

# A coalgebraic model of graphs

Christian Jäkel, *christian.jaekel(at)tu-dresden.de*  
Technische Universität Dresden

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## Abstract

In this note, we model various types of graphs, relational systems and multisets as coalgebras over  $\mathbf{Set} \times \mathbf{Set}$  and use the theory of coalgebras over arbitrary categories to conclude properties of the category of graphs. This point of view forces the formulation of a Co-Birkhoff like theorem for graphs.

**Keywords:** universal coalgebra, graph theory.

## 1 INTRODUCTION

For a covariant endofunctor  $F$  in  $\mathbf{Set}$ , we can model various types of graphs through comma categories over  $\mathbf{Set}$ . This yields a triple  $G = (V, E, g)$ , where  $V$  is the *vertex set*,  $E$  the *edge set* and  $g : E \rightarrow FV$  the *structure map* of the  $F$ -graph  $G$ . For two  $F$ -graphs  $G^{(1)} = (V^{(1)}, E^{(1)}, g^{(1)})$  and  $G^{(2)} = (V^{(2)}, E^{(2)}, g^{(2)})$ , a homomorphism is a pair of maps  $\varphi = (\varphi_v, \varphi_e)$ , where  $\varphi_v : V^{(1)} \rightarrow V^{(2)}$  defines the *vertex map* and  $\varphi_e : E^{(1)} \rightarrow E^{(2)}$ , the *edge map*, such that  $g^{(2)} \circ \varphi_e = F(\varphi_v) \circ g^{(1)}$  holds.

Next, we list examples for choices of  $F$  and resultant graph structures.

**Example 1.1.** • The identity functor yields a model for multisets, *i.e.*, a triple  $(\mathbb{N}, E, ar)$ , where the structure map  $ar$  assigns an arity to each element of  $E$ .

- By considering the functor  $F : V \mapsto \mathfrak{P}_{1,2}V$ , which assigns to a set its singleton and two-element subsets, we can model undirected graphs  $g : E \rightarrow \mathfrak{P}_{1,2}V$  and the induced homomorphisms are incidence preserving maps.
- Directed graphs  $g : E \rightarrow V \times V$  can be represented with  $F : V \mapsto V \times V$  and homomorphisms preserve source- and target nodes.
- Let  $\mathfrak{P}$  be the powerset functor. In this case,  $g : E \rightarrow \mathfrak{P}V$  or  $g : E \rightarrow V \times \mathfrak{P}V$  yields a model of hypergraphs or directed hypergraphs respectively.
- Hybrid graphs can be described by taking the sum of different type functors, as for instance,  $FV = (V \times V) + \mathfrak{P}_{1,2}V$  for a model of graphs with directed and undirected edges. With this framework, relational systems (see [3]) can be described too.

- Let  $X_v$  represent a set of vertex colors and  $X_e$  a set of edge colors. For every type functor  $F$ , we define colored graphs  $g : E \rightarrow X_e \times F(X_v \times V)$  by the functor  $\tilde{F}(V) := X_e \times F(X_v \times V)$ . For example, with  $X_v = X_e = [0, 1]$ , we get fuzzy graphs.
- Generally, this model allows for simple graphs and for graphs with multiple edges. The former ones can be easily recognized by their injective structure map.

Hence, various mathematical structures are generalized and can be described through  $F$ -graphs.

In [11], it was noticed that directed graphs can be considered as coalgebras over  $\mathbf{Set} \times \mathbf{Set}$ , by the functor  $\tilde{F} : (V, E) \mapsto (1, V \times V)$ , together with the structure map  $g : (V, E) \rightarrow (1, V \times V)$ . The induced homomorphisms are source- and target node preserving.

Inspired by this example, we developed a purely coalgebraic model of various types of graphs. We will present it in this note and apply the theory of coalgebras over arbitrary base categories (as in [6, 2]), to conclude structural results about the category of graphs. This yields a very economical approach to the category of graphs. Thereby, we generalize results of [10, 5, 12, 14], as for example the construction of cofree graphs and results about the category of multisets [8, 7, 13]. For a general reference on category theory see [1].

## 2 GRAPHS AS COALGEBRAS

We will define graphs as coalgebras and conclude immediate consequences for the category of graphs.

**Definition 2.1.** Let  $F : \mathbf{Set} \rightarrow \mathbf{Set}$  be a covariant endofunctor. We extend  $F$  to be a covariant endofunctor  $\tilde{F} : \mathbf{Set} \times \mathbf{Set} \rightarrow \mathbf{Set} \times \mathbf{Set}$ , via  $(V, E) \mapsto (1, FV)$  and  $(\varphi_v, \varphi_e) \mapsto (!, F(\varphi_v))$ , where  $1$  is the terminal object in  $\mathbf{Set}$  and  $! : V \rightarrow 1$  is the induced unique map.

An  $F$ -graph is a coalgebra  $((V, E), (!, g))$ , where  $g : E \rightarrow FV$  determines the graphs internal structure. A homomorphism  $\varphi = (\varphi_v, \varphi_e)$  is defined through the following commutative diagram.

$$\begin{array}{ccc}
 (V^{(1)}, E^{(1)}) & \xrightarrow{(\varphi_v, \varphi_e)} & (V^{(2)}, E^{(2)}) \\
 (\downarrow !, g^{(1)}) & & \downarrow !, g^{(2)} \\
 (1, FV^{(1)}) & \xrightarrow{(!, F(\varphi_v))} & (1, FV^{(2)})
 \end{array}
 \cong
 \begin{array}{ccc}
 (V^{(1)}, E^{(1)}) & \xrightarrow{(\varphi_v, \varphi_e)} & (V^{(2)}, E^{(2)}) \\
 g^{(1)} \downarrow & & \downarrow g^{(2)} \\
 \tilde{F}(V^{(1)}, E^{(1)}) & \xrightarrow{\tilde{F}(\varphi_v, \varphi_e)} & \tilde{F}(V^{(2)}, E^{(2)})
 \end{array}$$

For a simplified notation, we will write  $(V, E, g)$ , instead of  $((V, E), (!, g))$ . The category of graphs will be denoted by  $\mathbf{Graph}_F$ . It is equipped with the forgetful functor  $U : \mathbf{Graph}_F \rightarrow \mathbf{Set} \times \mathbf{Set}$ .

By applying the definition of  $\tilde{F}$ , the following can be shown straight forward.

**Proposition 2.2.** *The extension  $\tilde{F}$  preserves all limits that  $F$  preserves and  $\tilde{F}$  is bounded iff  $F$  is bounded.*

Next, we will list structural results for  $\mathbf{Graph}_F$ , which follow immediately from the theory of coalgebras. Thereby, we also use the fact that  $\mathbf{Set} \times \mathbf{Set}$  inherits most of its properties from  $\mathbf{Set}$ .

**Proposition 2.3.** *The forgetful functor  $U$  creates colimits. Hence, colimits in  $\mathbf{Graph}_F$  are constructed as pairs of colimits in  $\mathbf{Set}$ . Furthermore,  $U$  creates any limit preserved by  $F$  (see [6, Theorem 1.2.4, Theorem 1.2.7]).*

Colimit constructions, especially pushouts, play an important role in [9].

**Proposition 2.4.** *Isomorphisms are bijective<sup>1</sup> maps  $\varphi = (\varphi_v, \varphi_e)$  and epimorphisms are the surjective ones (see [2, 4.18 Corollary]). Regular monomorphisms are the injective maps (see [4, Theorem 3.4]).*

**Proposition 2.5.** *Subgraphs are defined as regular subobjects (see [6, Definition 2.2.1]) and the category  $\mathbf{Graph}_F$  has an epi-regular mono factorization system, which is created by  $U$  (see [2, 4.23 Remark]).*

**Proposition 2.6.** *The category  $\mathbf{Graph}_F$  has all equalizers and the equalizer object is cogenerated from the respective equalizer object in  $\mathbf{Set} \times \mathbf{Set}$  (see [6, Theorem 2.4.1]).*

The cofree graph for a color set  $X$  can be constructed as follows:

**Theorem 2.7.** *For a set of colors  $X = (X_v, X_e)$ , the cofree graph over  $X$  is given through  $(X_v, X_e \times FX_v, \pi_{FX_v})$  together with the map  $\varepsilon_X = (\text{id}_{X_v}, \pi_{X_e})$ , where  $\pi_{FX_v}$  and  $\pi_{X_e}$  are the canonical projections of the product  $X_e \times FX_v$ .*

*Proof.* We construct the transfinite cochain as in [2, 2.23 Cofree Coalgebra Construction] and see that it stops after two steps. We define:  $X_{\#}^0 = (1, 1)$ . Then,

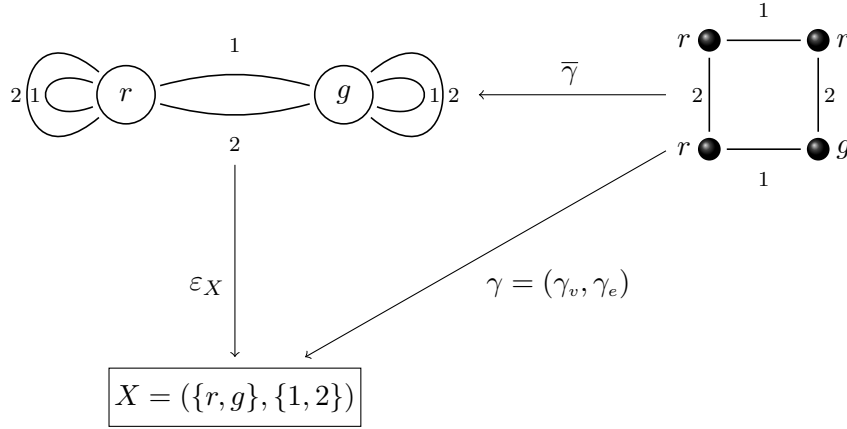
$$\begin{aligned} X_{\#}^1 &= (X_v, X_e) \times \bar{F}(1, 1) = (X_v \times 1, X_e \times F1) = (X_v, X_e \times F1), \\ X_{\#}^2 &= (X_v, X_e) \times \bar{F}(X_v, X_e \times F1) = (X_v, X_e) \times (1, FX_v) = (X_v, X_e \times FX_v), \\ X_{\#}^3 &= (X_v, X_e) \times \bar{F}(X_v, X_e \times FX_v) = (X_v, X_e) \times (1, FX_v) = (X_v, X_e \times FX_v). \end{aligned}$$

□

The above construction gives rise to a right adjoint functor  $U \dashv C$  with  $C : \mathbf{Set} \times \mathbf{Set} \rightarrow \mathbf{Graph}_F$ . Consequently, it holds that for every graph  $G$  and every coloring  $\gamma : UG \rightarrow (X_v, X_e)$  there exists a unique homomorphism  $\bar{\gamma}$ , such that  $\gamma = \varepsilon_X \circ U(\bar{\gamma})$ .

**Example 2.8.** We consider undirected graphs of type  $\mathfrak{P}_{1,2}$ . Let  $X_v = \{r, g\}$  and  $X_e = \{1, 2\}$ . The resulting cofree graph is pictured below (left-hand), together with the graph coloring  $\gamma$  and the assigned colors are labels on the graph  $G$  (right-hand).

<sup>1</sup>By injective, surjective and bijective maps  $\varphi = (\varphi_v, \varphi_e)$ , we mean that  $\varphi_v$  and  $\varphi_e$  are injective, surjective and bijective respectively.



*Remark 2.9.* For a graph  $G = (V, E, g)$  and a coloring  $\gamma : UG \rightarrow (X_v, X_e)$ , we can define colored graphs with respect to the functor  $(X_v, X_e) \times \bar{F}(V, E) \cong (X_v, X_e \times FV)$ . The structure map of the colored graph is given as  $(\gamma_v, (\gamma_e, g)) : (V, E) \rightarrow (X_v, X_e \times FV)$ .

*Remark 2.10.* For  $F = \mathfrak{P}$ , the extension  $\bar{\mathfrak{P}}$  yields a non-accessible covariator.

Using the dual of [2, 4.13 Theorem], we can conclude that **Graph<sub>F</sub>** is complete. Additionally to this fact, we will present an explicit construction of products, which does, together with proposition 2.6, also imply the completeness of **Graph<sub>F</sub>**.

**Theorem 2.11.** *Let  $(G^{(i)} = (V^{(i)}, E^{(i)}, g^{(i)}))_{i \in I}$  be a family of  $F$ -graphs. Their product is given as  $\prod G^{(i)} = (\prod V^{(i)}, E_{prod}, g_{prod})$ , where  $E_{prod}$  and  $g_{prod} := \text{pb}(\alpha)$  are defined through the following pullback square.*

$$\begin{array}{ccc} E_{prod} & \xrightarrow{\text{pb}(\alpha)} & F(\prod V^{(i)}) \\ \text{pb}(\beta) \downarrow & & \downarrow \beta \\ \prod E^{(i)} & \xrightarrow{\alpha} & \prod FV^{(i)} \end{array}$$

The projection homomorphisms are  $\pi^{(i)} := (\pi_v^{(i)}, \pi_e^{(i)} \circ \text{pb}(\beta)) : \prod G^{(i)} \rightarrow G^{(i)}$ , where  $\pi_v^{(i)}$  and  $\pi_e^{(i)}$  are the projections of  $\prod V^{(i)}$  and  $\prod E^{(i)}$  respectively.

*Proof.* It is straight forward to show that the  $\pi^{(i)}$  are homomorphisms. Next, we notice that every  $F$ -graph  $G^{(i)} = (V^{(i)}, E^{(i)}, g^{(i)})$  is a subgraph of  $C(V^{(i)}, E^{(i)})$  (see [6, Theorem 2.1.16]) and  $\prod C(V^{(i)}, E^{(i)}) = C(\prod V^{(i)}, \prod E^{(i)}) = (\prod V^{(i)}, \prod E^{(i)} \times F(\prod V^{(i)}))$  holds. According to [6, Theorem 2.4.3], the product of the  $G^{(i)}$  is cogenerated, with respect to  $\prod C(V^{(i)}, E^{(i)})$ , by the following pullback in **Set**  $\times$  **Set**.

$$\begin{array}{ccc} (\prod V^{(i)}, \tilde{E}) & \longrightarrow & (\prod V^{(i)}, \prod E^{(i)} \times F(\prod V^{(i)})) \\ \downarrow & & \downarrow \\ (\prod V^{(i)}, \prod E^{(i)}) & \longrightarrow & (\prod V^{(i)}, \prod E^{(i)} \times \prod FV^{(i)}) \end{array}$$

By diagram chasing, one can show that  $\tilde{E} \cong E_{prod}$ . Hence, the largest subgraph contained in  $(\prod V^{(i)}, \tilde{E})$  is  $(\prod V^{(i)}, E_{prod}, \text{pb}(\alpha))$ .  $\square$

It follows that the edge set of the product is a subset of  $E^{(1)} \times E^{(2)} \times F(V^{(1)} \times V^{(2)})$ . This yields the usual product of undirected graphs.

**Example 2.12.** In the product of two  $\mathfrak{P}_{1,2}$ -graphs  $G$  and  $\tilde{G}$ , for each  $e \in E$  with  $g(e) = \{v, w\}$  and  $\tilde{e} \in \tilde{E}$  with  $\tilde{g}(\tilde{e}) = \{\tilde{v}, \tilde{w}\}$ , there are two edges in  $G \times \tilde{G}$ , namely  $(e, \tilde{e}, \{(v, \tilde{v}), (w, \tilde{w})\})$  and  $(e, \tilde{e}, \{(v, \tilde{w}), (w, \tilde{v})\})$ .

At last, we define covarieties and coequation satisfaction in the usual way.

**Definition 2.13.** Let  $G = (V, E, g)$  be an  $F$ -graph and  $CX$  the cofree graph over a color set  $X = (X_v, X_e)$ . We define  $\text{Col}_X(G) = \{\gamma \mid \gamma : UG \rightarrow X\}$  to be the collection of all colorings of  $G$ . A subset  $P \subseteq (X_v, X_e)$  is called *pattern* over  $CX$ . We say that a pattern  $P$  holds in  $G$  if for all  $\gamma \in \text{Col}_X(G)$ , we have that the image  $\bar{\gamma}[G]$  is a subgraph of  $[P]_g$ , the graph cogenerated by  $P$ . For that, we write  $G \models P$ . This means that for all colorings, the induced homomorphism  $\bar{\gamma}$  factors through the largest in  $P$  contained subgraph, or in other words that  $G$  is regular-projective with respect to the inclusion morphism  $\iota_{([P]_g, CX)}$ .

$$\begin{array}{ccc} G & \xrightarrow{\bar{\gamma}} & CX \\ & \searrow & \uparrow \iota_{([P]_g, CX)} \\ & & [P]_g \end{array}$$

Let  $\mathcal{P}$  be a collection of patterns and  $\mathcal{K}$  a class of  $F$ -graphs. We say  $G \models \mathcal{P}$  if every  $P \in \mathcal{P}$  holds in  $G$ . Analogously,  $\mathcal{K} \models P$  provided  $G \models P$  for every  $G \in \mathcal{K}$ .

**Theorem 2.14.** A class  $\mathcal{K}$  of  $F$ -graphs is a co-variety (i.e., closed under subgraphs, homomorphic images and coproducts) if and only if there is a collection  $\mathcal{P}$  of patterns, such that for all  $G$ , it holds that  $G \in \mathcal{K} \Leftrightarrow G \models \mathcal{P}$  (see [6, Theorem 3.6.21]).

We define:  $\text{Graph}(\mathcal{P}) := \{G \in \mathbf{Graph}_F \mid G \models \mathcal{P}\}$  and  $\text{Pat}(\mathcal{K}) := \{P \mid \mathcal{K} \models P\}$ .

**Theorem 2.15** (Co-Birkhoff Theorem For  $F$ -Graphs). For a class  $\mathcal{K}$  of  $F$ -graphs, we have:  $\text{SH}\Sigma(\mathcal{K}) = \text{Graph}(\text{Pat}(\mathcal{K}))$ .<sup>2</sup>

Furthermore, it holds that:

**Theorem 2.16.** If  $F$  is bounded, then every covariety can be defined through a set of coequations (see [6, Theorem 3.7.22]).

<sup>2</sup>Here  $S$  is the class of all subgraphs,  $H$  the class of all homomorphic images and  $\Sigma$  the class of all coproducts of elements from  $\mathcal{K}$ .

## 3 CONCLUSION

We showed that various types of graphs and mathematical structures like multisets or relational systems, together with their structure preserving homomorphisms, can be considered as coalgebras. This yields new examples for coalgebras. Furthermore, from this point of view, the categorical properties of graphs follow immediately from the theory of universal coalgebras.

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