

# A Metric for Data Set Comparison Via QR-Factorization

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**Short Research  
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## Abstract

A new metric is introduced for comparing matrix data sets based on QR-factorizations. The metric measures the degree of similarity between the bases of any two matrices, with measure values in the interval  $[0, 1]$ . The metric is initially developed in a general framework, for any  $m \times n$  matrix. The proposed measure is then discussed specifically in reference to matrices representing theoretical and physical data sets. The measure has direct applications for projects involving particle travel, and image recognition, comparison, and registration, particularly in instances involving intermodal data acquisition.

The measure is shown to satisfy the requirements for a metric and is demonstrated to calculate rotation angle between two data sets that differ by angle,  $\theta$ . The results show that the metric works well in assessing differences between data sets that differ by small perturbations, assuming proper alignment prior to comparison. It is also shown to be an effective tool for 1) determining optimal rotational alignment of test  $2 - D$  images and 2) identifying an unknown sample in a data base in a single blind test.

**Keywords:** QR-decomposition; QR-factorization; basis comparison; trajectory comparison; matrix comparison; similarity measure

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## 1 Introduction

Object recognition is one of the fundamental problems of computer vision, and shape is recognized as one of the most powerful object descriptors ([Uijlings (2013)], [Shotton (2013)], [Pan (2012)], [Selvarajah (2011)]); and as the digital image storage and retrieval industry has grown, so has the need for shape matching methods ([Bai (2014)], [JonesMartin (2014)], [Belorgie (2002)], [Zhang (2004)]). While

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many methods have been developed, most are job-specific with advantages as well as disadvantages [Nawrat (2008)], and the need for more general methods remains. Existing methods proceed in two main steps: shape representation, followed by recognition ([Selvarajah (2011)]). Shape representation often involves the sampling of large data sets to form smaller data sets which may be easier to handle computationally. While singular value decompositions are often used, the calculation of a  $QR$ -decomposition is an interim step to finding the singular value decomposition. ([Nakatsukasa (2013)]) Increases in computational efficiency can be found if a  $QR$ -decomposition can be used rather than a singular value decomposition. Shape recognition typically consists of some measure of similarity between a known template and the unknown shape normally dictated by the representation. While the multi-variate classic theory of shape ([Goodall (1993)]) is built on the analysis of matrix representations, no universal shape comparison metric appears to exist across representations; the purpose of this paper is to describe one such measure based on  $QR$ -factorizations of matrix representations.

Where matrix comparisons are used, whether it be in shape recognition or other general matrix data setting, ([Fowlkes (1983)], [Sebastian (2003)]), such comparisons normally use  $L$ -norms to quantify direct matrix disparities, such as in linear least squares fitting problems ([Abdullah (2015)], [Arun (1987)], [Pratt (1987)]), where the typical matrix comparison is of the form

$$\|A_2 - A_1\|_L = \epsilon. \quad (1.1)$$

where  $\|\cdot\|_L$  is an appropriately chosen  $L$ -norm. For data that represents an Eulerian flow field, for example, this measures the difference in intensity values, while in Lagrangian trajectory data, it captures the difference in physical distance ([Jones (2004)]). Such  $L$ -norm measures therefore ultimately yield comparisons only from a physical distance or size perspective. In such studies, the goal is usually to draw conclusions about the inherent nature of a system based on these physical comparisons, and larger differences in the measurement are assumed to imply larger differences in the system as a whole. While this is certainly valid from the distance perspective, it is not true from a shape perspective. Consider a rock dropped into a lake. If one were to measure the difference in ripples at time  $t_1$  versus time  $t_2$ , the result would be a large error, appropriate for the time frame. Concurrently, however, the ripples themselves would still have largely the same shape, so to draw conclusions on the inherent nature of the system based on a distance argument could be misleading. This is also true for many biological shape applications, such as fingerprints or tree rings, where the shape is invariant while size grows, or in medical applications. ([Siong (2015)]) A different metric is therefore in order that can convey these shape similarities, rather than the physical distance between them; the  $Q$ -metric is introduced to address the issue.

Where  $QR$ -decomposition has been used in shape theory ([Diaz (2010)]), the focus has been on the shape-and-size distributions for which the  $R$ -matrix is central, along with the reduction of the original matrices so as to simplify direct comparison of their 'sizes', and not on a comparison measure. The  $Q$ -matrix of a  $QR$ -decomposition is in the Stiefel manifold, that is,  $Q^*Q = I$ , where  $()^*$  denotes conjugate transposition, and is used to construct a direct measure. To motivate the method, consider the curves given in Figure (1), representing an image that has been scaled by constant,  $k$ .

While the images are different in scale, they are images of the same figure: while their sizes differ, their shape is the same. However, direct comparison of the matrices of points that create them using equation (1) results in a value that is dependent upon the scaling factor,  $k$ ,

$$\|A_2 - A_1\|_L = \|kA_1 - A_1\|_L = |k - 1|\|A_1\|_L. \quad (1.2)$$

The measure clearly captures the difference in size, but not the same-ness in shape: a different metric is required to measure the equality of the shapes. Realizing that equality in this context means that the same basis shape template produced both shapes, and since a matrix's  $QR$ -factorization

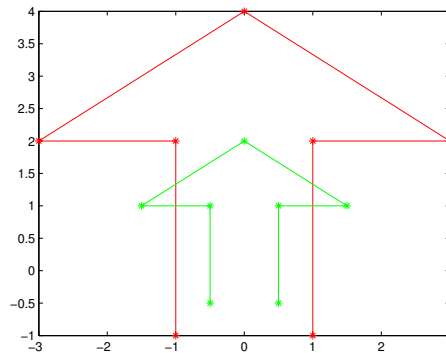


Figure 1: Images of two data sets that differ by scalar,  $k$ . Both figures have their centroids located at the origin.

produces a basis for its column space, a natural metric is suggested. The connection is explored and exploited to define a measure which captures the equality of the shapes.

The rest of the paper is organized as follows: In section 2, QR-factorization is briefly reviewed, and the new metric, the Q-metric, is defined for general  $n \times m$  QR-factorable matrices. The metric properties of the new measure are verified, and its rigid transformation properties are discussed with illustrations in section 3. Applications to shape rotation and registration and to image recovery using Hispaniola Amazon palate data are presented in section 4, followed by a brief discussion in section 5.

## 2 QR-factorization and the Q-metric

QR-factorizations have been used to great effect in efficiently solving the least squares problem in many areas or in reducing large data sets while maintaining representative information. ([Yu (2014)], [Xiao (2013)]) There are several properties which make them ideal for computationally solving certain problems. It is well-known that every matrix  $A \in R^{m \times n}$  ( $m \geq n$ ) has a QR-factorization, and it is unique under given circumstances. A full discussion of QR-factorizations is beyond the scope of this article; the reader is referred to the literature for in depth analysis of the features and formation of QR-factorizations (see [Stewart (1977)], [Stewart (1993)], [Sun (1991)]), and recall here only a few facts from Stewart (2000). A QR-factorization decomposes a matrix  $A \in R^{m \times n}$  into an orthonormal matrix,  $Q_A$ , and an upper triangular matrix,  $R_A$ , such that  $A = Q_A R_A$ . If  $A$  is of full rank, then this factorization is unique provided the diagonal elements of the  $R$  matrix are positive,  $r_{jj} > 0$ . The columns of  $Q_A$  form an orthonormal basis for the column space of  $A$ ,  $S(A)$  and  $Q_A^* Q_A = I$ , where  $Q^*$  is the matrix transpose of  $Q$ , is the projection of  $A$  onto  $S(A)$ : the features of the factorization are exploited to create the metric.

While matrices are widely used both as a means of shape representation and digital image storage, typical matrix comparisons using (1.1) are ill-suited for shape comparisons. The comparison in (1.1) is inferred from the measure of the difference between their representations and physical distances: there is an implicit assumption that the representations of the objects must be close in size and position in order for the objects to be similar. The same assumption underlies many shape recognition methods. A viewpoint more consistent with the observation of Figure(1) is offered,

requiring a metric which allows the relative representations to be different, but the shape assessment, as defined by human view, to be similar. The work is conducted under the perspective that two matrices  $A$  and  $B$  will be considered as effectively the same if their ranges are the same. The feature occurs provided that each matrix can be perfectly mapped onto the other, that is, if and only if  $S(A)=S(B)$ . Equality of the original matrices  $A$  and  $B$  is neither valid nor implied. The assessment is based upon a comparison of orthonormal bases for the matrices under consideration. As  $Q_A$  represents the column space ( $S(A)$ ) of  $A$ , it is proposed that the difference between the matrices  $Q_A$  and  $Q_B$  can be used to compare the spaces  $S(A)$  and  $S(B)$  in the same manner that the norm of the difference between matrices  $A$  and  $B$ ,  $(\|B - A\|)_p$  can, albeit with an upper bound. Let  $A \in R^{m \times n}$ ,  $B \in R^{m \times n}$  for  $m \geq n$ , be two shape matrices of full rank, with  $r_{jj} > 0$ . The matrices have guaranteed unique  $QR$ -factorizations  $A = Q_A R_A$  and  $B = Q_B R_B$ , and with respective column spaces denoted by  $S(A)$  and  $S(B)$ . In general, the expression  $Q_B Q_A$  is the mapping of  $A$  onto  $S(B)$  and the expression  $Q_A Q_B$  is the mapping of  $B$  onto  $S(A)$ . If  $Q_B^* Q_A = I$ , then  $Q_B^* = Q_A^{-1}$  and  $Q_B = Q_A^*$  or  $Q_B = Q_A$ . The result is  $A \in S(B)$  and  $B \in S(A)$ . The relationship drives the formulation of the  $Q$ -metric:

$$\mu(S(A), S(B)) = \frac{1}{2} \|Q_B^* Q_A - I\| \quad (2.1)$$

which provides a measure,  $\mu$ , of how different  $A$  is from  $S(B)$ , the column space of  $B$ , rather than how far apart  $A$  and  $B$  are from a distance perspective. It represents a mapping of one orthonormal basis onto another, with the expected result to be  $I$  if the two bases are equivalent. In the case when  $Q_A = Q_B$ , the weighted measure returns a value assessing the difference between the mapping and unit vector space. In this context, any residual found from the measure can be seen as an assessment of the deviation between the space of the mapping and unit vector space.

### 3 Properties of $Q$ -metric

Based on the orthonormal properties of the  $Q$ -matrices in an orthogonal decomposition,  $\|Q_B^* Q_A\| = 1$ . It follows from the triangle inequality that

$$0 \leq \mu(S(A), S(B)) \leq 1 \quad (3.1)$$

#### 3.1 Metric Properties

The  $Q$ -metric satisfies all metric properties:

1. Non-negativity
2. Identity
3. Symmetry
4. Triangle Inequality

regardless of matrix norm chosen.

### 3.1.1 Non-negativity

$$\mu(S(A), S(B)) \geq 0 \quad (3.2)$$

The non-negativity property is satisfied by definition. For all norms,  $\|\bullet\| \geq 0$ , thus it follows that

$$\mu(S(A), S(B)) = \frac{1}{2} \|Q_B^* Q_A - I\| \geq 0 \quad (3.3)$$

for all matrices  $A$  and  $B$ .

### 3.1.2 Identity

$$\mu(S(A), S(B)) = 0 \iff S(A) = S(B)$$

A feature of the orthonormal matrices of a  $QR$ -factorization is that  $Q_A^* Q_A = I$  (or  $Q_A^* = Q_A^{-1}$ ). First, assume that  $\mu(S(A), S(B)) = 0$ . It follows that

$$\mu(S(A), S(B)) = \frac{1}{2} \|Q_B^* Q_A - I\| = 0 \quad (3.5)$$

which can only be satisfied if  $Q_B^* Q_A = I$ . By definition,  $Q_B^*$  must be equivalent to  $Q_A^{-1}$ . Also from the properties of orthonormal matrices,  $Q_A^* = Q_A^{-1}$ , thus  $Q_B^* = Q_A^*$  and  $Q_B = Q_A$ . Conversely, assume that  $S(A) = S(B)$ . Then

$$\begin{aligned} \mu(S(A), S(B)) &= \mu(S(A), S(A)) \\ &= \frac{1}{2} \|Q_A^* Q_A - I\| \\ &= \frac{1}{2} \|I - I\| \\ &= 0 \end{aligned} \quad (3.6)$$

From construction of the  $Q$  matrices and uniqueness of inverses,  $Q_B = Q_A$  is the only solution, verifying uniqueness of the proposed measure.

### 3.1.3 Symmetry

$$\mu(S(A), S(B)) = \mu(S(B), S(A)) \quad (3.7)$$

The properties of the matrix transpose as applied to a product of matrices are also used. For square matrices,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times n}$ ,

$$(AB)^* = B^* A^* \quad (3.8)$$

where  $*$  denotes the conjugate transpose. Relative to a general  $p$ -norm,  $\|A^*\| = \|A\|$ . Given data

sets  $A \in R^{m \times n}$ ,  $m \geq n$ , the matrix  $Q_A \in R^{m \times m}$  and  $Q_B \in R^{m \times m}$  taken from a full  $QR$ -factorization:

$$\begin{aligned}
 \mu(S(A), S(B)) &= \frac{1}{2} \|Q_B^* Q_A - I\| \\
 &= \frac{1}{2} \|Q_B^* Q_A - Q_A^* Q_A\| \\
 &= \frac{1}{2} \|(Q_B^* - Q_A^*) Q_A\| \\
 &= \frac{1}{2} \| [Q_A^* (Q_B - Q_A)]^* \| \\
 &= \frac{1}{2} \| [Q_A^* (Q_B - Q_A)] \| \\
 &= \frac{1}{2} \| [Q_A^* Q_B - I]^* \| \\
 &= \mu(S(B), S(A))
 \end{aligned} \tag{3.9}$$

### 3.1.4 Triangle Inequality

$$\mu(S(A), S(C)) \leq \mu(S(A), S(B)) + \mu(S(B), S(C)) \tag{3.10}$$

To establish this property, consider 5 cases in turn:

(a) Case 1:  $A = B = C$

The inequality is trivially satisfied in this case since  $S(A) = S(B) = S(C)$ , so that  $\mu(S(A), S(B)) = \mu(S(A), S(C)) = \mu(S(B), S(C)) = \mu(S(A), S(A)) = 0$  by the identity property (3.1.2).

(b) Case 2:  $A = C \neq B$  In this case,  $S(A) = S(C)$ , so that  $\mu(S(A), S(C)) = 0$  by property (3.1.2). The inequality holds since the right-hand side of the triangle inequality is non-negative from the properties of  $p$ -norms. The case can be proved directly using property (3.1.2).

$$\begin{aligned}
 \mu(S(A), S(C)) &\leq \mu(S(A), S(B)) + \mu(S(B), S(C)) \\
 \mu(S(A), S(A)) &\leq \mu(S(A), S(B)) + \mu(S(B), S(A)) \\
 0 &\leq \mu(S(A), S(B)) + \mu(S(B), S(A)) \\
 0 &\leq 2\mu(S(A), S(B))
 \end{aligned} \tag{3.11}$$

which is true for all norms and is identically equal to 0 only in the case in which  $A = B$  from property (3.1.2).

(c) Case 3:  $A = B \neq C$  In this case,  $S(A) = S(B)$ , so that  $\mu(S(A), S(B)) = 0$ . Starting from the right side of the inequality:

$$\begin{aligned}
 \mu(S(A), S(C)) &\leq \mu(S(A), S(B)) + \mu(S(B), S(C)) \\
 &\leq \mu(S(A), S(A)) + \mu(S(A), S(C)) \\
 &\leq 0 + \mu(S(A), S(C)) \\
 &\leq \mu(S(A), S(C))
 \end{aligned} \tag{3.12}$$

(d) Case 4:  $C = B \neq A$  Proceeding as in Case 3, it is shown that

$$\begin{aligned}
 \mu(S(A), S(C)) &\leq \mu(S(A), S(B)) + \mu(S(B), S(C)) \\
 &\leq \mu(S(A), S(C)) + \mu(S(C), S(C)) \\
 &\leq \mu(S(A), S(C)) + 0 \\
 &\leq \mu(S(A), S(C))
 \end{aligned} \tag{3.13}$$

which establishes equality in the case.

(e) Case 5:  $A \neq B \neq C$  Case 5 is shown to satisfy the property through direct calculation. For simplicity, the measure is scaled by 2 to remove the  $\frac{1}{2}$  scaling factor.

$$\begin{aligned}
 2\mu(S(A), S(C)) &= \|Q_C^* Q_A - I\| \\
 &= \|Q_C^* Q_B Q_B^* Q_A - I\| \\
 &= \|(Q_C^* Q_B - Q_A^* Q_B)(Q_B^* Q_A)\| \\
 &= \|(Q_C^* Q_B - Q_A^* Q_B)\| \|Q_B^* Q_A\| \\
 &= \|(Q_C^* Q_B - Q_A^* Q_B) - I + I\| \\
 &= \|(Q_C^* Q_B - I) - (Q_A^* Q_B - I)\| \leq \|(Q_C^* Q_B - I)\| + \|(Q_A^* Q_B - I)\| \\
 &\leq 2(\mu(S(A), S(B)) + \mu(S(B), S(C)))
 \end{aligned} \tag{3.14}$$

The equivalence of  $\mu(S(A), S(B))$  with  $\mu(S(B), S(A))$  was established in (3.1.3).

## 3.2 Invariance Properties and Relationship with Rigid Transformations

General observations are presented about the effect of rigid transformations, namely scaling, rotation, and translation, on the  $Q$ -metric. For simplicity of exposition, observations are limited to rotations and translations of only  $2 \times 2$  matrices; the general  $n \times 2$  case will be demonstrated experimentally in the next section.

### 3.2.1 Scaling

Note that scale invariance is an intrinsic property of the measure; but for completeness, it is observed that if  $B$  is a scaling of a given matrix,  $A$ , then  $B \in S(A)$ , so that  $Q_B^* Q_A = I$ , and thus  $\mu(S(A), S(B)) = \mu(S(B), S(A)) = \frac{1}{2} \|Q_B^* Q_A - I\| = 0$ , and the measure is scale-invariant.

To illustrate scale invariance (assuming proper, centered alignment), additional data sets of the type in figure (1) were generated. In addition to the base image, scalings of the base image represented as  $kA = k(x_i, y_i)$ ,  $i \in [1 : n]$ , were generated for constant,  $k$ .

While physically different from a distance perspective, these perturbations have the same basic shape. The results of the norm and  $Q$ -metric comparisons are summarized in figure (2).

The figure provides an example of how the  $Q$ -metric is scale invariant; in particular, while the actual difference between the representative matrices ( $\|A_2 - A_1\|_2$ , where  $A_2 = kA_1$ , increases with  $k$ , the  $Q$ -metric measure of their difference remains constant at machine zero due to the scaling component being represented in the  $R$  matrix of the decomposition rather than the  $Q$  matrix.

### 3.2.2 Rotation

The proposed metric is based upon the orthonormal matrices of a  $QR$ -factorization. Orthonormal matrices are by definition also rotation matrices. The  $Q_i$  matrices of a factorization should then be able to be used to identify the angle between two data sets, and by extension, the orientation for optimal alignment. Singular value decompositions ( $SVD$ ) can be used to identify the rotation angle of optimal alignment between two  $3 - D$  data sets. ([Fitzpatrick (2000)]) **Relative to efficiency, however, using a  $QR$ -decomposition instead of an  $SVD$  has been shown to improve computational efficiency by an order of magnitude, potentially making it a preferred option in certain applications. ([Cheng (2016)])**

Consider orthonormal matrices  $Q_1$  and  $Q_2$  taken from matrices  $A_1, A_2 \in \mathbb{R}^{2 \times n}$  consisting of data points  $(x, y)$ , centered at the origin. Take  $A_2$  to be a rotation of  $A_1$  around the origin by some angle. For a vector,  $\mathbf{x}$ , both  $Q_1$  and  $Q_2$  will rotate the vector around the origin in a counterclockwise direction,

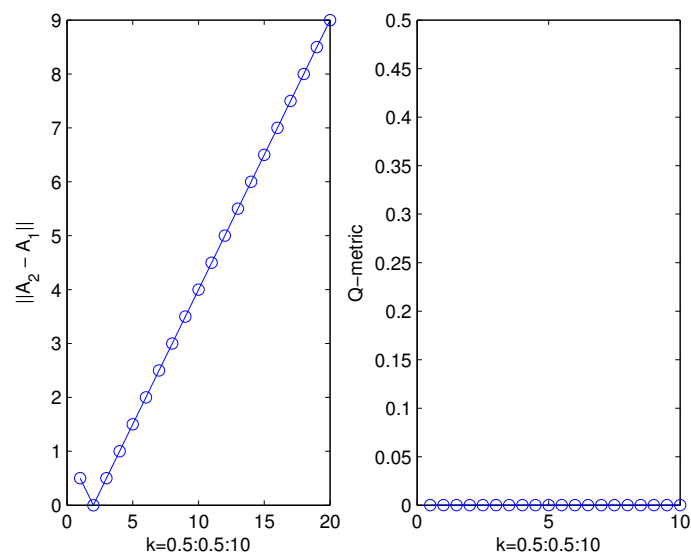


Figure 2: L-norm and Q-metric comparisons between  $A_2$  and  $A_1$ , where  $A_2 = kA_1$ . As the value of  $k$  increases, the  $L$ -norm also increases, resulting in a growing error measure. Conversely, the error measure generated by the  $Q$ -metric does not grow with increasing  $k$ .

while the inverses of the matrices will rotate the vector around the origin in the clockwise direction. Assume that  $Q_1$  rotates around the origin by an angle of  $\alpha$  and that  $Q_2$  rotates around the origin by an angle of  $\beta$ . Based on the proposed metric:

$$\mu(sp(A_1), sp(A_2)) = \frac{1}{2} \|Q_2^* Q_1 - I\|_2 \quad (3.15)$$

also, since the orthonormal matrices,  $Q_1$  and  $Q_2$  are also rotation matrices, they can be represented as

$$Q_1 = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

and

$$Q_2 = \begin{bmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{bmatrix}$$

If  $\alpha = \beta$  then  $Q_2 = Q_1$  and  $Q_2^* = Q_1^{-1}$ , resulting in an assessment of  $\mu = 0$  from the measure. Physically, if  $\alpha = \beta$ , the matrices rotate a vector in one direction and then back the other, realigning the vector in its original location. If  $\alpha \neq \beta$ , then the  $Q_1$  matrix will rotate a vector around the origin by  $\alpha$ , while the  $Q_2^*$  matrix will rotate the vector back by  $\beta$ . The residual between the product of the rotation matrices and the identity can be exploited to calculate the difference between the rotations. If the  $Q_i$  rotation matrices are taken from two different  $A_i$  matrices, the difference can be used to calculate the rotation angle of optimal alignment. Performing the assessment assumes that either the centroids the data sets have been previously aligned, or the axis of rotation between the data sets is known.

$$\begin{aligned} Q_2^* Q_1 &= \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta) \\ -\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha - \beta) & \sin(\alpha - \beta) \\ -\sin(\alpha - \beta) & \cos(\alpha - \beta) \end{bmatrix} \end{aligned} \quad (3.20)$$

note that  $Q_1^* Q_2$  is the transpose of the matrix  $((Q_2^* Q_1)^* = Q_1^* Q_2)$ . To calculate the  $L_2$  norm of the residual matrix:

$$\begin{aligned} \frac{1}{2} \|Q_2^* Q_1 - I\|_2 &= \frac{1}{2} \sqrt{\max(\lambda((Q_2^* Q_1 - I)^* (Q_2^* Q_1 - I)))} \\ \mu(sp(A_1), sp(A_2)) &= \frac{1}{2} (\sqrt{\max(\lambda(Q_1^* Q_2 Q_2^* Q_1 - Q_1^* Q_2 - Q_2^* Q_1 + I))}) \\ &= \frac{1}{2} \sqrt{\max(\lambda(2I - Q_1^* Q_2 - Q_2^* Q_1))} \\ &= \frac{1}{2} \sqrt{\max(\lambda(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2\cos(\alpha - \beta) & 0 \\ 0 & 2\cos(\alpha - \beta) \end{bmatrix}))} \\ &= \frac{1}{2} \sqrt{2\max(\lambda(\begin{bmatrix} 1 - \cos(\alpha - \beta) & 0 \\ 0 & 1 - \cos(\alpha - \beta) \end{bmatrix}))} \end{aligned} \quad (3.21)$$

Which results in one repeated eigenvalue,  $\lambda = (1 - \cos(\alpha - \beta))$ . Based on the definition of the metric, a closed-form representation can be found for the difference between the angles of rotation:

$$\begin{aligned}\mu &= \frac{1}{2}(\sqrt{2(1 - \cos(\alpha - \beta))}) \\ \frac{2\mu}{\sqrt{2}} &= \sqrt{(1 - \cos(\alpha - \beta))} \\ \frac{4\mu^2}{2} &= 1 - \cos(\alpha - \beta) \\ 1 - 2\mu^2 &= \cos(\alpha - \beta) \\ \cos^{-1}(1 - 2\mu^2) &= (\alpha - \beta)\end{aligned}\tag{3.22}$$

The result is a measure of the rotational difference between the two data sets. If the orientation of one of the data sets is known or assumed, the angle that results in the optimal alignment between the data sets can be calculated.

### 3.2.3 Translation

In its current form, the  $Q$ -metric is not translation invariant, and the problem of its normalization is not a trivial one. Based on the limitation, an alignment of the center of mass coordinate will be assumed in applications, so that comparisons are only made between representations on the same coordinate system. The practice is a standard one, found in many methods of registration. Typically, the center of mass is located and aligned *before* the registration is performed. ([JonesMartin (2014)], [Fitzpatrick (2000)])

## 4 Applications

In this section, two applications of the  $Q$ -metric are presented, first to calculating the measure of the angle that aligns two data sets and then to digital image registration and retrieval. In the second example, data sets representing palates of the Hispaniolan Amazona parrot will be used to demonstrate image registration and identification. All computations were conducted in Matlab, making extensive use of the built-in 'qr' function.

### 4.1 Image Alignment

Consider two data sets  $A_1$  and  $A_2$  that differ only by rotation angle,  $\theta$ , such that

$$A_2 = R_\theta A_1\tag{4.1}$$

where

$$R_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}\tag{4.2}$$

For testing purposes  $A_2$  is found by rotating  $A_1$  around the origin by an angle of  $50^\circ$ . The data being compared form the matrices:

$$A_1 = \begin{bmatrix} 1.3066 & 0.1585 & 2.0063 & -1.5307 & -3.35370 & -1.6892 & -0.5412 \\ -0.5412 & 2.2304 & 2.9958 & 3.6955 & 0.6997 & 1.4651 & -1.3066 \end{bmatrix}\tag{4.3}$$

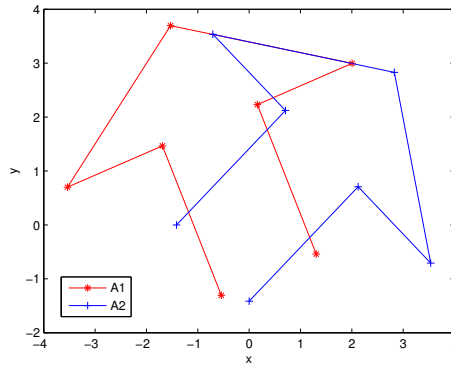


Figure 3: Data sets used for comparison, with aligned centroids, differing by angle,  $\theta$ . The data sets are identical, with the exception of orientation. A direct comparison would not identify the images as being similar, though the images are the same and only misaligned due to a rotation.

$$A_2 = \begin{bmatrix} 0.0000 & 2.1213 & 3.5355 & 2.8284 & -0.7071 & 0.7071 & -1.4142 \\ -1.4142 & 0.7071 & -0.7071 & 2.8284 & 3.5355 & 2.1213 & -0.0000 \end{bmatrix} \quad (4.4)$$

The metric value is calculated by first finding the  $QR$ -factorizations of each matrix and then plugging into the formula:

$$\begin{aligned} A_1 &= Q_1 R_1 \\ A_2 &= Q_2 R_2 \\ \mu &= \frac{1}{2} \|Q_2^* Q_1 - I\| \end{aligned} \quad (4.5)$$

Based on the value of  $\mu$ , the angle between the two data sets can be calculated as found in section 3.2.2.

$$\Delta\theta = \cos^{-1}(1 - 2\mu^2). \quad (4.6)$$

## 4.2 Image Retrieval

The  $Q$ -metric is particularly applicable to biological shape applications, such as for fingerprints, tree rings, or bird palates, where the pattern shape is effectively invariant while the size grows. We consider a data set that represents not trajectories, but fiducial landmarks as used in point based, rigid registration of images, and in particular data sets representative of the soft palates of parrots. It is believed that, while there is considerable variation in the pattern of ridges and sensory papillae in the cornified tissue of the upper palate in members of Hispaniolan Amazona, the integrated pattern of the cornified oral surface of the upper beak is likely to be as unique an identifier for parrots as fingerprints are for humans [Homerger (2008)]. In a previous study, Homerger, et. al. compiled casts of the palates of several parrots, digitized them and stored them in a database. In a follow up study, Redd, et. al. devised a method for extracting uniquely identifying data sets for each member of the palate database and created a separate, pared-down database. The information was stored as a matrix data set in which the columns of the matrices contained the  $(x, y)$  coordinates for the points that were proposed to be representative of the individual parrots. [Redd (2012)] In that study, a rigid

registration of an unknown sample was compared to a database of previously digitized samples. The registration aligned the data in a manner that minimized the distances between corresponding points in the two data sets. The minimum values were then compared to determine the sample number of the unknown birds.

A single blind test was conducted, in which an unknown image from a data set was analyzed to determine its identity. Representative landmarks (fiducial markers) were selected on the image using a prescribed criteria (which matched the criteria used for forming the database). The fiducial markers of the unknown data set were then compared to the fiducial markers of each member of the database in an effort to identify the unknown sample. An image of the unknown palate and its fiducial markers, along with some sample database palates are provided in Figures 4 and 5. The  $Q$ -metric is a measure of the similarity of matrices. Smaller values suggest matrices that are more similar, while larger values suggest matrices that are not very similar. It is expected that the sample that returns the smallest value should be the best match.

## 5 Results and Discussion

### 5.1 Angle of Optimal Alignment

After calculating the  $Q$ -metric value, the result was substituted into the angle formula 4.6. The angle between the data sets was found to be  $50^\circ$ . The value is exactly the angle that was used to generate the second data set. Applying a rotation matrix to the original data set,  $A_1$ , it is rotated and superimposed onto the plot of the original data sets (Figure 6). In this example, the images were known to be rotated around the same axis. Without *a priori* knowledge of the axis, it would be necessary to center each data set by locating the centroid and translating it to the origin. While the method works effectively in  $R^2$ , application to data sets in  $R^3$  has not been resolved. The additional directional axis will require that the single metric,  $\mu$ , be able to calculate rotation around up to three different axes.

### 5.2 Identification within a database

A single blind test similar to the one performed in the previous study (Homberger (2008)) was conducted. The test entailed using a data set representing an unknown, unregistered palate pattern to compare to a database of known palate patterns. Both data sets are known to be of full rank by nature, and using the  $Q$ -metric, it is possible to determine the best match to the data set without performing the registration process (involving a singular value decomposition) used in the previous study. The results from the study are summarized in Figure (8). Using the same data sets, the  $Q$ -metric was used to measure similarity between the unknown sample and all of the other members of the data set. The results are summarized in Table (1) and in Figure (7); Table (1) displays the  $Q$ -metric values for each sample number, while Figure (7) shows the fiducial markers from the unknown data set with the fiducial markers from the best match in the database, based on the  $Q$ -metric values. Based on the results in Figure (8), the sample in the database that best matches the unknown sample in a lowest  $Q$ -metric sense, is number 234. Sample number 234 is verified to be the correct sample. A limitation of this application over the previously mentioned method is that it does not actually perform a registration. If a registration is needed, the method presented in section (4.1) can be used to align the images and then a registration error can be calculated to assess the match. If the data are not well aligned to begin with, or at least known to have the same orientation, then the information gleaned from this could be misleading. In the situation under consideration here however, where the data acquisition process was known to be uniform, then the  $Q$ -metric is very appropriate for comparison. While it should not be used in a blindly definitive manner, it can be successfully used to narrow the

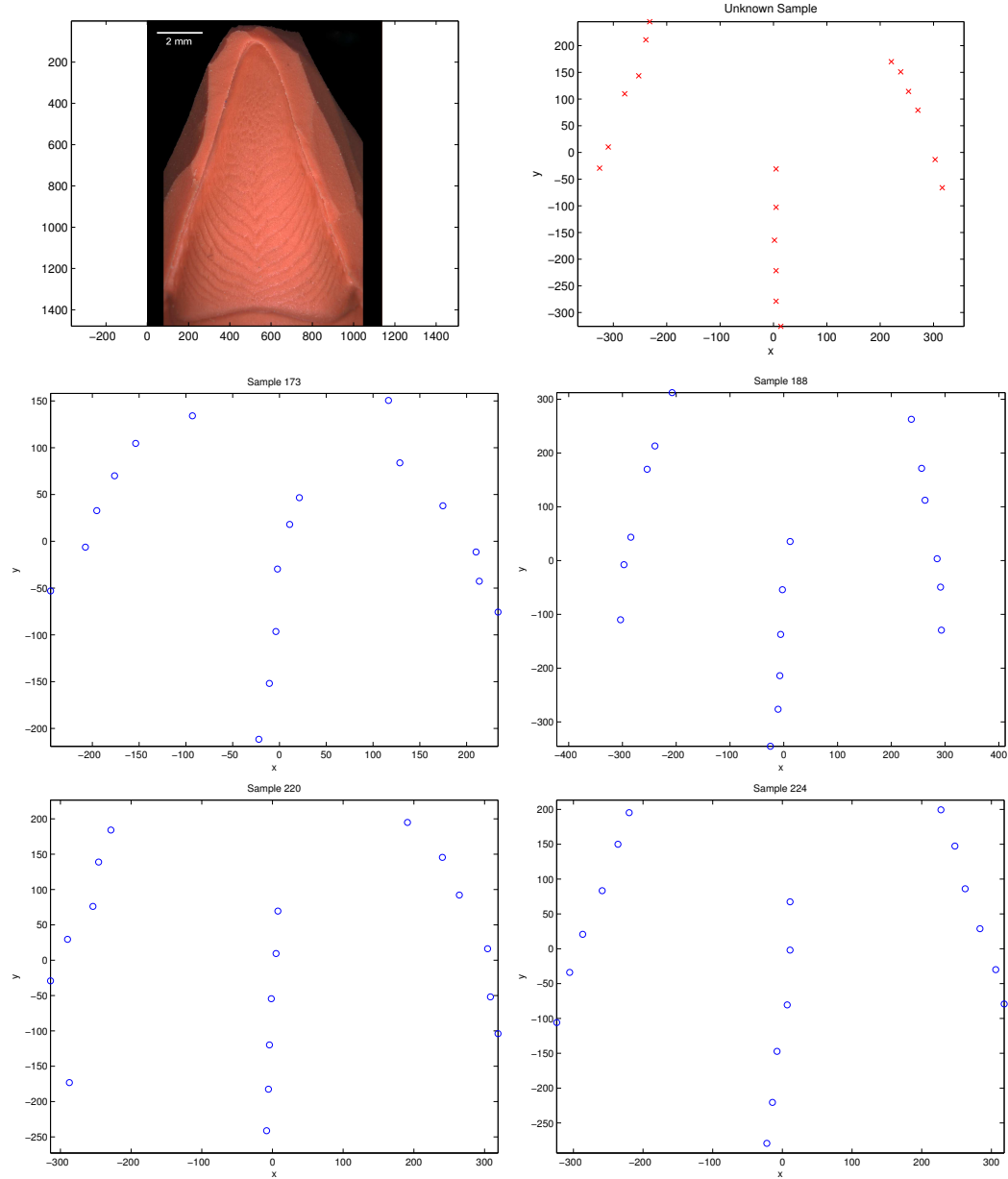


Figure 4: A sample of the unidentified palate image and the representative fiducial markers of the unidentified palate image. Members of the data sets that make up the search database are presented (with centroids translated to the origin).

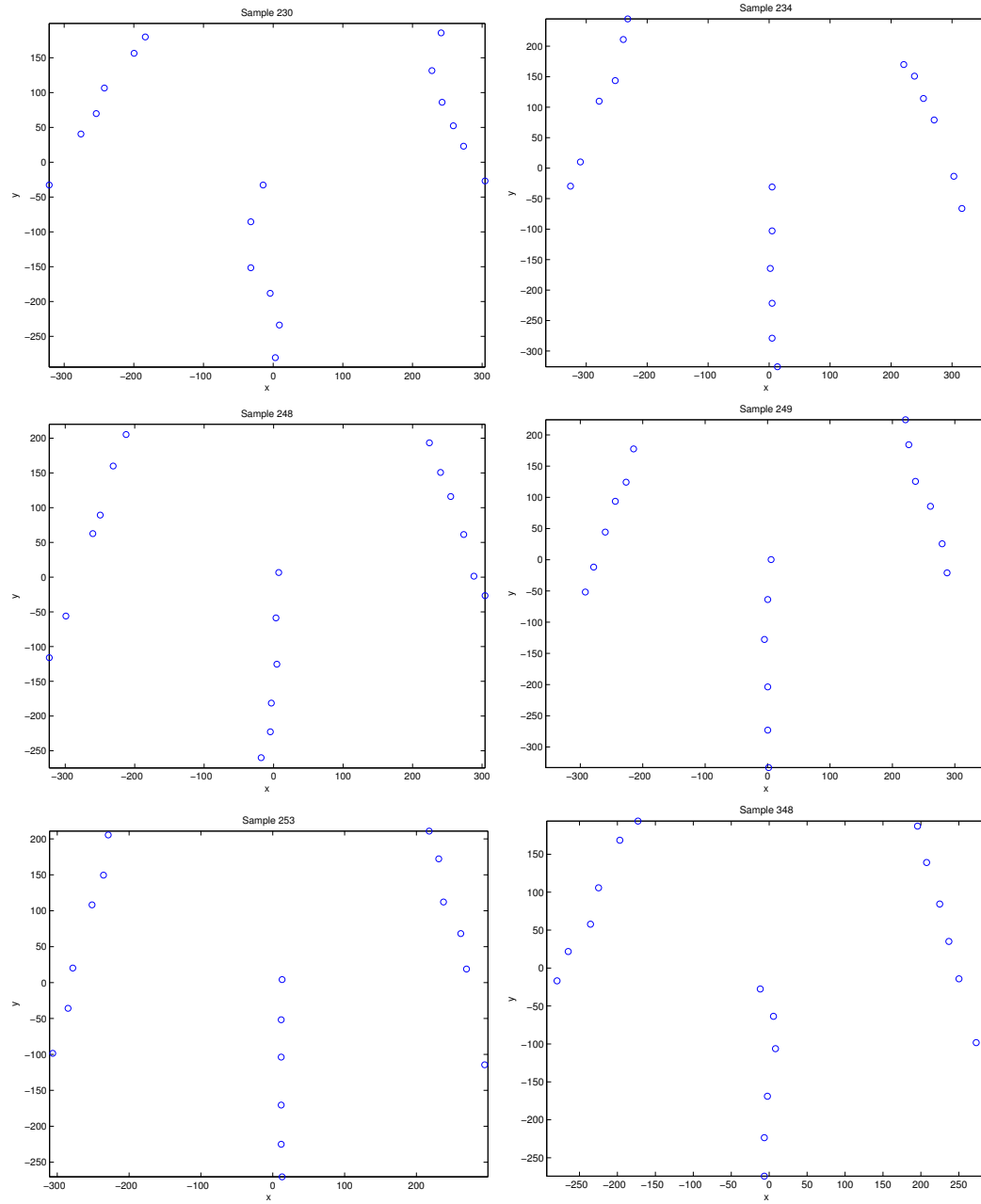


Figure 5: A sample of additional members of the search database (with centroids translated to the origin)

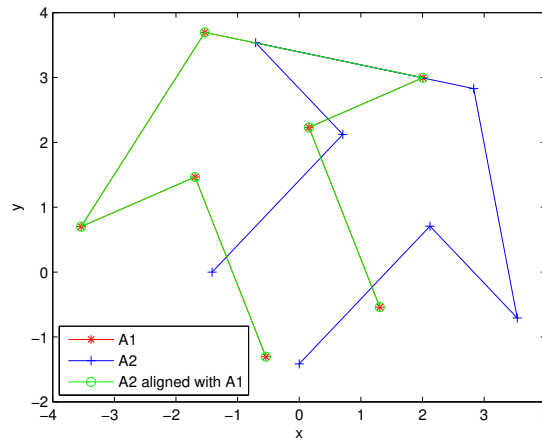
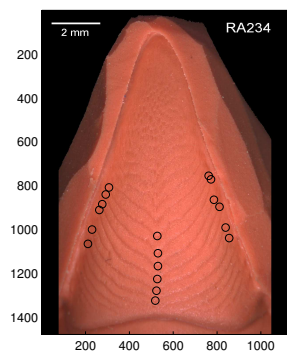
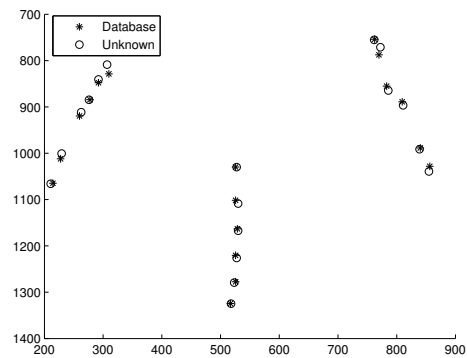


Figure 6: Data sets  $A_1$  and  $A_2$  and the aligned  $A_2$ . The angle calculated using the  $Q$ -metric has been applied to  $A_2$ , rotating it counterclockwise, resulting in excellent alignment with the  $A_1$  matrix



(a)



(b)

Figure 7: Example of a) a palate and sample points that represent its ridge structures, and b) a registration of an unknown data set (stars) with the optimal member of the database (circles) from palate project from a minimum distance perspective.

search window.

Relative to database searches, the  $Q$ -metric only conveys which specimen an unknown specimen is *most like*. If the correct match is not known to be in a database, a small, non-zero value cannot necessarily be used to definitively claim a match. It *can*, however, be used to effectively remove members of a database whose values are very large.

## 6 CONCLUSION

A method for comparing matrix data sets is presented based on  $QR$ -factorizations. In general this method applies to any  $m \times n$  of full column rank, and has direct applications in image recognition, comparison, and registration, particularly in instances involving intermodal data acquisition. The measure is defined in terms of the components of the  $QR$ -factorizations of the matrices under consideration. The metric satisfies the properties of the measure. Based on the inherent features of the orthonormal matrix,  $Q$ , of a  $QR$ -factorization, the introduced metric is bounded and scale invariant, and has predictable rotation variance; its application assumes the use of the center of mass of the objects being compared. The metric is a measure of the difference between the bases of two given matrices, which is advantageous in situations in which the space spanned by the bases of the matrices is of more interest than the difference between matrices themselves. In addition to comparing data sets with known alignment, the metric can be used to identify optimal alignment angle. In  $2 - D$  data sets, the metric can be used to determine the measure of the angle of rotation between corresponding data sets. The result is a closed form solution for aligning data sets in  $R^2$ . Relative to image registration and identification of a data set with a database, the metric performs well. It routinely returns a similarity assessment that is shown to accurately identify unknown specimens in single blind tests. While the metric has been shown to be a viable tool for calculating rotation angle between  $2 - D$  data sets, future efforts will be directed towards finding a closed form solution for calculating rotation angles of optimal alignment for a  $3 - D$  data set. An assessment of the efficiency of the process as compared to registration techniques using singular value decompositions will also be investigated. Additional applications under consideration include search algorithms for audio data and alignment of data sets with intermodal acquisition processes.

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| Sample | Q-metric |
|--------|----------|
| 115    | 0.0703   |
| 135    | 0.0881   |
| 167    | 0.0588   |
| 173    | 0.1531   |
| 188    | 0.0830   |
| 220    | 0.1565   |
| 224    | 0.0837   |
| 226    | 0.1565   |
| 227    | 0.0861   |
| 229    | 0.0713   |
| 230    | 0.0714   |
| 234    | 0.0130   |
| 238    | 0.0687   |
| 240    | 0.1280   |
| 242    | 0.1001   |
| 244    | 0.0566   |
| 245    | 0.1515   |
| 246    | 0.0862   |
| 247    | 0.0604   |
| 248    | 0.0648   |
| 249    | 0.0612   |
| 253    | 0.0627   |
| 348    | 0.0624   |
| 405    | 0.1052   |
| 407    | 0.0698   |
| 420    | 0.0555   |
| 531    | 0.0991   |

Table 1: Q-metric values for unknown palate versus database. The  $Q$ -metric value for each sample was calculated using (2.1), where the unknown sample is  $A_1$  and each known member of the database is taken to be  $A_2$  in turn. Sample 234 returns the lowest Q-metric value of 0.0130, resulting in an accurate identification of the unknown sample.

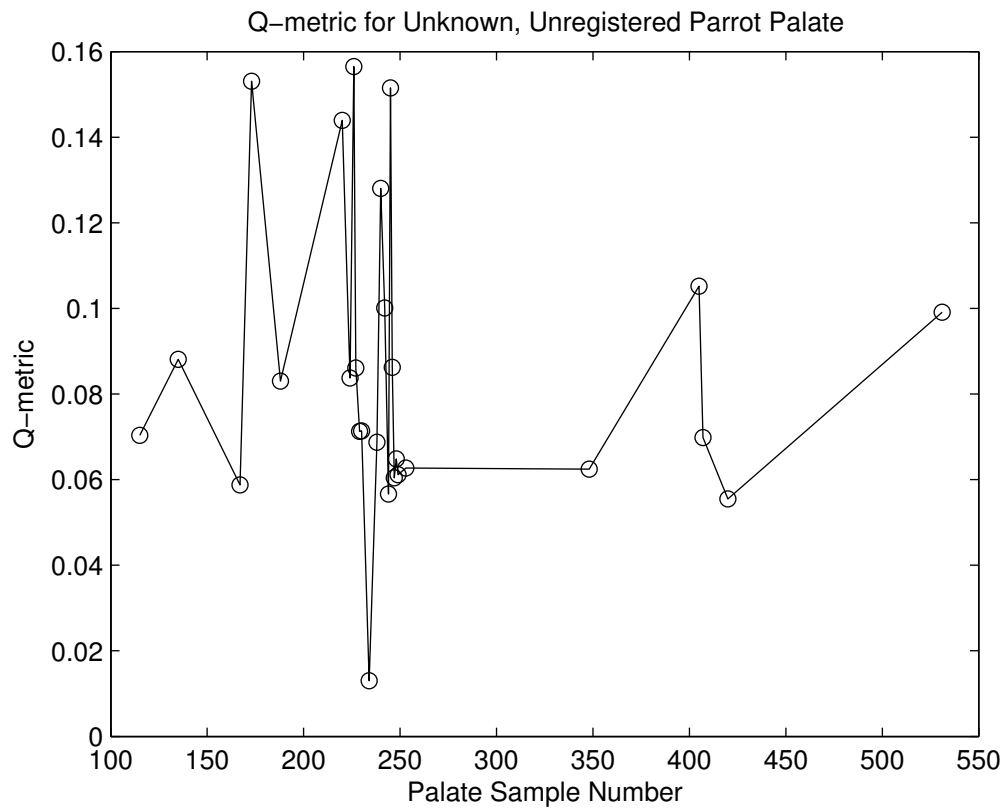


Figure 8: Q-metric values for comparison of unidentified sample with the members of the database. The calculated Q-metric value is plotted versus the sample number of the specimen that was compared with the unknown palate. Smaller Q-metric values suggest a better match between the database sample and the unknown sample. Sample number 234 returns the lowest Q-metric value and is the correctly identified specimen.

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