

ON SIMILAR PARTNER CURVES IN DUAL SPACE

FAİK BABADAĞ

ABSTRACT. In this paper, we give the new family of curves and call it a family of similar curves in three-dimensional dual space ID^3 with variable transformation. Also we give some characterizations of this family and some theorems. We show that the family of curves with vanishing curvatures forms a family of similar curves in three-dimensional dual space ID^3 .

1. INTRODUCTION

In the differential geometry, special curves which satisfy some relationships between their curvatures and torsions have an important role. The most fascinating of these curves are general helix, slant helix, B_2 -slant helix, Salkowski and anti-Salkowski curves [1, 2, 3, 4]. Another way to classification and characterization of curves is the relation between the Frenet vectors of the curves. For example (in 1845) Saint Venant [5] proposed the question upon the surface generated by the principal normal of a curve, a second curve can exist which has for its principal normal of a curve. This question was answered by Bertrand in 1850 in a paper [6] in which he solved that a linear relationship with constant coefficients shall exist between the first and second curvatures of the given original curve. In other words, if we denote first and second curvatures of a given curve by k_1 and k_2 respectively, then for $\lambda, \mu \in \mathbb{R}$ we have $\lambda k_1 + \mu k_2 = 1$. Since the time of Bertrand's paper, pairs of curves of this kind have been called Conjugate Bertrand Curves, or more commonly Bertrand Curves [7, 8, 9,].

In 1888, C. Bioche give a new theorem to obtaining Bertrand curves by using the given two curves C_1 and C_2 in Euclidean 3-space [10]. Later, in 1960, J. F. Burke give a theorem related with Bioche's theorem on Bertrand curves, [11].

Recently, similar curves have been introduced a new type of special curves in E^3 for which the arc-length parameters have a relationship and between the space curves ξ and η such that, at the corresponding points of the curves, the tangent lines of ξ coincides with the tangent of η , then ξ is called a similar curve, and η similar partner curve of ξ [12].

Dual numbers had been introduced by W.K. Clifford (1849-1870) as a tool for his geometrical investigations. After him E. Study used dual numbers and dual vectors in his research on line geometry and kinematics. He devoted special attention to the representation of oriented lines by dual unit vectors and defined the famous mapping: The set of oriented lines in an Euclidean three-dimension space E^3 is one to one correspondence with the points of a dual space ID^3 of triples of dual numbers, [13, 14].

In this work, we give some characterizations of these family and some theorems. We show that the family of curves with vanishing curvatures forms a family of similar curves in three-dimensional dual space ID^3 .

2. PRELIMINARIES

A dual number is a number of the form $\vec{x} = x + \varepsilon x^*$, where x, x^* are real numbers called the components of $\vec{x}$, and $\varepsilon = (0, 1)$ are arbitrary "dual unit" satisfy the relation $\varepsilon^2 = 0$, [17, 15]. Let $ID = \left\{ \vec{x} \mid \vec{x} = x + \varepsilon x^*, x, x^* \in IR \right\}$. Addition of two elements in ID and scalar multiplication of an element in ID by a real number are defined in the usual way. ID is a vector space with respect to addition and scalar multiplication. The product $\vec{x} \otimes \vec{y}$ is the element in ID obtained by multiplying $x + \varepsilon x^*$ and $y + \varepsilon y^*$ as if they were polynomials and then using the relation $\varepsilon^2 = 0$ to simplify the result

$$\vec{x} \otimes \vec{y} = \hat{x} \hat{y} = (x + \varepsilon x^*) \cdot (y + \varepsilon y^*) = xy + \varepsilon (xy^* + x^*y).$$

The dual number $x + \varepsilon x^*$ divided by the dual number $y + \varepsilon y^*$ provided $y \neq 0$ can be defined as

$$\frac{\vec{x}}{\vec{y}} = \frac{x + \varepsilon x^*}{y + \varepsilon y^*} = \frac{x}{y} + \varepsilon \frac{x^*y - xy^*}{y^2}.$$

The set of

$$\begin{aligned} ID^3 &= ID \times ID \times ID \\ &= \left\{ \vec{x} \mid \vec{x} = (x_1 + \varepsilon x_1^*) + (x_2 + \varepsilon x_2^*) + (x_3 + \varepsilon x_3^*) \right\} \\ &= \left\{ \vec{x} \mid \vec{x} = (x_1, x_2, x_3) + \varepsilon (x_1^*, x_2^*, x_3^*) \right\} \\ &= \left\{ \vec{x} \mid \vec{x} = x + \varepsilon x^* \right\} \end{aligned}$$

is a module on the ring ID . For any $\vec{x} = x + \varepsilon x^*, \vec{y} = y + \varepsilon y^* \in ID^3$, the scalar or inner product and the vector product of $\vec{x}$ and $\vec{y}$ are defined by, respectively,

$$\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \varepsilon \left(\langle \vec{x}, \vec{y}^* \rangle + \langle \vec{x}^*, \vec{y} \rangle \right),$$

$$\vec{x} \wedge \vec{y} = (\hat{x}_2 \hat{y}_3 - \hat{x}_3 \hat{y}_2, \hat{x}_3 \hat{y}_1 - \hat{x}_1 \hat{y}_3, \hat{x}_1 \hat{y}_2 - \hat{x}_2 \hat{y}_1),$$

where $\hat{x}_i = x_i + \varepsilon x_i^*, \hat{y}_i = y_i + \varepsilon y_i^* \in ID, 1 \leq i \leq 3$. If $x \neq 0$, the norm $\|\vec{x}\|$ of $\hat{x} = x + \varepsilon x^*$ is defined by

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \|\vec{x}\| + \varepsilon \frac{\langle \vec{x}, \vec{x}^* \rangle}{\|\vec{x}\|}.$$

A dual vector $\vec{x}$ with $norm = 1$ is called a dual unit vector. Let $\vec{x} = \vec{x} + \varepsilon \vec{x}^* \in ID$. The set

$$S^2 = \left\{ \vec{x} = \vec{x} + \varepsilon \vec{x}^* \mid \|\vec{x}\| = (0, 1); \vec{x}, \vec{x}^* \in IR^3 \right\}$$

is called the unit sphere with the center $\hat{O}$ in ID^3 . If every $x_i(t)$ and $x_i^*(t), 1 \leq i \leq 3$ real valued functions, are differentiable the dual space curve

$$\begin{aligned} \hat{x} &: I \subset IR \rightarrow ID^3 \\ t &\rightarrow \vec{x}(t) = x_1(t) + \varepsilon x_1^*(t) + x_2(t) + \varepsilon x_2^*(t) + x_3(t) + \varepsilon x_3^*(t) \\ &= \vec{x}(t) + \varepsilon \vec{x}^*(t) \end{aligned}$$

in ID^3 is differentiable. We call the real part $x(t)$ the indicatrix of $\vec{x}(t)$. The dual arc-length of the curve $\vec{x}(t)$ from t_1 to t is defined as

$$\widehat{s} = \int_{t_1}^t \left\| \vec{x}'(t) \right\| dt = \int_{t_1}^t \left\| \vec{x}(t)' \right\| dt + \varepsilon \int_{t_1}^t \langle \vec{t}, (\vec{x}^*)' \rangle ds = s + \varepsilon s^*,$$

where $\vec{t}$ is a unit tangent vector of $\vec{x}(t)$. From now on we will take the arc-length s of $\vec{x}(t)$ as the parameter instead of t . Now we will obtain equations relatively to the derivatives of dual Frenet vectors throughout in ID^3 . Let

$$\begin{aligned} \widehat{x} &: I \subset IR \rightarrow ID^3 \\ s &\rightarrow \vec{x}(s) = \vec{x}(s) + \varepsilon \vec{x}^*(s) \end{aligned}$$

be a dual curve with the arc-length parameter s of the indicatrix. Then ,

$$\frac{d\vec{x}}{d\widehat{s}} = \frac{d\vec{x}}{ds} \frac{ds}{d\widehat{s}} = \vec{t}$$

is called the dual unit tangent vector of $\vec{x}(s)$. With the aid of (1), we have

$$\widehat{s} = s + \varepsilon \int_{s_1}^s \langle \vec{t}, (\vec{x}^*)' \rangle ds$$

and from this $\frac{d\widehat{s}}{ds} = 1 + \varepsilon \Delta$, where the prime denotes differentiation with respect to the arc length s of indicatrix and $\Delta = \langle \vec{t}, (\vec{x}^*)' \rangle$. Since $\vec{t}$ has constant length 1, its differentiation with respect to $\widehat{s}$, which is given by

$$\frac{d\vec{t}}{d\widehat{s}} = \frac{d\vec{t}}{ds} \frac{ds}{d\widehat{s}} = \widehat{\kappa} \vec{n},$$

measure the way the curve is turning in ID^3 . The norm of the vector $\frac{d\vec{t}}{d\widehat{s}}$ is called curvature function of $\vec{x}$. We impose the restriction that the function $\widehat{\kappa}: I \rightarrow ID$ is never pure dual. Then , the dual unit vector $\vec{n} = \frac{1}{\widehat{\kappa}} \frac{d\vec{t}}{d\widehat{s}}$ is called the principal normal of $\vec{x}(s)$. The dual vector $\vec{b} = -\frac{\widehat{\tau}}{\widehat{\kappa}} \frac{d\vec{t}}{d\widehat{s}}$ is called the binormal of $\vec{x}$. The dual vectors $\vec{t}$, $\vec{n}$, $\vec{b}$ are called the dual Frenet frame of $\vec{x}$ at the point $\widehat{x}(s)$. The equalities relative to derivative of dual Frenet vectors $\vec{t}$, $\vec{n}$, $\vec{b}$ throughout the dual space curve are written in the matrix form

$$(2.1) \quad \frac{d}{d\widehat{s}} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix} = \begin{bmatrix} 0 & \widehat{\kappa} & 0 \\ -\widehat{\kappa} & 0 & \widehat{\tau} \\ 0 & -\widehat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix}$$

where $\widehat{\kappa} = \kappa + \varepsilon \kappa^*$ is nowhere pure dual curvature and $\widehat{\tau} = \tau + \varepsilon \tau^*$ is nowhere pure dual torsion. The formulae (2.1) are called the Frenet formulae of dual curve in ID^3 , [15].

3. THE SIMILAR CURVES IN ID^3 WITH VARIABLE TRANSFORMATIONS

Definition 1. Let ID^3 be 3-dimensional dual space with the standard inner product $\langle \cdot, \cdot \rangle$ and $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$, $\widehat{\xi}_\beta(\widehat{s}_\beta)$ be dual curves in ID^3 parameterized by arclengths $\widehat{s}_\alpha$ and $\widehat{s}_\beta$ with non-zero curvatures $\{\widehat{\kappa}_\alpha, \widehat{\tau}_\alpha\}$ and $\{\widehat{\kappa}_\beta, \widehat{\tau}_\beta\}$, and Frenet frame

$\{\widehat{t}_\alpha, \widehat{n}_\alpha, \widehat{b}_\alpha\}$ and $\{\widehat{t}_\beta, \widehat{n}_\beta, \widehat{b}_\beta\}$ respectively. If there exists variable transformation

$$(3.1) \quad \widehat{s}_\alpha = \int \widehat{\lambda}_\beta^\alpha(\widehat{s}_\beta) d\widehat{s}_\beta, \quad \widehat{\lambda}_\beta^\alpha \widehat{\lambda}_\alpha^\beta = 1$$

and a corresponding relationship between dual curves $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$ and $\widehat{\xi}_\beta(\widehat{s}_\beta)$ such that, at the corresponding points of dual curves, the tangent lines of $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$ is equal to the tangent lines of $\widehat{\xi}_\beta(\widehat{s}_\beta)$, then $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$ is called a dual similar curves, and $\widehat{\xi}_\beta(\widehat{s}_\beta)$ a dual similar partner curve of $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$. The pair $\{\widehat{\xi}_\alpha(\widehat{s}_\alpha), \widehat{\xi}_\beta(\widehat{s}_\beta)\}$ is said to be a dual similar partner curve. In this case the tangent vectors are the same for the two curves i.e.,

$$(3.2) \quad \widehat{t}_\alpha(\widehat{s}_\alpha) = \widehat{t}_\beta(\widehat{s}_\beta)$$

for all corresponding values of parameters under the transformation $\widehat{\lambda}_\beta^\alpha$. Where $\widehat{\lambda}_\beta^\alpha$ is arbitrary function of the arclength. If we integrate the equation (3.2), we have the following important theorem:

Theorem 1. The position vectors of the family dual similar curves in ID^3 with variable transformation can be written in the following form.

$$(3.3) \quad \widehat{\xi}_\beta(\widehat{s}_\beta) = \int \widehat{t}_\beta(\widehat{s}_\alpha(\widehat{s}_\beta)) d(\widehat{s}_\beta) = \int \widehat{t}_\alpha(\widehat{s}_\alpha) \widehat{\lambda}_\beta^\alpha d(\widehat{s}_\beta).$$

Theorem 2. Let $\widehat{\xi} = \widehat{\xi}(\widehat{s})$ be a dual curve with arc-length parameter $\widehat{s}$. If $\widehat{\xi}(\widehat{\varphi})$ is another parametrization of the curve by the parameter $\widehat{\varphi} = \int \widehat{\kappa}(\widehat{s}) d(\widehat{s})$. Then the unit tangent vector $\widehat{t}$ of $\widehat{\xi}(\widehat{s})$ satisfies a dual vector differential equation of third order as follows:

$$(3.4) \quad \left[\frac{1}{\gamma} \left[\frac{\widehat{t}}{\widehat{t}} \right]'' \right]' + \left[\frac{1+\gamma^2}{\gamma} \right] (\widehat{t})' + \frac{\gamma'}{\gamma} \widehat{t} = \mathbf{0},$$

where

$$\gamma(\widehat{\varphi}) = \frac{\widehat{\tau}(\widehat{\varphi})}{\widehat{\kappa}(\widehat{\varphi})}, \quad (\widehat{t})' = (d\widehat{t}) / d\widehat{\varphi}, \quad (\widehat{t})'' = (d^2\widehat{t}) / d\widehat{\varphi}^2.$$

Proof. If we write derivatives given in (2.1) according to φ , we have

$$(3.5) \quad \begin{aligned} \frac{d\widehat{t}}{d\varphi} &= \widehat{\kappa} \frac{1}{\widehat{\kappa}} \widehat{n} = \widehat{n} \\ \frac{d\widehat{n}}{d\varphi} &= -\widehat{\kappa} \frac{1}{\widehat{\kappa}} \widehat{t} + \widehat{\tau} \frac{1}{\widehat{\kappa}} \widehat{b} = -\widehat{t} + \gamma \widehat{b} \\ \frac{d\widehat{b}}{d\varphi} &= -\widehat{\tau} \frac{1}{\widehat{\kappa}} \widehat{b} = -\gamma \widehat{n} \end{aligned}$$

respectively, where $\gamma(\varphi) = \frac{\widehat{\tau}(\widehat{\varphi})}{\widehat{\kappa}(\widehat{\varphi})}$. Then, corresponding matrix form of (3.5) can be obtained by

$$(3.6) \quad \begin{bmatrix} \overrightarrow{t}' \\ \overrightarrow{n}' \\ \overrightarrow{b}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \widehat{\gamma} \\ 0 & -\widehat{\gamma} & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix}.$$

From the first and second equation of new Frenet derivatives (3.6) we obtain

$$(3.7) \quad \overrightarrow{b} = \frac{1}{\gamma(\widehat{\varphi})} \left(\overrightarrow{t}'' + \overrightarrow{t} \right).$$

Substituting the above equation in the last equation (3.6), we obtain a dual vector differential equation of third order (3.4) as desired. Where the representation of the position vector of an arbitrary space can be determined as follows

$$\widehat{\xi}(\widehat{s}) = \int \frac{1}{\widehat{\kappa}(\widehat{\varphi})} \overrightarrow{t}(\widehat{\varphi}) d\widehat{\varphi} + C \quad \text{and} \quad \varphi = \int \widehat{\kappa}(\widehat{\varphi})$$

is a constant vector. $\square$

Theorem 3. Let $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$, $\widehat{\xi}_\beta(\widehat{s}_\beta)$ be dual curves in ID^3 . Then $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$, $\widehat{\xi}_\beta(\widehat{s}_\beta)$ are dual similar curves in ID^3 with variable transformation if and only if the principal normal vectors are the same for all curves

$$(3.8) \quad \overrightarrow{n}_\alpha(\widehat{s}) = \overrightarrow{n}_\beta(\widehat{s}_\beta)$$

under the particular variable transformation

$$(3.9) \quad \widehat{\lambda}_\alpha^\beta = \frac{d\widehat{s}_\beta}{d\widehat{s}_\alpha} = \frac{\widehat{\kappa}_\alpha(\widehat{s}_\alpha)}{\widehat{\kappa}_\beta(\widehat{s}_\beta)}$$

of arc-lengths.

Proof. Let $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$, $\widehat{\xi}_\beta(\widehat{s}_\beta)$ be dual curves in ID^3 with variable transformation. Then differentiating the equation (3.8) with respect to $(\widehat{s}_\beta)$ we obtain

$$\widehat{\kappa}_\alpha(\widehat{s}_\alpha) \overrightarrow{n}_\alpha(\widehat{s}_\alpha) \frac{d\widehat{s}_\alpha}{d\widehat{s}_\beta} = \widehat{\kappa}_\beta(\widehat{s}_\beta) \overrightarrow{n}_\beta(\widehat{s}_\beta).$$

$\square$

From the above equation, we obtain equations (3.8) and (3.9).

Conversly, Let $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$, $\widehat{\xi}_\beta(\widehat{s}_\beta)$ be dual curves in ID^3 with variable transformation satisfying (3.8) and (3.9). By multipling (3.8) with $\widehat{\kappa}_\beta(\widehat{s}_\beta)$ and integrate the result equality with respect to $\widehat{s}_\beta$, we obtain

$$(3.10) \quad \int \widehat{\kappa}_\beta(\widehat{s}_\beta) \overrightarrow{n}_\beta(\widehat{s}_\beta) d\widehat{s}_\beta = \int \widehat{\kappa}_\beta(\widehat{s}_\beta) \overrightarrow{n}_\beta(\widehat{s}_\beta) \frac{d\widehat{s}_\beta}{d\widehat{s}_\alpha} d\widehat{s}_\alpha.$$

From the equations (3.8) and (3.9), equation (3.10) take the form

$$\int \widehat{\kappa}_\beta(\widehat{s}_\beta) \overrightarrow{n}_\beta(\widehat{s}_\beta) d\widehat{s}_\beta = \int \widehat{\kappa}_\beta(\widehat{s}_\beta) \overrightarrow{n}_\beta(\widehat{s}_\beta) d\widehat{s}_\alpha$$

which means that $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$ and $\widehat{\xi}_\beta(\widehat{s}_\beta)$ are dual similar curves in ID^3 with variable transformation and proof is completed.

Theorem 4. Let $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$ and $\widehat{\xi}_\beta(\widehat{s}_\beta)$ be dual curves in ID^3 . Then $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$ and $\widehat{\xi}_\beta(\widehat{s}_\beta)$ dual similar curves in ID^3 with variable transformation if and only if the binormal vectors are the same for all curves

$$(3.11) \quad \overrightarrow{\widehat{b}}_\beta(\widehat{s}_\beta) = \overrightarrow{\widehat{b}}_\alpha(\widehat{s}_\alpha),$$

under arbitrary variable transformation $\widehat{s}_\beta = \widehat{s}_\beta(\widehat{s}_\alpha)$ of the arc-lengths.

Proof. Let $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$ and $\widehat{\xi}_\beta(\widehat{s}_\beta)$ be dual curves in ID^3 with variable transformation. Then there are a variable transformation of arc-length such that the dual tangent vectors and the dual principal normal vectors are the same. From equations (3.2) and (3.8), we obtain

$$(3.12) \quad \widehat{b}_\beta(\widehat{s}_\beta) = \widehat{t}_\beta(\widehat{s}_\beta) \times \widehat{n}_\beta(\widehat{s}_\beta) = \widehat{t}_\alpha(\widehat{s}_\alpha) \times \widehat{n}_\alpha(\widehat{s}_\alpha) = \widehat{b}_\alpha(\widehat{s}_\alpha).$$

□

Conversly, Let $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$, $\widehat{\xi}_\beta(\widehat{s}_\beta)$ be dual curves in ID^3 which the same dual binormal vector under the arbitrary variable transformation $\widehat{s}_\beta = \widehat{s}_\beta(\widehat{s}_\alpha)$ of the arc-lengths. Differentiating the equation (3.11) with respect to $\widehat{s}_\beta$ we obtain

$$(3.13) \quad -\widehat{\tau}_\beta \overrightarrow{\widehat{n}}_\beta(\widehat{s}_\beta) = -\widehat{\tau}_\alpha \overrightarrow{\widehat{n}}_\alpha(\widehat{s}_\alpha) \frac{d \widehat{s}_\alpha}{d \widehat{s}_\beta}$$

from equation (3.13) we obtain

$$\begin{aligned} \overrightarrow{\widehat{n}}_\alpha(\widehat{s}) &= \overrightarrow{\widehat{n}}_\beta(\widehat{s}_\beta), \\ \widehat{b}_\beta(\widehat{s}_\beta) &= \widehat{t}_\beta(\widehat{s}_\beta) \times \widehat{n}_\beta(\widehat{s}_\beta) = \widehat{t}_\alpha(\widehat{s}_\alpha) \times \widehat{n}_\alpha(\widehat{s}_\alpha) = \widehat{b}_\alpha(\widehat{s}_\alpha). \end{aligned}$$

Theorem 5. Let $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$, $\widehat{\xi}_\beta(\widehat{s}_\beta)$ be dual curves in ID^3 . Then $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$, $\widehat{\xi}_\beta(\widehat{s}_\beta)$ are dual similar curves with variable transformation if and only if ratios of curvature S are the same i.e.,

$$(3.14) \quad \frac{\widehat{\tau}_\beta(\widehat{s}_\beta)}{\widehat{\kappa}_\beta(\widehat{s}_\beta)} = \frac{\widehat{\tau}_\alpha(\widehat{s}_\alpha)}{\widehat{\kappa}_\alpha(\widehat{s}_\alpha)},$$

under the particular variable transformations $\left(\widehat{\lambda}_\beta^\beta = \frac{d \widehat{s}_\beta}{d \widehat{s}_\alpha} = \frac{\widehat{\kappa}_\alpha(\widehat{s}_\alpha)}{\widehat{\kappa}_\beta(\widehat{s}_\beta)} \right)$ keeping equal total curvatures, i.e.,

$$(3.15) \quad \widehat{\varphi}_\beta(\widehat{s}_\beta) = \int \widehat{\kappa}_\beta(\widehat{s}_\beta) d \widehat{s}_\beta = \int \widehat{\kappa}_\alpha(\widehat{s}_\alpha) d \widehat{s}_\alpha = \widehat{\varphi}_\alpha(\widehat{s}_\alpha) \text{ of the arc-lengths.}$$

Proof. Let $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$ and $\widehat{\xi}_\beta(\widehat{s}_\beta)$ be dual curves in ID^3 . Then there exists a variable transformation of the arc-lengths such that the tangent and the binormal vectors are the same (definition.1 and theorem.4). Differentiating the equations (3.2) and (3.11), we have

$$\begin{aligned} \widehat{\kappa}_\beta(\widehat{s}_\beta) \overrightarrow{\widehat{n}}_\beta(\widehat{s}_\beta) &= \widehat{\kappa}_\alpha(\widehat{s}_\alpha) \overrightarrow{\widehat{n}}_\alpha(\widehat{s}) \frac{d \widehat{s}_\alpha}{d \widehat{s}_\beta} \\ -\widehat{\tau}_\beta(\widehat{s}_\beta) \overrightarrow{\widehat{n}}_\beta(\widehat{s}_\beta) &= -\widehat{\tau}_\alpha(\widehat{s}_\alpha) \overrightarrow{\widehat{n}}_\alpha(\widehat{s}) \frac{d \widehat{s}_\alpha}{d \widehat{s}_\beta} \end{aligned}$$

which leads to the following two equations

$$(3.16) \quad \widehat{\kappa}_\beta(\widehat{s}_\beta) = \widehat{\kappa}_\alpha(\widehat{s}_\alpha) \frac{d\widehat{s}_\alpha}{d\widehat{s}_\beta}$$

$$(3.17) \quad \widehat{\tau}_\beta(\widehat{s}_\beta) = \widehat{\tau}_\alpha(\widehat{s}_\alpha) \frac{d\widehat{s}_\alpha}{d\widehat{s}_\beta}.$$

□

The variable transformation (3.14) is the equation (3.16) after integration. Dividing the above two equations (3.16) and (3.17), we obtain the equation (3.14) under the variable transformations (3.15).

Conversely, Let $\widehat{\xi}_\alpha(\widehat{s}_\alpha)$ and $\widehat{\xi}_\beta(\widehat{s}_\beta)$ be dual curves, such that the equation (3.14) is satisfied under the variable transformation (3.15) of the arclengths. Let now consider *Theorem.2*, the dual tangent vectors $\vec{t}_\alpha(\widehat{s}_\alpha)$ and $\vec{t}_\beta(\widehat{s}_\beta)$ of the curves satisfy the following vector differential equations of third order as follows:

$$(3.18) \quad \left\{ \begin{aligned} &= \left[\frac{1}{\gamma_\alpha(\widehat{\varphi}_\alpha(\widehat{s}_\alpha))} \left[\vec{t}_\alpha(\widehat{\varphi}_\alpha(\widehat{s}_\alpha)) \right]'' \right]' + \left[\frac{1+(\gamma_\alpha(\widehat{\varphi}_\alpha(\widehat{s}_\alpha)))^2}{\gamma_\alpha(\widehat{\varphi}_\alpha(\widehat{s}_\alpha))} \right] (\vec{t}_\alpha(\widehat{\varphi}_\alpha(\widehat{s}_\alpha)))' \\ &\quad + \frac{(\gamma_\alpha(\widehat{\varphi}_\alpha(\widehat{s}_\alpha)))'}{\gamma_\alpha(\widehat{\varphi}_\alpha(\widehat{s}_\alpha))} \vec{t}_\alpha(\widehat{\varphi}_\alpha(\widehat{s}_\alpha)) = 0. \end{aligned} \right\}$$

$$(3.19) \quad \left\{ \begin{aligned} &= \left[\frac{1}{\gamma_\beta(\widehat{\varphi}_\beta(\widehat{s}_\beta))} \left[\vec{t}_\beta(\widehat{\varphi}_\beta(\widehat{s}_\beta)) \right]'' \right]' + \left[\frac{1+(\gamma_\beta(\widehat{\varphi}_\beta(\widehat{s}_\beta)))^2}{\gamma_\beta(\widehat{\varphi}_\beta(\widehat{s}_\beta))} \right] (\vec{t}_\beta(\widehat{\varphi}_\beta(\widehat{s}_\beta)))' \\ &\quad + \frac{(\gamma_\beta(\widehat{\varphi}_\beta(\widehat{s}_\beta)))'}{\gamma_\beta(\widehat{\varphi}_\beta(\widehat{s}_\beta))} \vec{t}_\beta(\widehat{\varphi}_\beta(\widehat{s}_\beta)) = 0. \end{aligned} \right\}$$

where

$$\gamma_\alpha(\widehat{\varphi}_\alpha(\widehat{s}_\alpha)) = \frac{\widehat{\tau}_\alpha(\widehat{\varphi}_\alpha(\widehat{s}_\alpha))}{\widehat{\kappa}_\alpha(\widehat{\varphi}_\alpha(\widehat{s}_\alpha))}, \quad \gamma_\beta(\widehat{\varphi}_\beta(\widehat{s}_\beta)) = \frac{\widehat{\tau}_\beta(\widehat{\varphi}_\beta(\widehat{s}_\beta))}{\widehat{\kappa}_\beta(\widehat{\varphi}_\beta(\widehat{s}_\beta))},$$

$$\widehat{\varphi}_\alpha = \int \widehat{\kappa}_\alpha(\widehat{s}_\alpha) \text{ and } \widehat{\varphi}_\beta = \int \widehat{\kappa}_\beta(\widehat{s}_\beta).$$

The equation (3.14) causes

$$\gamma_\alpha(\widehat{\varphi}_\alpha(\widehat{s}_\alpha)) = \gamma_\beta(\widehat{\varphi}_\beta(\widehat{s}_\beta))$$

under the variable transformations $\widehat{\varphi}_\alpha = \widehat{\varphi}_\beta$. So that the two equations (3.18) and (3.19) under the equation (3.14) and the transformation (3.15) are the same. Hence the solution is the same. **The proof of the theorem is completed.**

Example 1. Euler Spirals, were discovered **independently** by three researchers. In 1694, Bernoulli wrote the equations for the Euler spiral for the first time, but did not draw the spirals or compute them numerically. In 1774, Euler re-described their properties, and derived a series expansion to the curve's integrals. Later, In 1871, he also computed the spiral's end points. The curves were re-discovered 1891 for the third time by Talbot, who used them to design railway tracks. An Euler spiral is curve whose curvature changes linearly with its curve length (the curvature of a circular curve is equal to the reciprocal of the radius). Euler spirals are also commonly referred to as spiro, clothoids or Cornu spirals. Moreover, Euler spiral in railroad/highway engineering for connecting and transiting the geometry between a tangent and a circular curve [16].

Let us consider the Euler Spiral $\vec{\gamma}(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ of D^3

$$\begin{cases} \gamma_1(\hat{s}) = \frac{3}{5} \int \sin(\hat{s}^2 + 1) d\hat{s} \\ \gamma_2(\hat{s}) = \frac{3}{5} \int \cos(\hat{s}^2 + 1) d\hat{s} \\ \gamma_3(\hat{s}) = \frac{4}{5} \hat{s} \end{cases}$$

Then the dual unit tangent vector $\vec{t}$ of $\vec{\gamma}(s)$ satisfies a vector differential equation of third order given by $\left[\frac{1}{\gamma(\hat{\theta})} \left[\vec{t}(\hat{\theta}) \right]'' \right]' + \left[\frac{1+\gamma^2(\hat{\theta})}{\gamma(\hat{\theta})} \right] (\vec{t})' + \frac{\gamma'(\hat{\theta})}{\gamma(\hat{\theta})} \vec{t}(\hat{\theta}) = 0$. We are calculated this curve's curvature **function** with help of Mathematica Programme $\hat{\kappa} = \frac{6\hat{s}}{5}$ and $\hat{\tau} = -\frac{8\hat{s}}{5}$. The Frenet-Serret frame of the curve $\vec{\gamma} = \vec{\gamma}(s)$ may be written by the aid Mathematica Programme as follows:

$$\begin{cases} \vec{t}(\hat{s}) = (\frac{3}{5} \sin(\hat{s}^2 + 1), \frac{3}{5} \cos(\hat{s}^2 + 1), \frac{4}{5}), \\ \vec{n}(\hat{s}) = (\cos(\hat{s}^2 + 1), -\sin(\hat{s}^2 + 1), 0), \\ \vec{b}(\hat{s}) = (\frac{4}{5} \sin(\hat{s}^2 + 1), \cos(\hat{s}^2 + 1), -\frac{3}{5}) \end{cases}.$$

Lemma 1. The family of Dual Euler Spiral forms a family of dual similar curves with variable transformations. We can deduce the position vector of Euler Spiral curve using the definition of dual similar curves with variable transformation as follows:

$$\begin{aligned} \vec{\gamma}_\alpha(\hat{u}) &= (\gamma_1(\hat{u}), \gamma_2(\hat{u}), \gamma_3(\hat{u})) \\ &= (\frac{3}{5} \int \sin(\hat{u}^2 + 1) d\hat{u}, \frac{3}{5} \int \cos(\hat{u}^2 + 1) d\hat{u}, \frac{4}{5} \hat{u}) \end{aligned}$$

where $\hat{s} = \hat{u}$ is the arc-length of Euler Spiral and the curvature is $\hat{\kappa}(\hat{u}) = \frac{6\hat{u}}{5}$. The dual unit tangent vectors of Euler Spiral takes the form:

$$\vec{t}(\hat{u}) = (\frac{3}{5} \sin(\hat{u}^2 + 1), \frac{3}{5} \cos(\hat{u}^2 + 1), \frac{4}{5}),$$

from theorem.1, we can write as the following,

$$\vec{\gamma}_\beta(\hat{u}) = \int \left[\frac{3}{5} \sin[(\hat{u}(\hat{s}_\beta))^2 + 1], \frac{3}{5} \cos[(\hat{u}(\hat{s}_\beta))^2 + 1], \frac{4}{5} \right] d\hat{s}_\beta$$

where $\hat{s}_\beta = \hat{s}$ and we obtain

$$d\hat{s}_\alpha = \widehat{\lambda}_\beta^\alpha d\hat{s}_\beta = \frac{\widehat{\kappa}_\beta}{\widehat{\kappa}_\alpha} d\hat{s}_\beta$$

or

$$\widehat{s}_\alpha(\hat{s}_\beta) = \int \frac{\widehat{\kappa}_\alpha}{\widehat{\kappa}_\beta} d\hat{s}_\beta.$$

If we put the curvature $\widehat{\kappa}_\beta = \widehat{\kappa}_\alpha(\hat{s})$, ($\hat{s}_\beta = \hat{s}$), we have $\widehat{u}(\hat{s}_\beta) = \int \widehat{\kappa}(\hat{s}) d\hat{s}$. Then the position vector of dual Euler Spiral with arbitrary curvature $\widehat{\kappa}(\hat{s})$ takes the following form:

$$\vec{x}(\hat{u}) = \int \left(\frac{3}{5} \sin\left(\int \widehat{\kappa}(\hat{s}) d\hat{s} + 1\right), \frac{3}{5} \cos\left(\int \widehat{\kappa}(\hat{s}) d\hat{s} + 1\right), \frac{4}{5} \right) d\hat{s}.$$

2000 Mathematics Subject Classification. 53A04, 53A17, 53A25, 53A40

Key words and phrases: Dual Quaternionic frame, Similar curve, Dual space, Vector differential equation.

REFERENCES

- [1] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press Inc., London, 1983.
- [2] Izumiya, S., Takeuchi, N., New special curves and developable surfaces, Turk. J. Math. Vol. 28 (2004), 153-163.
- [3] Salkowski, E., Zur Transformation von Raumkurven, Mathematische Annalen 66(4), (1909), 517-557.
- [4] Monterde, J., Salkowski curves revisited: A family of curve with constant curvature and non-constant torsion, Comput. Aided Geo. Design, (2009), 26-271-278.
- [5] B. Saint Venant, Memoire sur les lignes courbes non planes, Journal de l'Ecole Polytechnique, (1845), 18, 1-76.
- [6] J. M. Bertrand, Memoire sur la theorie des courbes a double courbure, comptes Rendus, vol. 36, 1850.
- [7] Do Carmo, M.P., Differential Geometry of Curves and Surface, Pearson Education, 1976.
- [8] Hacisalihpoglu, H. H., On the pitch of a closed ruled surface, Mech. Mach. Theory, 7, no. 3, (1972), 291-305.
- [9] Kuhnel, W., Differential geometry: curves-surfaces-manifolds. Braunschweig, Wiesbaden, 1999.
- [10] Ch. Bioche, Sur les courbes de M. Bertrand, Bull. Soc. Math. France, (1899), 17, 109-112.
- [11] John F. Burke, Bertrand Curves Associated with a Pair of Curves, Mathematics Magazine, Vol. 34, No. 1, (1960), 60-62.
- [12] A.T. Ali, Position vectors of general helices in Euclidean 3-space, Bull. Math. Anal. Appl. 3(2), (2010), 198-205.
- [13] Guggenheimer, W., Differential Geometry, McGraw-Hill, New York, 1963.
- [14] Hamilton, W.R., Element of Quaternions, I, II and III, Chelsea, New YORK, 1899.
- [15] Köse, Ö., Nizamoğlu, Ş., and Sezer, M., An explicit characterization of dual spherical curves, Doğa Mat. 12, no. 3, (1998), 105-113.
- [16] G. Harary, A. Tal, 3D Euler Spirals for 3D Curve Completion, Symposium on Computational Geometry, (2010), 107-108.
- [17] Veldkamp, G. R., On the use of dual numbers, vectors and matrices in instantaneous, spatial kinematics, Mech. Mach. Theory, 11 (1976), no. 2, 141-156.

KIRIKKALE UNIVERSITY, ART & SCIENCE FACULTY (TURKEY)

E-mail address: faik.babadag@kku.edu.tr