

A NEW ITERATIVE ALGORITHM FOR ZEROS OF GENERALIZED PHI-STRONGLY MONOTONE AND BOUNDED MAPS WITH APPLICATION

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ABSTRACT. Let E be a uniformly smooth and uniformly convex real Banach space and $A : E \rightarrow E^*$ be a generalized Φ -strongly monotone and bounded map with $A^{-1}(0) \neq \emptyset$. A new iterative process is constructed and proved to converge strongly to the unique solution of the equation $Au = 0$. An application to convex minimization problem is given. Furthermore, the technique of proof is of independent interest.

1. INTRODUCTION

Let H be a real inner product space. A mapping $A : D(A) \subset H \rightarrow H$ is called *monotone* if for each $x, y \in D(A)$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq 0.$$

The mapping A is called *generalized Φ -strongly monotone* if there exists a strictly increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \Phi(\|x - y\|) \quad \forall x, y \in D(A).$$

The mapping A is called *ϕ -strongly monotone* if there exists a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \phi(\|x - y\|)\|x - y\| \quad \forall x, y \in D(A).$$

Monotone mappings were studied in Hilbert spaces by Zarantonello [47], Minty [29], Kačurovskii [21] and a host of other authors. Interest in such mappings stems mainly from their importance in numerous applications. Consider, for example, the following: Let $f : H \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper convex function. The *subdifferential* of f at $x \in H$ is defined by

$$\partial f(x) = \{x^* \in H : f(y) - f(x) \geq \langle y - x, x^* \rangle \quad \forall y \in H\}.$$

Clearly, $\partial f : H \rightarrow 2^H$ is a *monotone operator* on H , and $0 \in \partial f(u_0)$ if and only if u_0 is a *minimizer* of f . Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of f .

In general, the following problem is of interest and has been studied extensively by numerous authors.

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- Let H be a real Hilbert space. Find $u \in H$ such that

$$(1.1) \quad 0 \in Au,$$

where $A : H \rightarrow 2^H$ is a monotone-type operator.

Several existence theorems have been proved for the equation $Au = 0$, where A is of the monotone-type (see e.g., Deimling [19], Pascali and Sburian [30] and the references contained therein):

The extension of the monotonicity definition to operators from a Banach space into its dual has been the starting point for the development of non-linear functional analysis ... The monotone maps constitute the most manageable class, because of the very simple structure of the monotonicity condition. The monotone mappings appear in a rather wide variety of contexts, since they can be found in many functional equations. Many of them appear also in calculus of variations, as sub-differential of convex functions (Pascali and Sburian [30], p. 101).

Let E be a real normed space, E^* its topological dual space. A map $J : E \rightarrow 2^{E^*}$ defined by

$$Jx := \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \cdot \|x^*\|, \|x\| = \|x^*\|\}$$

is called the *normalized duality map* on E .

A map $A : E \rightarrow E^*$ is called *monotone* if for each $x, y \in E$, the following inequality holds:

$$(1.2) \quad \langle Ax - Ay, x - y \rangle \geq 0.$$

A is called *generalized Φ -strongly monotone* if there exists a strictly increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \Phi(\|x - y\|) \quad \forall x, y \in D(A).$$

A map $A : E \rightarrow E$ is called *accretive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$(1.3) \quad \langle Ax - Ay, j(x - y) \rangle \geq 0.$$

A is called *generalized Φ - strongly accretive* if there exists a strictly increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ such that for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \Phi(\|x - y\|).$$

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, *monotonicity* and *accretivity* coincide.

In general, the following problem has been studied extensively by numerous authors:

- Let E be a real Banach space. Find $u \in E$ such that

$$(1.4) \quad Au = 0,$$

where $A : E \rightarrow E$ is an *accretive-type operator*.

Solutions to equation (1.4), in many cases, may correspond to the equilibrium states of some dynamical systems (see e.g., Browder [5], Chidume [7], p. 116).

For approximating a solution of $Au = 0$, assuming existence, where $A : E \rightarrow E$ is of *accretive-type*, Browder [5] defined an operator $T : E \rightarrow E$ by $T := I - A$, where I is the identity map on E . He called such an operator *pseudo-contractive*. It is trivial to observe that zeros of A correspond to fixed points of T . For strongly pseudo-contractive maps, Chidume [11] proved the following theorem.

Theorem C1. Let $E = L_p$, $2 \leq p < \infty$, and $K \subset E$ be non-empty closed convex and bounded. Let $T : K \rightarrow K$ be a strongly pseudo-contractive and Lipschitz map. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n$, $n \geq 0$, where $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique fixed point of T .

This theorem signalled the return to extensive research efforts on inequalities in Banach spaces and their applications to iterative methods for solutions of non-linear equations. Consequently, Theorem C1 has been generalized and extended in various directions, leading to flourishing areas of research, for the past thirty years or so, for numerous authors (see e.g., Censor and Riech [6], Chidume [9, 10], Chidume and Ali [8], Chidume and Chidume [12, 13], Chidume and Osilike [14], Deng [18], Moudafi [25, 26, 27, 28], Zhou and Jia [48], Liu [23], Qihou [31], Berinde *et al.* [4], Reich [33, 34, 35], Reich and Sabach [36, 37], Weng [39], Xiao [41], Xu [42, 45, 46], Xu and Roach [43], Xu [44], Zhu [49] and a host of other authors). Recent monographs emanating from these researches include those by Berinde [3], Chidume [7], Goebel and Reich [20], and William and Shahzad [40].

By replacing T by $I - A$ in Theorem C1, the following theorem for approximating the unique solution of $Au = 0$ when $A : E \rightarrow E$ is a strongly accretive and Lipschitz map is easily proved.

Theorem C2. Let $E = L_p$, $2 \leq p < \infty$. Let $A : E \rightarrow E$ be a strongly accretive and Lipschitz map. For arbitrary $x_1 \in K$, let a sequence $\{x_n\}$ be defined iteratively by $x_{n+1} = x_n - \alpha_n A x_n$, $n \geq 1$, where $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique solution of $Au = 0$.

The most general convergence theorem for approximating the solution of $Au = 0$, assuming existence where $A : E \rightarrow E$ is generalized Φ -strongly accretive seems to be the following.

Theorem 1.1 (see e.g., Chidume [7], p. 123). *Let E be a real normed linear space. Suppose $A : E \rightarrow E$ is a generalized Φ -quasi-accretive, uniformly continuous and bounded map. For arbitrary $x_1 \in E$, define the sequence $\{x_n\}$ iteratively by*

$$x_{n+1} := x_n - \alpha_n A x_n, \quad n \geq 1,$$

where $\lim \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n \leq d_0$, $\{x_n\}$ converges strongly to the unique solution of the equation $Ax = 0$.

For approximating a solution of inclusion (1.1) in real Banach spaces more general than real Hilbert spaces where $A : E \rightarrow E$ is of accretive type, geometric properties of Banach spaces developed from the mid 1980s to early 1990s played a crucial role. Unfortunately, these geometric properties seem not to be directly applicable to iterative methods for approximating zeros of A when $A : E \rightarrow E^*$ is of the monotone-type. Fortunately, new geometric properties of Banach spaces recently introduced by Alber and studied by Alber [1], are appropriate for approximating zeros of monotone-type mappings.

In this paper, we introduce an iterative algorithm of the Mann-type [24], and combining the new geometric properties of Banach spaces recently introduced by Alber with our technique, we prove the strong convergence of the algorithm to a zero of a generalized Φ -strongly monotone and bounded map in uniformly convex and uniformly smooth real Banach spaces. Our theorem which is an analogue of theorem 1.1 for monotone type mappings is also an extension of the theorems of Chidume *et al.*, [15], from L_p spaces, $1 < p < \infty$ to the more general class of uniformly convex and uniformly smooth real Banach spaces.

2. PRELIMINARIES

Definition 2.1. A continuous, strictly increasing function $\omega : (0, \infty) \rightarrow (0, \infty)$ is called **modulus of continuity** if $\omega(t) \rightarrow 0$ as $t \rightarrow 0$. It follows that a function is uniformly continuous if and only if it has a modulus of continuity.

In the sequel, we shall need the following definitions and results. Let E be a smooth real Banach space with dual E^* . The function $\phi : E \times E \rightarrow \mathbb{R}$, defined by,

$$(2.1) \quad \phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \text{ for } x, y \in E,$$

where J is the normalized duality mapping from E into 2^{E^*} will play a central role in the sequel. It was introduced by Alber and has been studied by Alber [1], Alber and Guerre-Delabriere [2], Kamimura and Takahashi [22], Reich [32] and a host of other authors. If $E = H$, a real Hilbert space, then equation (2.1) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(2.2) \quad (\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \text{ for } x, y \in E.$$

Define a map $V : X \times X^* \rightarrow \mathbb{R}$ by

$$(2.3) \quad V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2.$$

Then, it is easy to see that

$$(2.4) \quad V(x, x^*) = \phi(x, J^{-1}(x^*)) \quad \forall x \in X, x^* \in X^*.$$

Lemma 2.2 (Alber, [1]). *Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Then,*

$$(2.5) \quad V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*)$$

for all $x \in X$ and $x^*, y^* \in X^*$.

Remark 1 (e.g., see [1], p. 48). If $X = L_p$, $p \geq 2$, then, the normalized duality map $J : L_p \rightarrow L_p^*$ is Lipschitz. i.e., there exists $L > 0$ such that $\|Jx - Jy\| \leq L\|x - y\| \quad \forall x, y \in L_p$. Also, if $X = L_p$, $1 < p \leq 2$, then, $J : L_p \rightarrow L_p^*$ is Hölder continuous. i.e., there exists $\beta \in (0, 1]$ such that $\|Jx - Jy\| \leq H\|x - y\|^\beta \quad \forall x, y \in L_p$, for some Hölder constant $H > 0$.

Lemma 2.3 (Kamimura and Takahashi, [22]). *Let X be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X . If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 2.4 (Tan and Xu, [38]). *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$(2.6) \quad a_{n+1} \leq a_n + \sigma_n, \quad n \geq 0,$$

such that $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then, $\lim_{n \rightarrow \infty} a_n$ exists. If, in addition, the sequence $\{a_n\}$ has a subsequence that converges to 0, then the sequence $\{a_n\}$ converges to 0.

Lemma 2.5 (Chidume, [16]). *Let E be uniformly convex real Banach space. For arbitrary $d > 0$, let $B_d(0) := \{x \in E : \|x\| \leq d\}$. Then, for arbitrary $x, y \in B_d(0)$, the following inequality holds:*

$$(2.7) \quad \phi(x, y) \leq \|x - y\|^2 + \|x\|^2.$$

Since this lemma is new, we give a proof for completeness.

Proof: Since E is uniformly convex, the following inequality holds for arbitrary $p > 1$, $x, y \in B_d(0)$, (see e.g., Chidume, p. 43, inequality (4.31)):

$$(2.8) \quad \|x + y\|^p \geq \|x\|^p + p\langle y, J_p(x) \rangle + g(\|y\|),$$

where $g : [0, \infty) \rightarrow [0, \infty)$ is a continuous strictly increasing and convex function. In particular, we have

$$(2.9) \quad \|x + y\|^2 \geq \|x\|^2 + 2\langle y, J(x) \rangle + g(\|y\|)$$

Replace y by $(-y)$ to get:

$$(2.10) \quad \|x - y\|^2 \geq \|x\|^2 - 2\langle y, J(x) \rangle + g(\|y\|).$$

Interchanging x and y in this inequality yields:

$$\begin{aligned} \|x - y\|^2 &\geq \|y\|^2 - 2\langle x, J(y) \rangle + g(\|x\|) \\ &= \|x\|^2 - 2\langle x, J(y) \rangle + \|y\|^2 - \|x\|^2 + g(\|x\|) \\ &\geq \|x\|^2 - 2\langle x, J(y) \rangle + \|y\|^2 - \|x\|^2 \end{aligned}$$

so that

$$(2.11) \quad \phi(x, y) \leq \|x - y\|^2 + \|x\|^2,$$

establishing the lemma.

3. MAIN RESULTS

In theorem 3.1 below, the sequence $\{\lambda_n\}_{n=1}^\infty \subset (0, 1)$ satisfies the following conditions:

$$(i) \sum_{n=1}^\infty \lambda_n = \infty; (ii) \sum_{n=1}^\infty 2\lambda_n \omega(\lambda_n M) M < \infty; (iii) \omega(\lambda_n M) \leq \gamma_0,$$

where $\omega : (0, \infty) \rightarrow (0, \infty)$ is the modulus of continuity of J^{-1} on the bounded subsets of E^* and

$$M := \sup\{\|Au\| : \|u\| \leq \|u^*\| + \sqrt{r}\}$$

for some $r > 0$, $u^* \in A^{-1}(0)$.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $A : E \rightarrow E^*$ be a generalized Φ -strongly monotone and bounded map with $A^{-1}(0) \neq \emptyset$. For arbitrary $u_1 \in E$, define a sequence $\{u_n\}$ iteratively by:*

$$u_{n+1} = J^{-1}(Ju_n - \lambda_n Au_n), \quad n \geq 1.$$

Then, the sequence $\{u_n\}_{n=1}^\infty$ converges strongly to u^* , a solution of $Au = 0$.

Proof. The proof is in two steps:

Step 1: We prove that $\{u_n\}_{n=1}^\infty$ is bounded. Since $A^{-1}(0) \neq \emptyset$, let $u^* \in A^{-1}(0)$. Let $\delta > 0$ be arbitrary but fixed. Then, there exists $r > 0$ such that

$$r \geq \max\{\phi(u^*, u_1), 4\delta^2 + \|u^*\|^2\}.$$

We show that $\phi(u^*, u_n) \leq r \quad \forall n \geq 1$. This proof is by induction.

By construction, $\phi(u^*, u_1) \leq r$. Assume $\phi(u^*, u_n) \leq r$ for some $n \geq 1$. This implies, from (2.2) that

$$\|u_n\| \leq \sqrt{r} + \|u^*\|.$$

We now show that $\phi(u^*, u_{n+1}) \leq r$. Suppose for contradiction, i.e., $\phi(u^*, u_{n+1}) > r$. Define

$$\begin{aligned} M &:= \sup\{\|Au\| : \|u\| \leq \|u^*\| + \sqrt{r}\}, \\ \gamma_o &:= \min\{\Phi(\delta), \delta\}. \end{aligned}$$

Take $y^* = \lambda_n Au_n$ and using inequality (2.5), we expand as follows:

$$\begin{aligned} r < \phi(u^*, u_{n+1}) &= \phi(u^*, J^{-1}(Ju_n - \lambda_n Au_n)) = V(u^*, Ju_n - \lambda_n Au_n) \\ &\leq V(u^*, Ju_n) - 2\langle J^{-1}(Ju_n - \lambda_n Au_n) - u^*, \lambda_n Au_n \rangle \\ &= V(u^*, Ju_n) - 2\lambda_n \langle u_n - u^*, Au_n \rangle \\ &\quad + 2\lambda_n \langle J^{-1}(Ju_n) - J^{-1}(Ju_n - \lambda_n Au_n), Au_n \rangle. \\ &\leq \phi(u^*, u_n) - 2\lambda_n \langle u_n - u^*, Au_n - Au^* \rangle \\ &\quad + 2\lambda_n \|J^{-1}(Ju_n) - J^{-1}(Ju_n - \lambda_n Au_n)\| \|Au_n\|. \end{aligned}$$

Using the fact that A is generalized Φ -strongly monotone and that J^{-1} is uniformly continuous on bounded sets we obtain:

$$(3.1) \quad \phi(u^*, u_{n+1}) \leq \phi(u^*, u_n) - 2\lambda_n \Phi(\|u_n - u^*\|) + 2\lambda_n \omega(\lambda_n M) M.$$

But from the recursion formula, we have

$$\|Ju_{n+1} - Ju_n\| = \lambda_n \|Au_n\| \leq \lambda_n M.$$

Observe that by uniform continuity of J on bounded sets, Ju_n is bounded. So, from the equation

$$Ju_{n+1} = Ju_n - \lambda_n Au_n,$$

we have that $\|Ju_{n+1}\| \leq \|Ju_n\| + \lambda_n M$, which implies that Ju_{n+1} is bounded. So, by the uniform continuity of J^{-1} on bounded subsets of E^* , we have

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|J^{-1}(Ju_{n+1}) - J^{-1}(Ju_n)\| \\ &\leq \omega(\|Ju_{n+1} - Ju_n\|) \leq \omega(\lambda_n \|Au_n\|) \leq \omega(\lambda_n M). \end{aligned}$$

So,

$$\|u_{n+1} - u^*\| - \|u_n - u^*\| \leq \|u_{n+1} - u_n\| \leq \omega(\lambda_n M),$$

which yields

$$(3.2) \quad \|u_n - u^*\| \geq \|u_{n+1} - u^*\| - \omega(\lambda_n M).$$

From lemma 2.5, we have

$$(3.3) \quad r < \phi(u^*, u_{n+1}) \leq \|u_{n+1} - u^*\|^2 + \|u^*\|^2.$$

Using the choice of r , we obtain from (3.3) that

$$\|u_{n+1} - u^*\|^2 > r - \|u^*\|^2 \geq 4\delta^2 + \|u^*\|^2 - \|u^*\|^2.$$

Hence,

$$\|u_{n+1} - u^*\| \geq 2\delta.$$

From inequality (3.2), we have that

$$\|u_n - u^*\| \geq 2\delta - \omega(\lambda_n M) \geq 2\delta - \gamma_o \geq \delta.$$

Since Φ is strictly increasing, we have

$$(3.4) \quad \Phi(\|u_n - u^*\|) \geq \Phi(\delta).$$

From inequality (3.1), we have that

$$\begin{aligned} r < \phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) - 2\lambda_n \Phi(\delta) + 2\lambda_n \omega(\lambda_n M)M \\ &\leq r - 2\lambda_n \Phi(\delta) + \lambda_n \Phi(\delta) < r. \end{aligned}$$

This is a contradiction. Hence, the sequence $\{u_n\}$ is bounded.

Step 2: We show that the sequence $\{u_n\}$ converges strongly to u^* . From inequality (3.1) we have

$$\begin{aligned} \phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) - 2\lambda_n \Phi(\|u_n - u^*\|) + 2\lambda_n \omega(\lambda_n M)M \\ &\leq \phi(u^*, u_n) + 2\lambda_n \omega(\lambda_n M)M. \end{aligned}$$

By lemma 2.4 $\lim \phi(u^*, u_n)$ exists. Also, from the inequality above, we have that

$$\begin{aligned} \phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) - 2\lambda_n \Phi(\|u_n - u^*\|) + 2\lambda_n \omega(\lambda_n M)M \\ \implies 2\lambda_n \Phi(\|u_n - u^*\|) &\leq \phi(u^*, u_n) - \phi(u^*, u_{n+1}) + 2\lambda_n \omega(\lambda_n M)M. \end{aligned}$$

Claim: $\liminf \Phi(\|u_n - u^*\|) = 0$.

Suppose not. i.e., suppose $\liminf \Phi(\|u_n - u^*\|) := a > 0$. Then, there exists an integer $N_0 > 0$ such that for all integers $n \geq N_0$,

$$\Phi(\|u_n - u^*\|) > \frac{a}{2}.$$

Hence, using condition (ii) and summing the first two terms by telescoping, we have:

$$a \sum_{n=1}^{\infty} \lambda_n \leq \sum_{n=1}^{\infty} \left(\phi(u^*, u_n) - \phi(u^*, u_{n+1}) \right) + 2 \sum_{n=1}^{\infty} \lambda_n \omega(\lambda_n M) M < \infty,$$

contradicting the hypothesis that $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Hence,

$$\liminf \Phi(\|u_n - u^*\|) = 0.$$

So, there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\Phi(\|u_{n_k} - u^*\|) \rightarrow 0, k \rightarrow \infty.$$

From the property of Φ (i.e., Φ is strictly increasing and $\Phi(0)=0$), it follows that $\|u_{n_k} - u^*\| \rightarrow 0$ as $k \rightarrow \infty$. i.e., $u_{n_k} \rightarrow u^*$ as $k \rightarrow \infty$. Using the definition of ϕ and the continuity of J on bounded subsets of E , we have

$$\phi(u^*, u_{n_k}) = \|u^*\|^2 - 2\langle u^*, Ju_{n_k} \rangle + \|u_{n_k}\|^2 \rightarrow 0, k \rightarrow \infty.$$

Consequently, by lemma 2.4, $\phi(u^*, u_n) \rightarrow 0$ as $n \rightarrow \infty$. Thus by lemma 2.3, we obtain that $\lim \|u_n - u^*\| = 0$. This completes the proof. $\square$

EXAMPLE: Let X and Y be real normed spaces.

(a) If a map $S : X \rightarrow Y$ is Lipschitz, then the modulus of continuity of S is given by $\omega(t) = Lt$, where $L > 0$ is the Lipschitz constant of S .

(b) If $S : X \rightarrow Y$ is Hölder continuous, i.e., $\forall x, y \in X$

$$\|Sx - Sy\| \leq H\|x - y\|^\beta,$$

where $0 < \beta \leq 1$ and H is the Hölder constant, the modulus of continuity of S is given by $\omega(t) = Ht^\beta$.

It is known that in L_p spaces, $2 \leq p < \infty$, $J : L_p \rightarrow L_p^*$ is Lipschitz (see remark 1). So, in this case, the modulus of continuity of J is given by $\omega(t) = Lt$. It is also known that in L_p spaces, $1 < p < 2$, $J : L_p \rightarrow L_p^*$ is Hölder continuous. In this case, $\omega(t) = Ht^{p-1}$ (here $\beta = p - 1$).

Hence, if one choose $\lambda_n = \frac{1}{n}$ $\forall n \geq 1$ and $E = L_p$ ($1 < p < \infty$), all the conditions on our iteration parameter $\{\lambda_n\}$ in theorem 3.1 are satisfied.

With this example in mind, we have the following corollaries of theorem 3.1 where $\omega : [0, \infty) \rightarrow [0, \infty)$ will represent the modulus of continuity of $J_* = J^{-1}$.

In corollary 3.2 below, the sequence $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$ satisfies the following conditions which are the analogues of the conditions on $\{\lambda_n\}_{n=1}^\infty$ in theorem 3.1: (i) $\sum_{n=1}^\infty \alpha_n = \infty$; (ii) $2LM^2 \sum_{n=1}^\infty \alpha_n^2 < \infty$; (iii) $LM\alpha_n \leq \gamma_0$, for some $\gamma_0 > 0$.

Corollary 3.2. *Let $E = L_p$, $1 < p < 2$. Let $A : E \rightarrow E^*$ be a generalized Φ -strongly monotone and bounded map with $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}$ iteratively by:*

$$(3.5) \quad x_{n+1} = J^{-1}(Jx_n - \alpha_n Ax_n), \quad n \geq 1.$$

Then, the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to a solution of the equation $Ax = 0$.

Proof. This follows from Theorem 3.1. □

In corollary 3.3 below, $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$ satisfies the following conditions which are the analogues of the conditions on $\{\lambda_n\}_{n=1}^\infty$ in theorem 3.1:

(i) $\sum_{n=1}^\infty \alpha_n = \infty$; (ii) $2HM^p \sum_{n=1}^\infty \alpha_n^p < \infty$; (iii) $HM^{p-1}\alpha_n^{p-1} \leq \gamma_0$, for some $\gamma_0 > 0$.

Corollary 3.3. *Let $E = L_p$, $2 \leq p < \infty$. Let $A : E \rightarrow E^*$ be a generalized Φ -strongly monotone and bounded map with $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}$ iteratively by:*

$$(3.6) \quad x_{n+1} = J^{-1}(Jx_n - \alpha_n Ax_n), \quad n \geq 1.$$

Then, the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to a solution of the equation $Ax = 0$.

Proof. This follows from Theorem 3.1. □

A prototype of the parameter in corollary 3.2 and corollary 3.3 is the canonical choice,

$$\alpha_n = \frac{1}{n}, \quad n \geq 1.$$

4. APPLICATION TO CONVEX OPTIMIZATION PROBLEMS

In this section, we apply our theorem in solving the problem of finding a minimizer of a convex function f defined from a real Banach space E to $\mathbb{R}$.

The following results are well known.

Lemma 4.1. *(see e.g., Diop et al., [50]) Let E be a real Banach space and $f : E \rightarrow \mathbb{R}$ be a differentiable convex function. Let $df : E \rightarrow E^*$ denotes the differential map associated to f . Then, $x \in E$ is a minimizer of f if and only if $df(x) = 0$.*

Lemma 4.2. *Let E be a real normed space and $f : E \rightarrow \mathbb{R}$ be a convex function. Suppose f is bounded on bounded subsets of E . Then, for every $x_0 \in E$ and $r > 0$, there exists $\gamma > 0$ such that f is γ -Lipschitzian on $B(x_0, r)$. i.e.,*

$$|f(x) - f(y)| \leq \gamma \|x - y\| \quad \forall x, y \in B(x_0, r).$$

Lemma 4.3. *Let E be a real normed space and $f : E \rightarrow \mathbb{R}$ be a differentiable convex function. Assume that f is bounded, then, the differential map $df : E \rightarrow E^*$ is bounded.*

Proof. Let $x_0 \in E$ and $r > 0$. Set $B := B(x_0, r)$. We show that $df(B)$ is bounded. From lemma 4.2, there exists $\gamma > 0$ such that

$$|f(x) - f(y)| \leq \gamma \|x - y\| \quad \forall x, y \in B. \quad (*)$$

Let $z^* \in df(B)$ and $x^* \in B$ such that $z^* = df(x^*)$. Since B is open, for all $u \in E$, there exists $t > 0$ such that $x^* + tu \in B$. Using the fact that $z^* = df(x^*)$ and inequality (*), it follows that

$$\langle z^*, tu \rangle \leq f(x^* + tu) - f(x^*) \leq t\gamma \|u\|,$$

so that

$$\langle z^*, u \rangle \leq \gamma \|u\| \quad \forall u \in E.$$

Therefore, $\|z^*\| \leq \gamma$, which implies $df(B)$ is bounded. Hence, df is bounded. $\square$

Lemma 4.4. (see e.g., Chidume [7], p. 43) *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \leq r\}$. Then, there exists a continuous strictly increasing convex function*

$$\Phi : [0, \infty) \rightarrow [0, \infty), \Phi(0) = 0,$$

such that for every $x, y \in B_r(0)$, the following inequality is satisfied;

$$(4.1) \quad \langle x - y, Jx - Jy \rangle \geq \Phi(\|x - y\|),$$

where J is the single-valued normalized duality map.

Lemma 4.5. *Let E be a uniformly convex and uniformly smooth real Banach space and $f : E \rightarrow \mathbb{R}$ be a differentiable convex function. Then, the differential map $df : E \rightarrow E^*$ satisfies the following inequality:*

$$\langle df(x) - df(y), x - y \rangle \geq \langle Jx - Jy, x - y \rangle, \quad \forall x, y \in E.$$

Proof. Define $g := f - \frac{1}{2}\|\cdot\|^2 \implies f = g + \frac{1}{2}\|\cdot\|^2$.

Observe that since f and $\|\cdot\|^2$ are differentiable, then, g is differentiable and $df = dg + J$ which implies $dg = df - J$. Let $x \in E$. Then, by the definition of dg we have

$$(4.2) \quad \langle df(x) - Jx, y - x \rangle \leq f(y) - \frac{1}{2}\|y\|^2 - f(x) + \frac{1}{2}\|x\|^2 \quad \forall y \in E.$$

By swapping x and y , we obtain

$$(4.3) \quad \langle df(y) - Jy, x - y \rangle \leq f(x) - \frac{1}{2}\|x\|^2 - f(y) + \frac{1}{2}\|y\|^2.$$

Adding inequalities (4.2) and (4.3), we obtain

$$\langle df(x) - df(y), x - y \rangle \geq \langle x - y, Jx - Jy \rangle.$$

$\square$

Remark 2. If for any $R > 0$ and any $x, y \in E$ such that $\|x\| \leq R, \|y\| \leq R$, then the map $df : E \rightarrow E^*$ is generalized Φ -strongly monotone on $B := \{u \in E : \|u\| \leq R\}$. This can easily be seen from lemmas 4.4 and 4.5, i.e.,

$$\langle df(x) - df(y), x - y \rangle \geq \Phi(\|x - y\|) \quad \forall x, y \in B.$$

We now prove the following theorem.

Theorem 4.6. *Let E be a uniformly convex and uniformly smooth real Banach space and E^* be its dual. Let $f : E \rightarrow \mathbb{R}$ be a differentiable, convex, bounded and coercive function. For arbitrary $u_1 \in E$, let $\{u_n\}$ be the sequence defined iteratively by:*

$$u_{n+1} = J^{-1}(Ju_n - \lambda_n df(u_n)), \quad n \geq 1,$$

where J is the normalized duality mapping from E into E^* and $\{\lambda_n\} \subset (0, 1)$ is a sequence satisfying the following conditions:

$$(i) \sum_{n=1}^{\infty} \lambda_n = \infty; (ii) \sum_{n=1}^{\infty} \lambda_n \omega(\lambda_n M) < \infty; (iii) \omega(\lambda_n M) \leq \gamma_0,$$

for some $\gamma_0 > 0$. Then, f has a unique minimizer $u^* \in E$ and the sequence $\{u_n\}$ converges strongly to u^* .

Proof. Since f is lower semi-continuous, convex and coercive, then f has a minimizer $u^* \in E$. Using the same method of computation as in theorem 3.1, we obtain

$$\phi(u^*, u_{n+1}) \leq \phi(u^*, u_n) - 2\lambda_n \langle u_n - u^*, df(u_n) - df(u^*) \rangle + 2\lambda_n \omega(\lambda_n M)M. \quad (*)$$

By monotonicity of df , we have

$$\phi(u^*, u_{n+1}) \leq \phi(u^*, u_n) + 2\lambda_n \omega(\lambda_n M)M.$$

Using lemma 2.4 we have that limit of $\phi(u^*, u_n)$ exists. Thus, $\phi(u^*, u_n)$ is bounded and using inequality (2.2), $\{u_n\}$ is bounded. By lemma 4.3, df is bounded. Since $\{u_n\}$ and u^* are bounded, using remark 2 and $(*)$ we have that

$$\phi(u^*, u_{n+1}) \leq \phi(u^*, u_n) - 2\lambda_n \Phi(\|u_n - u^*\|) + 2\lambda_n \omega(\lambda_n M)M.$$

Therefore, the proof follows as in theorem 3.1. □

Remark 3. If $E = L_p$ spaces $1 < p < \infty$, the formulas for the normalized duality map $J : E \rightarrow E^*$ and $J^{-1} : E^* \rightarrow E$ are known precisely (see e.g., Alber [1], Cioranescu [17], Chidume [7]), and they are given by

$$J(f) = |f|^{p-1} \cdot \text{sign} \frac{f}{\|f\|^{p-1}},$$

$$J^{-1}(f) = |f|^{q-1} \cdot \text{sign} \frac{f}{\|f\|^{q-1}}.$$

Remark 4. Trivially, our theorems hold for ϕ -strongly monotone and bounded operators and for k -strongly monotone and bounded operators in uniformly convex and uniformly smooth real Banach spaces by simply setting $\Phi(s) = s\phi(s)$ and $\Phi(s) = ks^2$, respectively, in Theorem 3.1. Hence, theorem 3.1 generalizes and improves the results in Diop *et al.*, [50] in the sense that the result in Diop *et al.*, [50] is a special case of theorem 3.1 in which the space is 2-uniformly smooth and the operator studied there is k -strongly monotone. We remark that L_p spaces, $1 < p < 2$ are not 2-uniformly smooth. Our theorem is valid, in particular, in all

L_p spaces, $1 < p < \infty$.

Remark 5. Theorem 3.1 is the analogue of theorem 1.1 in uniformly convex and uniformly smooth real Banach spaces without the assumption that A is uniformly continuous which is central in the proof of theorem 1.1.

Remark 6. Theorem 3.1 again is a significant improvement of results of Chidume *et al.*, [15], corollaries 3.2 and 3.3 are the main results of [15].

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