofinite submodule of M , $U + X$ is also cofinite submodule of M . Then by $\frac{M_1}{M_1 \cap (U + X)} \cong \frac{M_1 + U + X}{U + X} = \frac{M}{U + X}$, $M_1 \cap (U + X)$ is a cofinite submodule of M_1 . Since M_1 is cofinitely g -supplemented, $M_1 \cap (U + X)$ has a g -supplement Y in M_1 , i.e. $M_1 \cap (U + X) + Y = M_1$ and $M_1 \cap (U + X) \cap Y \ll_g Y$. Following this, we have $M = M_1 + U + X = M_1 \cap (U + X) + Y + U + X = U + X + Y$ and $U \cap (X + Y) \leq X \cap (U + Y) + Y \cap (U + X) \leq X \cap (M_1 + U) + Y \cap M_1 \cap (U + X) \ll_g X + Y$. Hence $X + Y$ is a g -supplement of U in M . ■

Corollary 2.7 *Let M be an R -module, U be a cofinite submodule of M and $M_i \leq M$ for $i = 1, 2, \dots, n$. If $U + M_1 + M_2 + \dots + M_n$ has a g -supplement in M and M_i is a cofinitely g -supplemented module for every $i = 1, 2, \dots, n$, then U has a g -supplement in M .*

Proof. Clear from Lemma 2.6. ■

Lemma 2.8 *Any sum of cofinitely g -supplemented modules is cofinitely g -supplemented.*

Proof. Let $\{M_i\}_{i \in I}$ be family of cofinitely g -supplemented submodules of an R -module M and $M = \sum_{i \in I} M_i$. Let U be any cofinite submodule of M . Since U is cofinite submodule of M , there exists a finite subset $\{i_1, i_2, \dots, i_n\}$ of I such that $M = U + M_{i_1} + M_{i_2} + \dots + M_{i_n}$. Since $U + M_{i_1} + M_{i_2} + \dots + M_{i_n}$ has a g -supplement 0 in M and M_{i_k} is cofinitely g -supplemented for $k = 1, 2, \dots, n$, then by Corollary 2.7, U has a g -supplement in M . ■

Example 2.9 *Consider the $\mathbb{Z}$ -module ${}_Z\mathbb{Q}$. Since $\text{Rad}_Z \mathbb{Q} = {}_Z\mathbb{Q}$, ${}_Z\mathbb{Q}$ have no proper cofinite submodules. Hence ${}_Z\mathbb{Q}$ cofinitely g -supplemented. But it is well known that ${}_Z\mathbb{Q}$ is not g -supplemented.*

3 AMPLY COFINITELY G-SUPPLEMENTED MODULES

Definition 3.1 *Let M be an R -module. If every cofinite submodule of M has ample g -supplements in M , then M is called an amply cofinitely g -supplemented modules.*

Lemma 3.2 *Let M be an R -module. If every submodule of M is cofinitely g -supplemented, then M is amply cofinitely g -supplemented.*

Proof. Let $M = U + V$ with $V \leq M$ and U is a cofinite submodule of M . Since $\frac{V}{U \cap V} \cong \frac{U+V}{U} = \frac{M}{U}$, $U \cap V$ is a cofinite submodule of V . Since V is cofinitely g -supplemented, $U \cap V$ has a supplement T in V . Because of this $V = U \cap V + T$ and $U \cap V \cap T \ll_g T$. Thus $M = U + V = U + U \cap V + T = U + T$ and $U \cap T = U \cap V \cap T \ll_g T$. Hence U has ample g -supplements in M and M is amply cofinitely g -supplemented. ■

Corollary 3.3 *Every R -module is cofinitely g -supplemented if and only if every R -module is amply cofinitely g -supplemented.*

Proof. Clear from Lemma 3.2. ■

Lemma 3.4 *If M is a π -projective and cofinitely g -supplemented module, then M is a cofinitely amply g -supplemented module.*

Proof: Let $M = U + V$, U be a cofinite submodule of M and X be a g -supplement of U . Since M is π -projective and $M = U + V$, there exists an R -module homomorphism $f : M \rightarrow M$ such that $Im f \subset V$ and $Im(1 - f) \subset U$. So, we have $M = f(M) + (1 - f)(M) = f(U) + f(X) + U = U + f(X)$. Suppose that $a \in U \cap f(X)$. Since $a \in f(X)$, then there exists $x \in X$ such that $a = f(x)$. Since $a = f(x) = f(x) - x + x = x - (1 - f)(x)$ and $(1 - f)(x) \in U$ we have $x = a + (1 - f)(x)$ and $x \in U$. Thus $x \in U \cap X$ and so $f(x) \in f(U \cap X)$. Therefore we have $U \cap f(X) \leq f(U \cap X) \ll_g f(X)$. This means that $f(X)$ is a g -supplement of U in M with $f(X) \subset V$. Therefore M is amply g -supplemented.

Corollary 3.5 *If M is a projective and g -supplemented module, then M is an amply supplemented module.*

Proof. Clear from Lemma 3.4. ■

Lemma 3.6 *Let M be an amply cofinitely g -supplemented module. Then every factor module of M is amply cofinitely g -supplemented.*

Proof. Let M/T be any factor module of M and let U/T be a cofinite submodule of M/T . Assume $\frac{M}{T} = \frac{U}{T} + \frac{V}{T}$ with $V \leq M$. Since U/T is a cofinite submodule of M/T , U is a cofinite submodule of M . Since $\frac{M}{T} = \frac{U}{T} + \frac{V}{T}$, $M = U + V$. Since M is amply cofinitely g -supplemented, U has a g -supplement K in M with $K \leq V$. Then by Lemma 1.3, $(K + T)/T$ is a g -supplement of U/T in M/T with $(K + T)/T \leq V/T$. Hence M/T is amply cofinitely g -supplemented. ■

Corollary 3.7 *Let M be an amply cofinitely g -supplemented module. Then every homomorphic image of M is amply cofinitely g -supplemented.*

Proof. Clear from Lemma 3.6. ■

Proposition 3.8 *Let R be a ring. The following statements are equivalent.*

- (a) ${}_R R$ is g -supplemented.
- (b) ${}_R R$ is amply g -supplemented.
- (c) ${}_R R$ is cofinitely g -supplemented.
- (d) ${}_R R$ is amply cofinitely g -supplemented.
- (e) Every finitely generated R -module is g -supplemented.
- (f) Every finitely generated R -module is cofinitely g -supplemented.
- (g) $R^{(I)}$ is cofinitely g -supplemented for every index set I .
- (h) $R^{(I)}$ is amply cofinitely g -supplemented for every index set I .
- (i) Every R -module is cofinitely g -supplemented.
- (k) Every R -module is amply cofinitely g -supplemented.

Proof. (a) $\iff$ (b) Clear from [5] Corollary 8, since ${}_R R$ is projective.

(a) $\iff$ (c) Clear from Lemma 2.2.

(c) $\iff$ (d) Clear from Corollary 3.5, since ${}_R R$ is projective.

(a) $\implies$ (e) Assume M be a finitely generated R -module and let $M = \langle m_1, m_2, \dots, m_n \rangle$. Then $M = Rm_1 + Rm_2 + \dots + Rm_n$. Since ${}_R R$ is g -supplemented and Rm_i ($i = 1, 2, \dots, n$) is an homomorphic image of ${}_R R$, by [5] Corollary 4, Rm_i is g -supplemented. Then by [5] Corollary 3, M is g -supplemented.

(e) $\iff$ (f) Obtained from Lemma 2.2.

(f) $\implies$ (g) By hypothesis, ${}_R R$ is cofinitely g -supplemented. Because of this, by Lemma 2.8, $R^{(I)}$ is cofinitely g -supplemented for every index set I .

(g) $\iff$ (h) Clear from Corollary 3.5, since $R^{(I)}$ is projective for every index set I .

(g) $\implies$ (i) Clear from Corollary 2.4, since every R -module is ${}_R R$ -generated.

(i) $\iff$ (k) Obtained from Corollary 3.3.

(i) $\implies$ (c) Clear. ■

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