

Analytical approximate solutions of fractional convection-diffusion equation by means of local fractional derivative operators

Mehmet MERDAN

Gümüşhane University, Department of Mathematical Engineering,
29100-Gümüşhane, Turkey. mmerdan@gumushane.edu.tr

Abstract

In this article, local fractional decomposition method (LFDM) is applied to obtain approximate analytical solution of nonlinear fractional convection-diffusion. Numerical solutions obtained by local fractional decomposition method are compared with the exact solutions, revealing that the obtained solutions are of high accuracy. A new application of local fractional decomposition method (LFDM) was extended to reproduce analytical solutions to this equation in the form of a series. It is shown that the solutions obtained by the LFDM are reliable, simple and an effective method for strongly nonlinear partial equations.

Keywords: Local fractional decomposition method; fractional convection-diffusion equation; Riemann-Liouville derivative

1. Introduction

Local fractional calculus (LFC) was used in the modeling and processing of non-differentiable phenomena in various physical phenomena [1–15]. Some of these local fractional model can be listed as the local fractional Fokker-Planck equation (LFFPE) [1], the local fractional stress strain relations (LFSSR) [2], the local fractional heat conduction equation (LFHCE) [12], wave equations on the Cantor sets (WECSs) [14], local fractional Laplace equation (LFLE) [15], Newtonian mechanics (NM) on fractals subset of real-line [16]. An approximate solution of the fractional diffusion equation which includes an absorbent term and external force was given by Das et al. [17]. Similar studies were made by the following: Momani and Yildirim [18] examined Fractional convection-diffusion equation with nonlinear source term, Yildirim and Kocak [19] studied space-time fractional advection-dispersion equation, Yildirim and Gulkanat [20] worked on fractional Zakharov-Kuznetsov equations, El-Shahed [21] studied an integro-differential equation, Siddiqui et al [22] studied non-Newtonian flow, Yildirim [23] studied fractional PDEs in fluid mechanics and Kumar et al. [24] studied reaction-diffusion Brusselator system with fractional time derivative.

Successful applications of the homotopy perturbation method (HPM) [25–26], the Adomian decomposition method [27–29], the homotopy analysis method (HAM) [30] and the variational iteration method, which was given by Ji-Huan He, have been given on autonomous ordinary and partial differential equations and various other fields. The first application of variational iteration method on fractional differential equations was made by Ji-Huan He [34]. Jumarie [5] recently recommended a new modified Riemann-Liouville left derivative. Also, Adomian decomposition method, HPM and FVIM were used by Momani [35] to solve nonlinear fractional convection-diffusion equation. Finally, a recent effective

method called the Residual power series is used to solve important models, such as the time fractional Fisher [42] and other nonlinear fractional models arise in biology and physics [43-46].

In this study, the use of LFDM is extended to find analytical approximate solutions for the nonlinear fractional convection-diffusion problem

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - c \frac{\partial u(x,t)}{\partial x} + \Phi(u(x,t)) + f(x,t), \quad (1)$$

$$0 < x \leq 1, 0 < \alpha \leq 1, t > 0,$$

$$u(x, 0) = h(x), 0 < x \leq 1, \quad (2)$$

here $\Phi(u)$ is an appropriate nonlinear function of u selected as potential energy, α is the parameter that describes time-fractional derivative order and c is a constant. We consider the fractional derivative in the modified Riemann-Liouville derivative. $u(x, t)$ is a causal function of time which means it vanishes for $t < 0$. The fields of science and engineering have wide usage of the convection-diffusion equations for the mathematical modeling of computational simulations in oil reservoir simulations, mass and energy transport and global weather productions, where diffusion and convection propagates the initially discontinuous profile, the latter with a speed of c .

This study aims to extend the use of LFDM in solving fractional nonlinear convection-diffusion equations with modified Riemann-Liouville derivative.

The paper organization as follows:

Definitions related to local fractional calculus theory are given in Section 2. Solution procedure of local fractional decomposition method is defined to emphasize the inefficiency of this method in Section 3. The method is used on the problem (19)-(30) and graphs are given for numerical simulations, respectively in Section 4. Section 5 includes the conclusions.

2. Basic definitions

Let us recall some definitions and properties of local fractional continuity (LFC), local fractional derivative (LFD) and local fractional integral (LFI) of non-differential functions [1-10, 37-40].

Definition 1 Assume the relation below exists [9,37-40]

$$|f(u) - f(u_0)| < \varepsilon^\alpha \quad (3)$$

with $|u - u_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in \mathbb{R}$. Then $f(u)$ is local fractional continuous at $u = u_0$ which is denoted by $\lim_{u \rightarrow u_0} f(u) = f(u_0)$. If $f(u)$ is local fractional continuous on the interval (a, b) , it is denoted by

$$f(u) \in C_\alpha(a, b). \quad (4)$$

Definition 2 $f(x)$ is a non-differentiable function of exponent $0 < \alpha \leq 1$ if it is a Hölder function of exponent α , i.e.

$$|f(u) - f(v)| < C|u - v|^\alpha \quad (5)$$

for $u, v \in X$ [9, 37-40].

Definition 3 $f(x)$ is said to be continuous of order α (or α continuous) with $0 < \alpha \leq 1$ if the following relation is satisfied [9, 37-40], $|f(u) - f(u_0)| < \varepsilon^\alpha$,

$$f(u) - f(u_0) = o((u - u_0)^\alpha). \quad (6)$$

(3) is the standard definition for local fractional continuity (LFC) when compared with (6) and (5) is the unified local fractional continuity (ULFC).

Definition 4. For $f(u) \in C_\alpha(a, b)$, LFD of $f(x)$ of order α at $u = u_0$, is given by [9, 37-40]:

$$f^{(\alpha)}(u_0) = \left. \frac{d^\alpha f(u)}{du^\alpha} \right|_{u=u_0} = \lim_{u \rightarrow u_0} \frac{\Delta^\alpha(f(u) - f(u_0))}{(u - u_0)^\alpha}, 0 < \alpha \leq 1, \quad (7)$$

where $\Delta^\alpha(f(u) - f(u_0)) \cong \Gamma(1 + \alpha)\Delta(f(u) - f(u_0))$. For any $u \in (a, b)$, there exists $f^{(\alpha)}(u) = D_u^\alpha f(u)$, denoted by $f(u) \in D_u^\alpha(a, b)$. LFD of higher order can be expressed as follows:

$$f^{(k\alpha)}(u) = \underbrace{D_u^\alpha \dots D_u^\alpha f(u)}_{k \text{ times}},$$

and the higher ordered local fractional partial derivative (LFPD) is shown in the given as:

$$\frac{\partial^{(k\alpha)} f(u)}{\partial u^{k\alpha}} = \underbrace{\frac{\partial^\alpha}{\partial u^\alpha} \dots \frac{\partial^\alpha}{\partial u^\alpha} f(u)}_{k \text{ times}}.$$

Definition 5. For $f(u) \in C_\alpha(a, b)$, local fractional integral (LFI) of $f(u)$ with an order of α within interval $[a, b]$ is given by [37-40]:

$$\begin{aligned} {}_a I_b^\alpha f(u) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha, \quad 0 < \alpha \leq 1, \end{aligned} \quad (8)$$

Where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_3, \dots\}$ and $[\Delta t_j, \Delta t_{j+1}]$, $j = 0, 1, \dots, N - 1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a, b]$. For any $u \in (a, b)$, there exists ${}_a I_b^\alpha f(u)$, denoted by $f(u) \in I_u^{(\alpha)}(a, b)$. If $f(u) = D_u^\alpha f(a, b)$, or $I_u^{(\alpha)}(a, b)$, we have $f(u) \in C_\alpha(a, b)$.

For any $f(u) \in C_\alpha(a, b)$, $0 < \alpha \leq 1$, local fractional multiple integrals (LFMIs) have the form:

$${}_u I_u^{(k\alpha)} f(u) = \overbrace{{}_u I_u^{(\alpha)} \dots {}_u I_u^{(\alpha)} f(u)}^{k \text{ times}},$$

For $0 < \alpha \leq 1$, $f^{(k\alpha)}(u) \in C_\alpha^k(a, b)$, then we have

$$\left({}_u I_u^{(k\alpha)} f(u) \right)^{(k\alpha)} = f(u),$$

$$\text{where } {}_u I_u^{(k\alpha)} f(u) = \underbrace{{}_u I_u^{(\alpha)} \dots {}_u I_u^{(\alpha)} f(u)}_{k \text{ times}} \text{ and } f^{(k\alpha)}(u) = \underbrace{D_u^\alpha \dots D_u^\alpha f(u)}_{k \text{ times}}.$$

Definition 6. In the fractional space, the Mittag-Leffler function (MLF) is given by (see[9, 37-40])

$$E_\alpha(u^\alpha) = \sum_{n=0}^{\infty} \frac{u^{(n\alpha)}}{\Gamma(1+n\alpha)}, 0 < \alpha \leq 1. \quad (9)$$

Some useful formulas of LFD were summarized [9,40] as follows:

$$\frac{d^\alpha u^{n\alpha}}{du^\alpha} = \frac{\Gamma(1+n\alpha)u^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}, \quad (10)$$

$$\frac{d^\alpha E_\alpha(u^\alpha)}{du^\alpha} = E_\alpha(u^\alpha), \quad (11)$$

$$\frac{d^\alpha E_\alpha(nu^\alpha)}{du^\alpha} = nE_\alpha(nu^\alpha), \quad (12)$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b E_\alpha(u^\alpha) (du)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha), \quad (13)$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b u^{n\alpha} (du)^\alpha = \frac{\Gamma(1+n\alpha)(b^{(n+1)\alpha} - a^{(n+1)\alpha})}{\Gamma(1+(n+1)\alpha)}. \quad (14)$$

3. Local Fractional Decomposition Method

We demonstrate the local fractional decomposition method solution procedure by examining the following fractional differential equation [18, 39-40]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - c \frac{\partial u(x,t)}{\partial x} + \Phi(u(x,t)) + f(x,t), \quad (15)$$

$0 < x \leq 1, 0 < \alpha \leq 1, t > 0,$

A local fractional differential operator form for Eq. can be built by the LFDM (15) as follows

$$L_t^{(\alpha)} u(x,t) = u_{xx}(x,t) - u_x(x,t) + \Phi(u(x,t)) + f(x,t), \quad (16)$$

where $0 < \alpha \leq 1$, $u(x,t)$ is a local fractional continuous function. An application of the inverse operator $L_t^{(-\alpha)}$ on both sides of (16) gives

$$\begin{cases} u_{n+1}(x,t) = L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_n(x,t) - \frac{\partial}{\partial x} u_n(x,t) + \Phi(u_n(x,t)) + f(x,t) \right], n \geq 0 \\ u_0(x,t) = u(x,0). \end{cases} \quad (17)$$

We can obtain successive approximations $u_{n+1}(x,t)$, $n \geq 0$ of the solution $u(x,t)$ by the use of a selective function u_0 . Generally, the initial values are considered for the zeroth approximation u_0 . Thus, the exact solution can be obtained from

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \quad (18)$$

Hence, the condition

$$|f(x) - f(x_0)| < \varepsilon^\alpha,$$

can be found, where the fractional dimension of $f(x)$ is to α for any $x \in (a,b)$.

4. Applications

This section presents the solutions of nonlinear fraction differential equations by an application of the local fractional decomposition method (LFDM).

Examples 4.1. Let's examine the following nonlinear fractional convection-diffusion equation where $0 < \alpha \leq 1$, $c = 1$, $0 < x \leq 1$, $t > 0$,

and $\Phi(u) = u \frac{\partial^2 u}{\partial x^2} - u^2 + u$. We get

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + u(x,t) \frac{\partial^2 u(x,t)}{\partial x \partial t} - u(x,t)^2 + u(x,t) \quad (19)$$

with the initial conditions

$$u(x, 0) = e^x, [18,35-36]. \quad (20)$$

Using Eq.(20), the recurrence relation is found as:

$$u_0(x, t) = u(x, 0) \text{ and}$$

$$u_{n+1}(x, t) = L_t^{(-\alpha)} \left[\begin{aligned} & \frac{\partial^2}{\partial x^2} u_n(x, t) - \frac{\partial}{\partial x} u_n(x, t) \\ & + u_n(x, t) \frac{\partial^2}{\partial x^2} u_n(x, t) - u_n(x, t)^2 + u_n(x, t) \end{aligned} \right] n \geq 0. \quad (21)$$

With the recursive relation (21) and the conditions in (20), the following results are obtained:

$$u_0(x, t) = e^x, \quad (22)$$

$$u_1(x, t) = L_t^{(-\alpha)} \left[\begin{aligned} & \frac{\partial^2}{\partial x^2} u_0(x, t) - \frac{\partial}{\partial x} u_0(x, t) \\ & + u_0(x, t) \frac{\partial^2}{\partial x^2} u_0(x, t) - u_0(x, t)^2 + u_0(x, t) \end{aligned} \right] \\ = \frac{e^{x t^\alpha}}{\Gamma(1+\alpha)}, \quad (23)$$

$$u_2(x, t) = L_t^{(-\alpha)} \left[\begin{aligned} & \frac{\partial^2}{\partial x^2} u_1(x, t) - \frac{\partial}{\partial x} u_1(x, t) \\ & + u_1(x, t) \frac{\partial^2}{\partial x^2} u_1(x, t) - u_1(x, t)^2 + u_1(x, t) \end{aligned} \right] \\ = \frac{e^{x t^{2\alpha}}}{\Gamma(1+2\alpha)}, \quad (24)$$

$$u_3(x, t) = L_t^{(-\alpha)} \left[\begin{aligned} & \frac{\partial^2}{\partial x^2} u_2(x, t) - \frac{\partial}{\partial x} u_2(x, t) \\ & + u_2(x, t) \frac{\partial^2}{\partial x^2} u_2(x, t) - u_2(x, t)^2 + u_2(x, t) \end{aligned} \right] \\ = \frac{e^{x t^{3\alpha}}}{\Gamma(1+3\alpha)}, \quad (25)$$

$\vdots$,

$$u_n(x, t) = L_t^{(-\alpha)} \left[\begin{aligned} & \frac{\partial^2}{\partial x^2} u_{n-1}(x, t) - \frac{\partial}{\partial x} u_{n-1}(x, t) \\ & + u_{n-1}(x, t) \frac{\partial^2}{\partial x^2} u_{n-1}(x, t) - u_{n-1}(x, t)^2 + u_{n-1}(x, t) \end{aligned} \right] = \frac{e^{x t^{n\alpha}}}{\Gamma(1+n\alpha)} \quad (26)$$

Hence, the approximate solution can be given in a series form as

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = e^x \left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \dots \right) = e^x \sum_{n=0}^{\infty} \frac{t^{(n\alpha)}}{\Gamma(1+n\alpha)} \quad (27)$$

And the corresponding exact solution would be
 $u(x, t) = e^x E_\alpha(t^\alpha).$

(28)

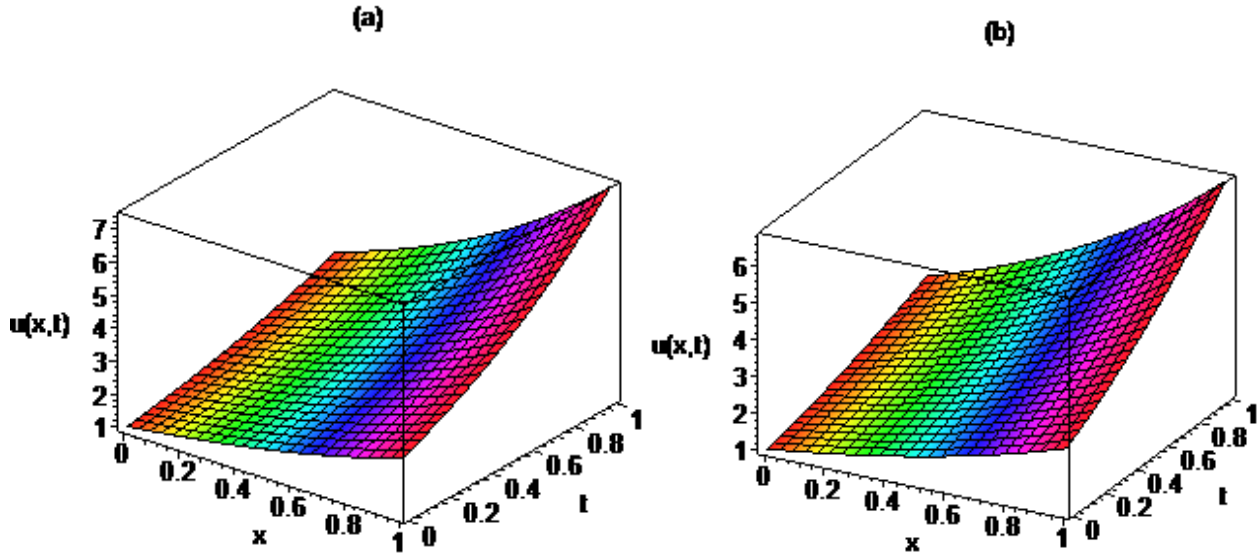
For $\alpha = 1$ we would have

$$u(x, t) = e^x \sum_{k=0}^{\infty} \frac{(t^k)}{\Gamma(k+1)} = e^{x+t} \quad (29)$$

as the exact solution to the nonlinear convection-diffusion equation. Fig. 1 shows the approximate solutions for the nonlinear convection-diffusion equation (19) obtained using local fractional decomposition method(LFDM).

The approximate solution of Eq. (19) is plotted in Figure 2 for $\alpha = 0.9, 0.8, 0.7, 0.6$. The effect of α on $u(x, t)$ is demonstrated in the Figures 3 and 4.

By considering $x = 0.5$ in the numerical results of Figures 3 and 4, it can be said that a decrease in the fractional order α results in an increase in the function. Five sequential values $\alpha = 1, 0.9, 0.8, 0.7, 0.6$ are demonstrated in Figures 3 and 4.



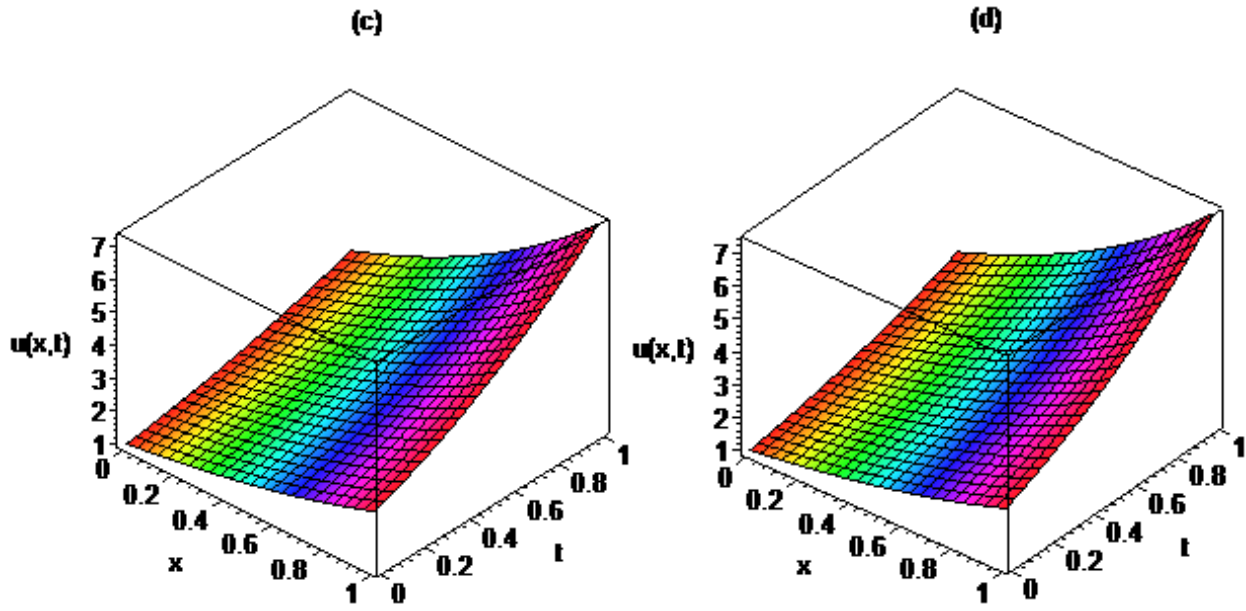
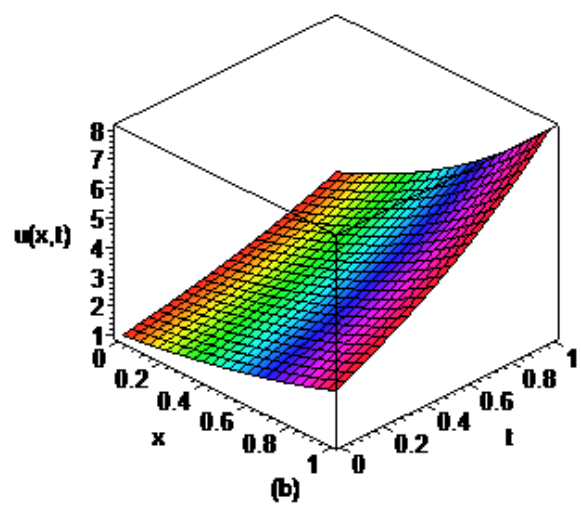
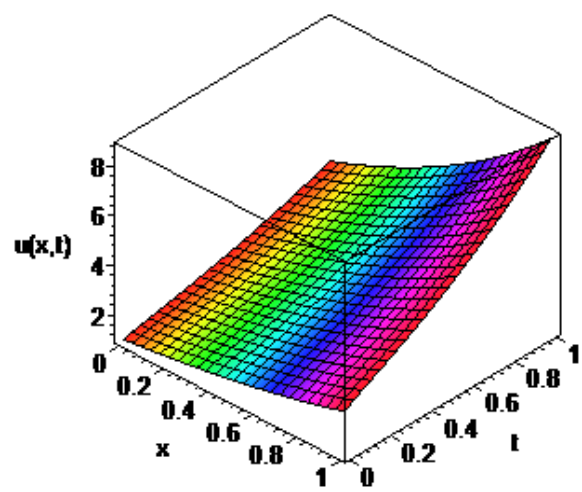


Fig. 1. The surface indicates the solution $u(x,t)$ of (19) for $\alpha=1$. (a) Exact solution (b) 2-iterate LFD approximate solution, (c) 3-iterate LFD approximate solution and (d) 4-iterate LFD approximate solution

(a)



(b)



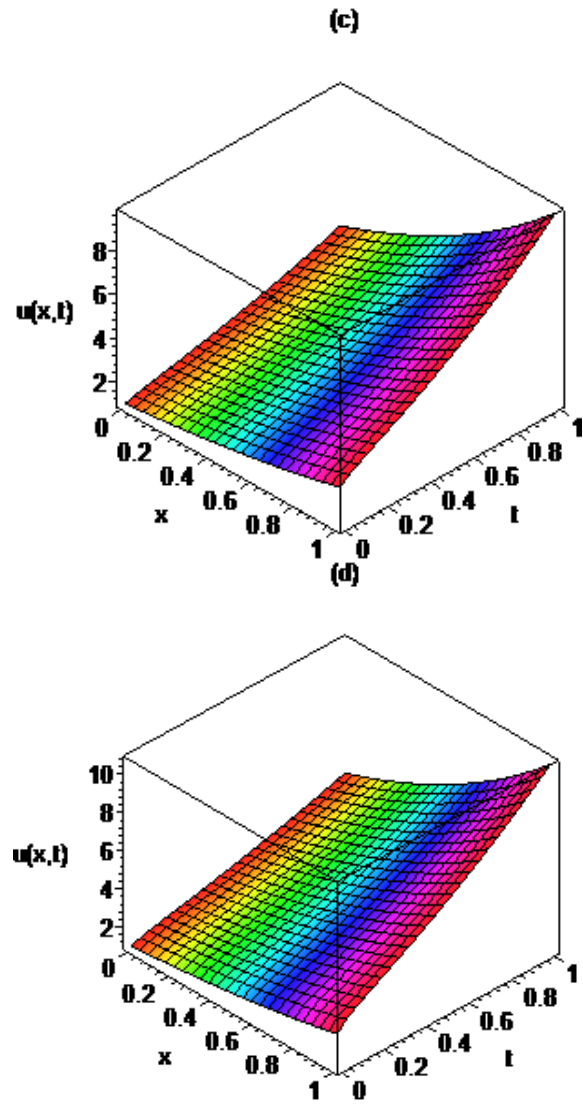


Fig. 2. The surface indicates the solution $u(x,t)$ of (19) (a) 4-iterate LFD approximate solution for $\alpha = 0.9$ (b) 4-iterate LFD approximate solution for $\alpha = 0.8$ (c) 4-iterate LFD approximate solution for $\alpha = 0.7$ and (d) 4-iterate LFD approximate solution for $\alpha = 0.6$

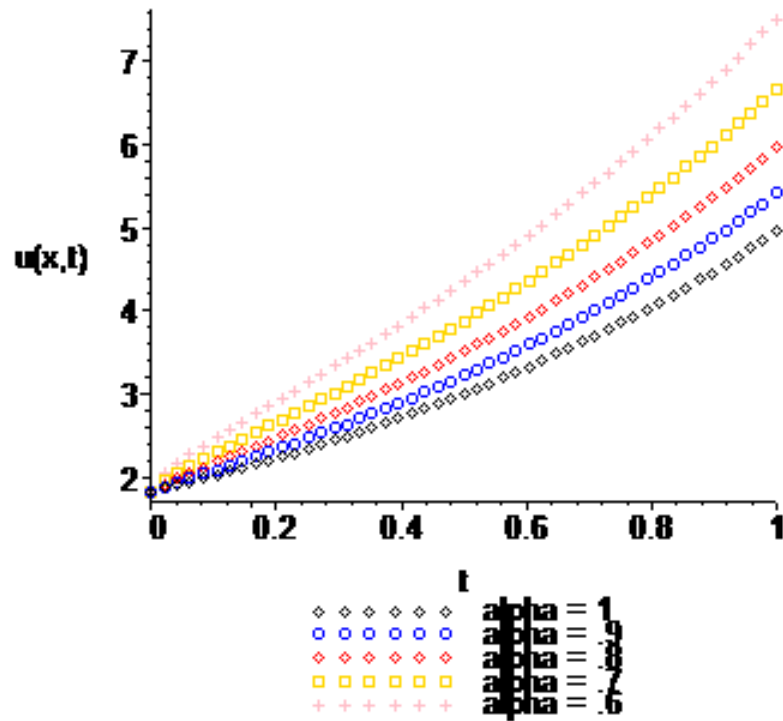


Fig. 3. 5-iterate LFD approximate solution for $x = 0.6$

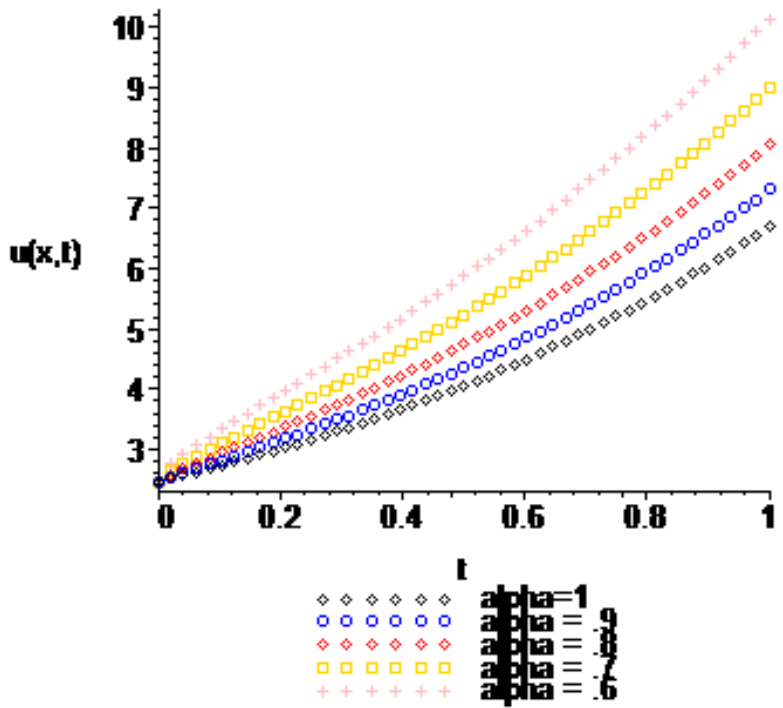


Fig. 4. 5-iterate LFD approximate solution for $x = 0.9$

Solution of (19) is found by using Adomian decomposition method in [35], FVIM in [36] and HPM in [18]. It is seen that the present algorithm for Local fractional decomposition method (LFDM) works with considerable efficiency, simplicity and reliability. The results obtained from LFDM are in full accordance with modified variational iteration method(FVIM), ADM and HPM.

Table 1 Numerical values when $\alpha = 0.5, 0.75, 1$ for $u(x, t)$ obtained using the LFDM

t	x	u_{LFDM}			u_{Exact}	u_{Error}
0.2	0	1.791910038	1.404517379	1.221400000	1.221402758	0.2758e-5
	0.25	2.300858034	1.803436013	1.568308644	1.568312185	0.3541 e-5
	0.50	2.954360195	2.315657678	2.013748160	2.013752707	0.4547e-5
	0.75	3.793473581	2.973363315	2.585703821	2.585709659	0.5838 e-5
	1	4.870916494	3.817874068	3.320109425	3.320116923	0.7498 e-5
0.4	0	2.383956218	1.798154972	1.491733334	1.491824698	0.000091364
	0.25	3.061060377	2.308876688	1.915423516	1.915540829	0.000117313
	0.50	3.930479326	2.964656351	2.459452478	2.459603111	0.000150633
	0.75	5.046835354	3.806694106	3.157999493	3.158192910	0.000193417
	1	6.480264866	4.887891984	4.054951614	4.055199967	0.000248353
0.6	0	3.003654242	2.251671269	1.821400000	1.822118800	0.000718800
	0.25	3.856768391	2.891203140	2.338723895	2.339646852	0.000922957
	0.50	4.952188640	3.712378316	3.002980923	3.004166024	0.001185101
	0.75	6.358736081	4.766788115	3.855903831	3.857425531	0.001521700
	1	8.164778744	6.120677093	4.951078522	4.953032424	0.001953902

Examples 4.2. Next, consider the non-homogenous nonlinear fractional convection-diffusion equation for $0 < \alpha \leq 1, c = 1, 0 < x \leq 1, t > 0$,

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{\partial^2 u(x, t)}{\partial x^2} - c \frac{\partial u(x, t)}{\partial x} + \frac{\partial u(x, t)}{\partial t} \frac{\partial u(x, t)}{\partial x} + u(x, t) \frac{\partial^2 u(x, t)}{\partial x \partial t} - 2x \quad (30)$$

with initial condition [18,35-36]

$$u(x, 0) = x^2. \quad (31)$$

By LFAMD, the recurrence relation reads as follows:

$$u_0(x, t) = u(x, 0) \text{ and}$$

$$u_{n+1}(x, t) = L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_n(x, t) - \frac{\partial}{\partial x} u_n(x, t) + \frac{\partial}{\partial t} u_n(x, t) \frac{\partial}{\partial x} u_n(x, t) + u_n(x, t) \frac{\partial^2}{\partial x^2} u_n(x, t) - 2x \right] n \geq 0. \quad (32)$$

Using (32) and the condition (31), the followings results are obtained:

$$u_0(x, t) = x^2, \quad (33)$$

$$u_1(x, t) = L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_0(x, t) - \frac{\partial}{\partial x} u_0(x, t) + \frac{\partial}{\partial t} u_0(x, t) \frac{\partial}{\partial x} u_0(x, t) \right] + u_0(x, t) \frac{\partial^2}{\partial x^2} u_0(x, t) - 2x \quad (34)$$

$$= \frac{(2-4x)t^\alpha}{\Gamma(1+\alpha)},$$

$$u_2(x, t) = L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_1(x, t) - \frac{\partial}{\partial x} u_1(x, t) + \frac{\partial}{\partial t} u_1(x, t) \frac{\partial}{\partial x} u_1(x, t) \right] + u_1(x, t) \frac{\partial^2}{\partial x^2} u_1(x, t) - 2x \quad (35)$$

$$= \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{(32x-16)\alpha\Gamma(2\alpha)t^{3\alpha-1}}{\Gamma(1+\alpha)^2\Gamma(3\alpha)} - \frac{2xt^\alpha}{\Gamma(1+\alpha)},$$

$$u_3(x, t) = L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_2(x, t) - \frac{\partial}{\partial x} u_2(x, t) + \frac{\partial}{\partial t} u_2(x, t) \frac{\partial}{\partial x} u_2(x, t) \right] + u_2(x, t) \frac{\partial^2}{\partial x^2} u_2(x, t) - 2x \quad (36)$$

$$= -\frac{32\alpha\Gamma(2\alpha)t^{4\alpha-1}}{\Gamma(\alpha+1)\Gamma(4\alpha)} + \frac{256\alpha^2\Gamma(2\alpha)\Gamma(5\alpha-1)t^{6\alpha-2}}{\Gamma(\alpha+1)^2\Gamma(3\alpha)\Gamma(1+2\alpha)\Gamma(6\alpha-1)} - \frac{16\alpha\Gamma(3\alpha)t^{4\alpha-1}}{\Gamma(1+2\alpha)\Gamma(1+\alpha)\Gamma(4\alpha)} +$$

$$\frac{32(32x-16)\alpha^2(3\alpha-1)\Gamma(2\alpha)^2\Gamma(6\alpha-2)t^{7\alpha-3}}{\Gamma(\alpha+1)^4\Gamma(3\alpha)^2\Gamma(7\alpha-2)} - \frac{2(32x-16)\alpha(3\alpha-1)\Gamma(2\alpha)\Gamma(4\alpha-1)t^{5\alpha-2}}{\Gamma(\alpha+1)^3\Gamma(3\alpha)\Gamma(5\alpha-1)} -$$

$$\frac{64x\alpha^2\Gamma(2\alpha)\Gamma(4\alpha-1)t^{5\alpha-2}}{\Gamma(\alpha+1)^3\Gamma(3\alpha)\Gamma(5\alpha-1)} + \frac{4x\alpha\Gamma(2\alpha)t^{3\alpha-1}}{\Gamma(\alpha+1)\Gamma(3\alpha)} - \frac{2xt^\alpha}{\Gamma(\alpha+1)},$$

⋮,

Hence, the approximate solution can be given in a series form as

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = x^2 + \frac{(2-8x)t^\alpha}{\Gamma(1+\alpha)} + \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{(32x-16)\alpha\Gamma(2\alpha)t^{3\alpha-1}}{\Gamma(1+\alpha)^2\Gamma(3\alpha)} -$$

$$- \frac{32\alpha\Gamma(2\alpha)t^{4\alpha-1}}{\Gamma(\alpha+1)\Gamma(4\alpha)} + \frac{256\alpha^2\Gamma(2\alpha)\Gamma(5\alpha-1)t^{6\alpha-2}}{\Gamma(\alpha+1)^2\Gamma(3\alpha)\Gamma(1+2\alpha)\Gamma(6\alpha-1)} - \frac{16\alpha\Gamma(3\alpha)t^{4\alpha-1}}{\Gamma(1+2\alpha)\Gamma(1+\alpha)\Gamma(4\alpha)} +$$

$$\frac{32(32x-16)\alpha^2(3\alpha-1)\Gamma(2\alpha)^2\Gamma(6\alpha-2)t^{7\alpha-3}}{\Gamma(\alpha+1)^4\Gamma(3\alpha)^2\Gamma(7\alpha-2)} - \frac{2(32x-16)\alpha(3\alpha-1)\Gamma(2\alpha)\Gamma(4\alpha-1)t^{5\alpha-2}}{\Gamma(\alpha+1)^3\Gamma(3\alpha)\Gamma(5\alpha-1)} -$$

$$\frac{64x\alpha^2\Gamma(2\alpha)\Gamma(4\alpha-1)t^{5\alpha-2}}{\Gamma(\alpha+1)^3\Gamma(3\alpha)\Gamma(5\alpha-1)} + \frac{4x\alpha\Gamma(2\alpha)t^{3\alpha-1}}{\Gamma(\alpha+1)\Gamma(3\alpha)} \dots = \quad (37)$$

For $\alpha = 1$ we would have [40]

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) = x^2 + 2t - 4tx + 2t^2 + 4tx - 12x^2t + 16xt^2 - 8t^2 + \dots, \quad (38)$$

The exact solution of (38) can be found by omitting the noise terms and keeping the non-noise terms as

$$u(x, t) = x^2 + 2t \quad (39)$$

And this can be easily confirmed. Formal proof can be found in [41].

Lastly, Figure 5 shows the solution surfaces for the non-homogenous nonlinear fractional convection-diffusion equation with various α values. From the graphical results in Figs. 5 and 6, it can be seen the effect of α on the function $u(x, t)$. Certainly, $u(x, t)$ increases with an increase in t for $\alpha = 1, 0.9, 0.8, 0.7, 0.6 \dots$

Solution of 30 is found by using Adomian decomposition method in [35], FVIM in [36] and HPM in [18]. The results in Fig. 5 are in accordance with the results of the Adomian decomposition method and HPM.

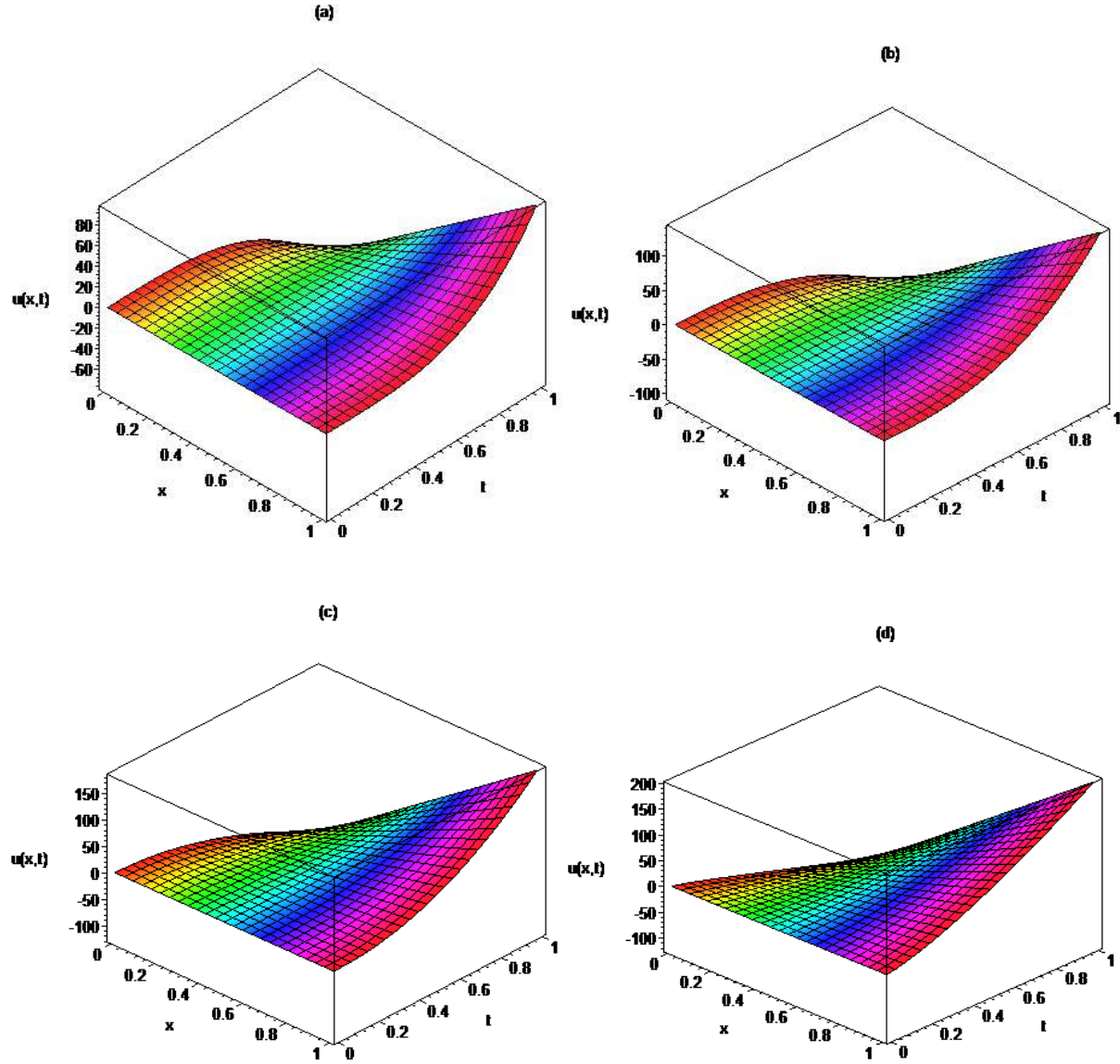


Fig. 5. The surface show the solution $u(x, t)$ of (30) (a) 3-iterate LFD approximate solution for $\alpha = 0.9$ (b) 3-iterate LFD approximate solution for $\alpha = 0.8$ (c) 3-iterate LFD approximate solution for $\alpha = 0.7$ and (d) 3-iterate LFD approximate solution for $\alpha = 0.6$

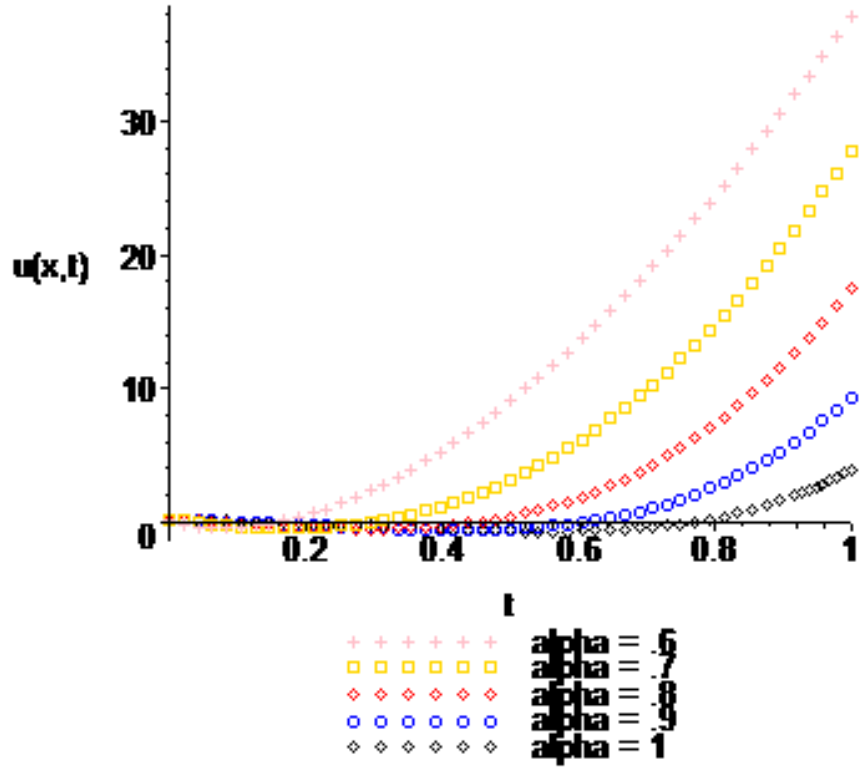


Fig. 6. 3-iterate LFD approximate for $x = 0.5$

Table 2

Approximate solution of (30) at $t=1$ for some values of the order α

x					u_{3LFD}			
	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 1$
0.0	5.00328	3.48585	2.97494	2.71828	-5.23681	-5.92718	-4.92124	-4
0.1	5.52948	3.85246	3.28782	3.00417	-3.60512	-4.34755	-3.65775	-2.99000
0.2	6.11102	4.25762	3.63360	3.32012	-1.95344	-2.74793	-2.37427	-1.96000
0.3	6.75372	4.70540	4.01575	3.66930	-0.28175	-1.12830	-1.07078	-0.91000
0.4	7.46401	5.20027	4.43809	4.05520	1.40993	0.511319	0.25269	0.160000
0.5	8.24901	5.74710	4.49048	4.48169	3.12162	2.17094	1.59618	1.25000
0.6	9.11656	6.35163	5.42069	4.95303	4.85330	3.85057	2.95966	2.36000
0.7	10.07540	7.01964	5.99079	5.47395	6.60499	5.55019	4.34315	3.49000
0.8	11.13500	7.75790	6.62085	6.04965	8.37668	7.26982	5.74663	4.64000
0.9	12.30812	8.57380	7.31717	6.68509	10.1683	9.00945	7.17012	5.81000

5. Conclusions

Local fractional decomposition method (LFDM) has been successfully employed for finding the solutions of nonlinear problems, ordinary, partial, fractional and integral equations. This study takes the Local fractional decomposition method having integral with respect to $(d\tau)^\alpha$ which had been used by Jumarie for the first time. Results show that the method is a powerful tool and is meaningful for the solutions of nonlinear fractional differential equations. The results of the two different examples obtained by local fractional decomposition method having integral with respect to $(d\tau)^\alpha$ are in perfect accordance with the results from classical VIM, MVIM, GDTM, HPM and Adomian decomposition method which can be found in the referred studies.

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