

On Minimal Non-Elementary Groups and Proper NF-Groups

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Abstract

Relations between a subgroup having F-supplement and a minimal non-elementary group are investigated and as a conclusion some conditions are given for a minimal non-elementary group to be an F-group. Then proper NF-groups which are also F-groups are studied.

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1 Introduction

In a group G , the notation we used for Frattini subgroup is $Frat(G)$, for the center of the group $Z(G)$. A complement of a subgroup H of G is a subgroup S of G such that $G = HS$ and $H \cap S = 1$. An nCS -group is a group in which every proper normal subgroup has a characteristic supplement in G . In [3] Tani Corsi called group a *minimal non - elementary group*, if every proper subgroup has trivial Frattini Subgroup but the group itself has nontrivial Frattini subgroup. She also called group a *proper NF - group*, if every

subnormal subgroup has trivial Frattini subgroup but the group itself has nontrivial Frattini subgroup. Kirtland studied minimal non-elementary groups and proper NF-groups in [1]. We call a group G is an F -group if for every subgroup H of G , there exists a subgroup S of G such that $G = HS$ and $H \cap S \leq \text{Frat}(S)$ in [2]. Here the subgroup S is called **an** F -supplement of H in G . The motivation of this study is to find relations between F-groups, minimal non-elementary groups and proper NF-groups. Firstly in Proposition 1 we concluded that if a group satisfies the both properties of being an F-group and minimal non-elementary group then every proper subgroup has a complement in the group. Then in Theorem 2, it is shown that for minimal non-elementary groups normal F-supplement is transitive and in the light of Theorem 2 in Corollary 3 we had the group is an F-group with extra conditions. In Proposition 4 and Corollary 5 proper NF-groups and F-groups are discussed. Finally in Theorem 6 and Corollary 7 we have shown that in proper NF-groups if a normal subgroup has the group as F-supplement then the normal subgroup is elementary abelian.

2 Main Results

Proposition 1. *Let G be a minimal non-elementary F-group. For any proper nontrivial subgroup H of G , if G is not F-supplement of H then H has **a** complement in G .*

Proof. Since G is an F-group for every $1 \neq H < G$, there exists a subgroup S of G such that $G = HS$ and $H \cap S < \text{Frat}(S)$. By hypothesis $S \neq G$ and since G is minimal non-elementary $\text{Frat}(S) = 1$. Hence $H \cap S < \text{Frat}(S) = 1$ and S is a complement of H in G . $\square$

Theorem 2. *Let G be a minimal non-elementary group, $M < N < K$ be sequence of proper subgroups M, N and $K \leq G$. If M has a normal F-supplement in N and N has an F-supplement in K , then M has **an** F-supplement in K .*

Proof. If $M = 1$, K is **a** complement of M in K . So we can take $M \neq 1$. By hypothesis, M has a normal F -supplement in N , say

S . Then $N = MS$ and $M \cap S = 1$ by Proposition 1. Now if $K = G$, then N has an F -supplement in G . So $K = G = NT$ and $N \cap T \leq \text{Frat}(T)$ for some $T \leq G$. If T is proper then T is a complement of N in G . In this case we have $K = G = NT = (MS)T = M(ST)$ since S is a normal subgroup of G . Now we will show that $M \cap ST = 1$. Let $a \in M \cap ST$. Since $a \in ST$ we write $a = st$ for some $s \in S$ and $t \in T$. $a = st \Rightarrow t = s^{-1}a \in SM = MS = N$ and hence $t \in N \cap T = 1$. Therefore, we have $a = s \in M \cap S = 1 \Rightarrow a = 1$. So ST is a complement of M in K . If $T = G$ then $G = NG$ and $N = N \cap G \leq \text{Frat}(G)$. Now, $G = NG = MSG = MG$ and $M \cap G = M < N < \text{Frat}(G) \Rightarrow M \cap G \leq \text{Frat}(G)$, hence G is an F -supplement of M in $G = K$.

If K is proper subgroup of G then N has a complement, say T , in K and ST is a complement of M in K in similar way as mentioned in previous paragraph. $\square$

Corollary 3. *Let G be a minimal non-elementary group, $M < N < K$ be sequence of proper subgroups M, N and $K \leq G$. If M has a normal F -supplement in N and N has a normal F -supplement in K , then G is an F -group.*

Proof. Let H be a subgroup of G . If $H = 1$ or $H = G$ then H has an F -supplement in G . So we can assume $H \neq 1$ and H is proper subgroup of G . Let M be a maximal subgroup of G such that $H \leq M < G$. Now H has a normal F -supplement in M and M has a normal F -supplement in G by hypothesis. Finally, H has an F -supplement in G , by Theorem 2. If G has no maximal subgroup then $\text{Frat}(G) = G$ and G is F -supplement of H in G . Hence G is an F -group. $\square$

Proposition 4. *Let G be a proper NF -group. If G is also an F -group then 1 and G are only subgroups which have normal F -supplement.*

Proof. Since G is an F -group, every subgroup of G has an F -supplement in G . 1 has G as F -supplement and G has 1 as F -supplement. For an $1 \neq H < G$, assume that H has a normal F -supplement in G . If N is a normal F -supplement of H in G then $G = NH$ and $N \cap H \leq \text{Frat}(N) = 1$ since G is a proper NF -group. It means that G has a direct decomposition, which

is a contradiction by [1] □

Corollary 5. *Let G be a proper NF – group. If G is not a simple F – group then G can not be an nCS – group.*

Proof. By Proposition 4 any normal subgroup N of G has normal F – supplement. So any of the normal subgroups has characteristic supplement. Hence G can not be an nCS – group. □

Theorem 6. *Let G be a finite proper NF – group. If G is the only F – supplement of proper normal subgroup N of G then N is elementary abelian .*

Proof. Let G be a finite proper NF – group, N is a proper normal subgroup of G and G is F – supplement of N in G . We know that G is the only F – supplement of N by [2]. So we can write $G = NG$ and $N = N \cap G \leq \text{Frat}(G)$. Since $\text{Frat}(G)$ is elementary abelian by [1], then N is elementary abelian. □

Corollary 7. *Let G be a finite nonabelian proper NF – group. Then G is the only F – supplement of $Z(G)$. In this case $Z(G)$ is elementary abelian.*

Proof. $Z(G) \cap G = Z(G) \leq \text{Frat}(G)$ by [1]. So G is the only F – supplement of $Z(G)$. Finally, if G is finite then $Z(G)$ is elementary abelian by Theorem 6. □

3 Conclusion

In this study relations between minimal non-elementary groups, proper NF -groups and F -groups which we studied in our former work.

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