

Original Research Article

An Application of the Maximum Theorem and the Distance Method in Set-Valued Optimization Related to Efficient Continuous Selections

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Abstract

In the present paper, we consider relationships between the basic solution concepts of set-valued optimization, the properties of continuous multifunctions with compact and convex values and the existence of efficient continuous selections using the Maximum Theorem and the Distance method. We also discuss two solution concepts in set-valued optimization problem and prove existence of two types of efficient continuous selections which correspond to the two set-valued optimization solutions, respectively.

Keywords: set-valued optimization, continuous multifunction, efficient continuous selection, Maximum Theorem, compact, convex.

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1. Introduction

Set-valued optimization appears naturally in many branches of science, such as game theory, economics, finance, physics, control theory, biomathematics and so on. The aim of this paper is to show some relationships between the basic solution concepts of set-valued optimization, the properties of continuous multifunctions with compact and convex values and the existence of efficient continuous selections in finite dimensional Euclidean spaces.

In present paper, we will consider an application of the Maximum Theorem and the Distance method in set-valued optimization problem. So, let $\|\cdot\|$ be the Euclidean norm on R^n , $d(x,y) = \|x - y\|$ for $x, y \in R^n$ be the Euclidean distance and τ be the topology induced by d . It is known that the Euclidean norm is strictly convex.

We will focus our attention on set-valued optimization. The general form of this optimization problem is to find a variable $x \in X \subset R^m$, $m \geq 1$, so as to maximize $F(x)$ subject to $x \in X$,

where the feasible domain X is nonempty and $F: X \Rightarrow R^n$, $n \geq 2$, is a given multifunction [1] [2] [3] [4].

It is known that the image of each point by a multifunction is nonempty and if the multifunction F is function with vector values, then the above problem reduces to a multi-criteria, multi-objective or vector optimization problem. Additionally, if $n=1$, then F is a real function and we are in the framework of classical optimization.

Definition 1. Let $M \subset R^n$ be a nonempty set. A point $x \in M$ is called an efficient point of M if and only if $\{x\} = M \cap (\{x\} + R_+^n)$. The set of the all efficient points of M is denoted by $Eff(M)$ and it is called an efficient set.

Remark 1. It is well-known that if M is nonempty compact and $x \in M$, then $Eff(M)$ is nonempty, $Eff(M) \subset \partial M$ and there exists a point $y \in Eff(M)$ such that $y \in (\{x\} + R_+^n)$, see also [5] and [6] for more details.

Recall that $z(z_1, z_2, \dots, z_n) \in R^n$ is an ideal point for compact set M if and only if $z_i = \max\{x_i \mid x(x_1, x_2, \dots, x_n) \in M\}$ for every $i \in \{1, 2, \dots, n\}$. It is easy to show that there are two cases: $z \in \partial M$ or $z \notin M$ [7].

The multifunction $F: X \Rightarrow R^n$ is usually identified with its graph

$$Graph(F) = \{(x, y) \in X \times F(X) \mid y \in F(x)\}.$$

For our set-valued optimization problem we study the following solution concepts [2].

Definition 2. (a) A pair $(x, y) \in Graph(F)$ is called an efficient solution if and only if $x \in X$ and $y \in Eff(F(x))$. The set of the all efficient solutions of $Graph(F)$ is denoted by $Eff(X, F)$ and it is called an efficient-graph set.
 (b) A pair $(x, y) \in Graph(F)$ is called a Pareto-optimal solution if and only if $x \in X$, $y \in F(x)$ and $y \in Eff(F(x))$. The set of the all Pareto-optimal solutions of $Graph(F)$ is denoted by $PO(X, F)$ and it is called a Pareto-optimal set.

From above definition it follows that $PO(X, F) \subset Eff(X, F)$.

Remark 2. In our set-valued optimization problem it can be shown that if the feasible domain X is compact and F is upper semi-continuous with compact and convex values, then $F(X)$ is compact [8 – Proposition 9.7]. As a result there exists a point $y \in F(X)$ such that $y \in Eff(F(X))$, i.e. there exists a point $x \in X$ such that $y \in F(x)$ and $(x, y) \in PO(X, F)$, see also [1] and [5]. Usually, a Pareto-optimal solution is not necessarily uniquely determined, but there are several Pareto-optimal solutions [9].

Remark 3. Consider a multifunction $F: X \Rightarrow R^n$.

(1) By a continuous selection f for F , we mean a continuous function $f: X \rightarrow F(X)$ such that $f(x) \in F(x)$ for all $x \in X$ [10].

(2) Let F have compact values. By an efficient selection f for F , we mean a function $f : X \rightarrow F(X)$ such that $f(x) \in \text{Eff}(F(x))$ for all $x \in X$.

In this paper, we study our set-valued optimization problem where the multifunction F is continuous on X and it has compact and convex values. In this case we get that $\text{Eff}(X, F)$ is nonempty and if X is compact, then $PO(X, F)$ is also nonempty. We prove the existence of two different efficient continuous selections which correspond to the two set-valued optimization solutions, respectively. Here we use the Maximum Theorem and the Distance method in set-valued optimization.

2. Main result

In this section, we start our analysis with the first basic theorem.

Theorem 1.

Let $X \subset R^m$ and $F : X \rightrightarrows R^n$ be a continuous multifunction with compact and convex values. The following statements are true:

- (a) There exists an efficient continuous selection $f : X \rightarrow F(X)$ for F such that $(x, f(x)) \in \text{Eff}(X, F)$ for all $x \in X$.
- (b) If $X \subset R^m$ is compact, then there exist an efficient continuous selection $f : X \rightarrow F(X)$ for F and a point $x \in X$ such that $(x, f(x)) \in PO(X, F)$.

Remark 4. We often use the Maximum Theorem (or Berge Maximum Theorem) as a mathematical tool in optimization and nonlinear analysis. In order to give the proof of Theorem 1 we will use this theorem in its classical variant: "Let $S \subset R^n$ and $\Theta \subset R^m$, $g : S \times \Theta \rightarrow R$ be a continuous function, and $D : \Theta \rightrightarrows S$ be a compact-valued and continuous multifunction. Then, the function $g^* : \Theta \rightarrow R$ defined by $g^*(\theta) = \max\{g(x, \theta) \mid x \in D(\theta)\}$ is continuous on Θ , and the multifunction $D^* : \Theta \rightrightarrows S$ defined by $D^*(\theta) = \{x \in D(\theta) \mid g(x, \theta) = g^*(\theta)\}$ is compact-valued and upper semi-continuous on Θ " [11] [12].

The above remark allows us to prove the theorem.

Proof of Theorem 1.

(a) Let $S = F(X)$ and $x \in X$. For each $i \in \{1, 2, \dots, n\}$ we define a function $m_i : X \rightarrow R$ by $m_i(x) = \max\{y_i \mid y = (y_1, y_2, \dots, y_n) \in F(x)\}$. It is easy to see that these functions $\{m_i\}_{i=1}^n$ are continuous on X , see Remark 4. As a result we get a continuous function $m : X \rightarrow R^n$ such that $m(x) = (m_1(x), m_2(x), \dots, m_n(x))$ for all $x \in X$. Observe that for each $x \in X$, $m(x)$ is an ideal point for $F(x)$.

Now, let us define a function $g : S \times X \rightarrow R$ by $g(y, x) = -\sqrt{\sum_{i=1}^n (y_i - m_i(x))^2}$ for all $y = (y_1, y_2, \dots, y_n) \in S$ and for all $x \in X$. Clearly, g is continuous on $S \times X$ and $F(x) \subset (\{m(x) - R_+^n\})$ for all $x \in X$.

Let $x \in X$. We also see that either $m(x) \notin F(x)$ or $m(x) \in \partial F(x)$.

First, let $m(x) \notin F(x)$. From $F(x)$ is compact and convex it immediately follows that there exists a unique $z \in F(x)$ such that $g(z, x) = \max\{g(y, x) \mid y \in F(x)\}$ and $g(z, x) < 0$.

Second, let $m(x) \in \partial F(x)$. This means that $d(m(x), F(x)) = 0$ and $d(m(x), y) > 0$ for all $y \in F(x) \setminus \{m(x)\}$. In this case, let $z = m(x)$. We get that $z \in F(x)$ is unique and $g(z, x) = \max\{g(y, x) \mid y \in F(x)\}$.

Finally, from the construction of $z \in F(x)$ it follows that $\{z\} = F(x) \cap (F(x) + R_+^n)$, i.e. $z \in \text{Eff}(F(x))$, see also [5].

From Remark 4 (or the Maximum Theorem) it follows the multifunction $F^*: X \Rightarrow Y$ defined by $F^*(x) = \{y \in F(x) \mid g(y, x) = g^*(x)\}$ is compact-valued and upper semi-continuous on X .

But, we also get that $z \in F^*(x)$ and $|F^*(x)| = 1$, because $F(x)$ is convex compact and the Euclidean norm is strictly convex, i.e. F^* is a function. As a result we deduce that $f = F^*$ is a continuous function on X because F^* is upper semi-continuous as a multifunction and $z = f(x)$, i.e. $f(x) \in \text{Eff}(F(x))$.

Finally, we obtain f is an efficient continuous selection for F and $(x, f(x)) \in \text{Eff}(X, F)$ for all $x \in X$.

(b) In Remark 2 we have shown that $S = F(X)$ is compact and $PO(X, F)$ is nonempty.

Let $v(v_1, v_2, \dots, v_n) \in R^n$ be an ideal point for S and define a function $g: S \times X \rightarrow R$ by $g(y, x) = -\sqrt{\sum_{i=1}^n (y_i - v_i)^2}$ for all $y(y_1, y_2, \dots, y_n) \in S$ and for all $x \in X$. Clearly, g is continuous on $S \times X$ and $F(x) \subset (\{v - R_+^n\})$ for all $x \in X$.

As in the proof of (a) we construct a unique point $z \in F(x)$ such that $z \in \text{Eff}(F(x))$ and an efficient continuous selection f for F such that $(x, f(x)) \in \text{Eff}(X, F)$ for all $x \in X$.

It is easy to show that $f(X)$ is compact; therefore, there exist a point $w \in f(X)$ and a point $x \in X$ such that $d(v, w) = d(v, F(X))$ and $w = f(x)$. Thus we obtain $(x, f(x)) \in PO(X, F)$.

The theorem is proven.

Note that the efficient selections in (a) and (b) are different, but we use one idea for their constructions.

Remark 5. We used the Maximum Theorem, the Distance method (Approximation of the Ideal Point or Compromise Method) and the Euclidean metric in the proof of Theorem 1 [13]. This allows us to transform our set-valued optimization problem into mono-objective optimization problem, see [5], [14], [15] and [16].

It is interesting to note that in Theorem 1(a) we do not prove that $(x, f(x)) \in PO(X, F)$. Recall that $PO(X, F)$ is nonempty when X is compact. If X is no compact, then $PO(X, F) = \emptyset$ is possible.

Remark 6. Theorem 1 shows that the following statement is true: "If $(x, y) \in \text{Graph}(F)$ and $(x, y) \in PO(X, F)$, then $y \in \text{Eff}(F(x))$ ". But, the converse is not true in general.

Definition 3. A multifunction $F: X \Rightarrow R^n$ is called strongly injective if and only if $F(x) \cap F(y) = \emptyset$ for any two distinct points $x, y \in X$.

Remark 7. Let $X \subset R^m$ and $F: X \Rightarrow R^n$ be a strongly injective multifunction. It is easy to show that there exists a function $v: F(X) \rightarrow X$ such that $v(F(z)) = z$ for all $z \in F(X)$. This function v is called an inverse function for F . In view into Definition 3 we understand that the condition $F(x) \cap F(y) = \emptyset$ for any two distinct points $x, y \in X$ is very important. By analogy, for an injective function f we have that $f(x) \neq f(y)$ [15].

It is easy to show that Theorem 1 implies the following corollary.

Corollary 1.

Let $X \subset R^m$ be compact and $F: X \Rightarrow R^n$ be a strongly injective continuous multifunction with compact and convex values. Define f as in Theorems 1(a) or 1(b). Then X and $f(X)$ are homeomorphic.

Proof.

According to Theorems 1(a) or 1(b) there exists an efficient continuous selection f for F . From $F(x) \cap F(y) = \emptyset$ for all $x, y \in X$ such that $x \neq y$ we obtain that f is bijective. The set X is compact and f is a continuous bijective function on X imply f^{-1} is continuous on $f(X)$, i.e. f is homeomorphism.

The corollary is proven.

3. Conclusion

In this paper, we have shown an application of the Maximum Theorem and the Distance method in set-valued optimization for the construction of the two continuous selections and the role of these selections to consider the two solution sets – efficient-graph set and Pareto-optimal set. As a future work, we intend to study the Maximum Theorem as a generator for continuous selections and solutions in multi-criteria and set-valued optimization problems.

Competing Interests

Authors have declared that no competing interests exist.

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