

On Rings where Certain Subsets are Multiplicatively Generated by Idempotents

Abstract

In a Boolean ring, every element is trivially multiplicatively generated by idempotent elements. In this paper, we study the structure of rings in which certain subsets are multiplicatively generated by idempotents or multiplicatively generated by idempotents and nilpotents.

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Introduction

In a Boolean ring every element is trivially a product of idempotents. This naturally motivates the study of the structure of a ring, which as a semi-group, is generated by its idempotents. That is, rings which are multiplicatively generated by idempotents. This also motivated the study of rings which are multiplicatively generated by idempotents and nilpotents in [2]. Theorems A and B below were proved by Putcha and Yaqub [13], and Theorem C below was proved by Abu-Khuzam in [2].

Theorem A: Let R be a ring which is multiplicatively generated by idempotents. If R has an identity element, then R is Boolean.

Theorem B: Let R be a ring which is multiplicatively generated by idempotents. If R is finite, then R is Boolean.

Theorem C: Let R be a ring which is multiplicatively generated by idempotents and nilpotents such that the set N of nilpotent elements is commutative, then N forms an ideal of R and R/N is Boolean.

In this paper, we have tried to generalize the above results by assuming that only certain subsets of R , such as the set of non-nilpotent elements ($R \setminus N$), or the set of elements that are not in the Jacobson radical ($R \setminus J$), or the set of non-central elements ($R \setminus C$), are multiplicatively generated by idempotents. The conclusion in each of Theorems 1, 2, 3, and 4 for the ring is the next best structure to being Boolean.

Main Results

Throughout, R is a ring, N is the set of nilpotents, C is the center, J is the Jacobson radical of R , $C(R)$ is the commutator ideal of R , and Z is the ring of integers. As usual $[x, y]$ will denote the commutator $xy - yx$.

Theorem 1. Let R be a finite ring such that
(1) The set $R \setminus N$ is multiplicatively generated by idempotents.
Then N is an ideal of R and R/N is Boolean.

The following Lemmas are needed for the proof of Theorem 1 .

Lemma 1. Any homomorphic image of a ring satisfying (1) must satisfy (1).

Proof. Let $\phi: R \rightarrow R'$ be a surjective ring homomorphism. Let $x' = \phi(x)$ be any element of R' . Using (1),

$x \in N$ or $x = e_1 \cdot e_2 \dots e_n$, where e_i is idempotent in R . Thus,

$x^n = 0$ or $x = e_1 \cdot e_2 \dots e_n$. So,

$x'^n = \phi(x^n) = 0$ or $x' = \phi(x) = \phi(e_1)\phi(e_2)\dots\phi(e_n)$.

Clearly, $\phi(e_i)^2 = \phi(e_i^2) = \phi(e_i)$. So R' satisfies (1).

The following Lemma is well known:

Lemma 2. Let R be a ring. If the commutator ideal $C(R)$ of R is nil, then the set N of nilpotents is an ideal of R .

Lemma 3. Let R be a ring with identity such that R satisfies (1). Then R is Boolean.

Proof:

Let $a \in N$. Then $a+1$ is invertible and $a+1 \notin N$. Hence, by (1),

$1 + a = e_1 \cdot e_2 \dots e_k$, e_i is idempotent.

Multiplying both sides by e_1 from the left, we get

$e_1(1 + a) = e_1 \cdot e_2 \dots e_k$, since e_1 is idempotent
 $= 1 + a$.

But $1 + a$ has an inverse in R , hence $e_1 = 1$.

Similarly, we show that $e_2 = \dots = e_k = 1$.

Hence $1 + a = 1$ and $a = 0$. So $N=0$, and hence R is multiplicatively generated by idempotents. So, to show that R is a Boolean ring, it is enough to prove that the idempotents are central.

Let e be an idempotent element in R and let $a = ex - exe$. Clearly,

$a^2 = (ex - exe)(ex - exe) = 0$.

So $a = 0$. Therefore, $ex = exe$ for every $x \in R$.

Following the same above argument, we consider $a = xe - exe$ and we show that $xe = exe$ for every x in R . So, $ex = xe$ for all x in R .

Thus e is central and R is Boolean.

97 Proof of Theorem 1:

98

99 Let R be a finite ring satisfying (1).

100

101 Case 1. If $J=0$, then R is a finite semisimple Artinian ring with identity satisfying (1).

102 So R is Boolean, by Lemma 3, and the theorem follows.

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104 Case 2. If $J \neq 0$, then R/J is a semisimple Artinian ring with identity satisfying (1), by

105 Lemma 1. So by Lemma 3, R/J is Boolean and hence commutative. Thus, $xy-yx \in J$

106 for all x, y in R . But J is nil, since R is finite (hence Artinian). Hence,

107 $xy-yx \in N$ for all x, y in R .

108 Since $xy-yx \in J$, therefore $C(R) \subseteq J$, and hence

109 $C(R) \subseteq J \subseteq N$.

110 Thus, The commutator ideal $C(R)$ is nil and N is an ideal of R , by Lemma 2.

111 Using (1) we have

112 $x \in R$ implies that $x \in N$ or $x = e_1 e_2 \dots e_n$, e_i is idempotent.

113 Moreover, $x \notin N$ implies that

114 $x+N = e_1 \cdot e_2 \dots e_n + N$

115 $= (e_1+N) \cdot (e_2+N) \dots (e_n+N)$

116 and clearly, $(e_i+N)^2 = e_i+N$.

117 So every element $x+N$ in R/N is multiplicatively generated by idempotents, and since

118 R/N is finite, R/N is Boolean, by Theorem B. This completes the proof of Theorem 1.

119

120 **Theorem 2.** Let R be a finite ring such that

121 (2) The set $R \setminus J$ is multiplicatively generated by idempotents.

122 Then N is an ideal of R and R/N is Boolean.

123

124 The following Lemmas are needed for the proof of Theorem 2.

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126 Lemma 4. Any homomorphic image of a ring satisfying (2) must satisfy (2).

127

128 *Proof.* Let $\phi: R \rightarrow R'$ be a surjective ring homomorphism. Let $x' \in R' = \phi(R)$,

129 then $x' = \phi(x)$ for some $x \in R$. Now using (2),

130 $x \in J$ or $x = e_1 \cdot e_2 \dots e_n$, where e_i is idempotent in R .

131 Let J' denote the Jacobson radical of R' . It is readily verified that $\phi(J) \subseteq J'$, and

132 thus, for all $x' = \phi(x) \in R' = \phi(R)$,

133 $x' = \phi(x) \in \phi(J) \subseteq J'$ or $\phi(x) = \phi(e_1 e_2 \dots e_n)$.

134 Thus for all $x' = \phi(x) \in R' = \phi(R)$

135 $x' \in J'$ or x' is a product of idempotents.

136 So, R' satisfies (2).

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138 Lemma 5. Let R be a ring with identity such that R satisfies (2).

139 Then R is Boolean.

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141 *Proof.* Let $a \in J$. Then $a+1 \notin J$, and hence

142 $a+1 = e_1 \cdot e_2 \dots e_n$, e_i is idempotent.

143 So, $e_1(a+1) = e_1 \cdot e_2 \dots e_n = a+1$, and thus $e_1=1$, since $a+1$ is invertible.

144 Continue this way to show that $e_2=e_3=\dots=e_n=1$, which implies that $a=0$.

145 So, $J=0$, and thus (2) implies that R is multiplicatively generated by idempotents.
146 Therefore, R is Boolean, by Theorem A.
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148 Proof of Theorem 2
149
150 Let R be a finite ring satisfying (2).
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152 Case 1. If $J=0$, then R is a semisimple Artinian ring and hence has an identity. Thus,
153 by Lemma 5, R is Boolean and the Theorem follows.
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155 Case 2. Suppose $J \neq 0$. Then R/J is a semisimple Artinian ring and satisfies (2) by
156 Lemma 4. Hence R/J is a ring with identity satisfying (2), and thus R/J is Boolean by
157 Lemma 5.
158 So, R/J is commutative, and thus
159 $xy - yx \in J$, for all x, y in R .
160 Hence $C(R) \subseteq J$. But J is nilpotent, since R is Artinian. So
161 $C(R) \subseteq J \subseteq N$,
162 and hence the commutator ideal $C(R)$ is nil, which implies that N is an ideal of R , by
163 Lemma 2.
164
165 To complete the proof, since $J \subseteq N$, $x \notin N$ implies that $x \notin J$ and by (2)
166 $x = e_1 \cdot e_2 \dots e_n$, e_i is idempotent.
167 Hence $x \notin N$ implies that
168 $x + N = e_1 \cdot e_2 \dots e_n + N$
169 $= (e_1 + N) \cdot (e_2 + N) \dots (e_n + N)$
170 where, $(e_i + N)^2 = e_i + N$.
171 Thus, R/N is multiplicatively generated by idempotents, and since R/N is finite, R/N
172 is Boolean, by Theorem B.
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175
176 **Theorem 3.** Let R be a ring with identity 1 such that
177 (3) The set $R \setminus C$ is multiplicatively generated by idempotents.
178 Then R is commutative.
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180 **Proof.** Let e be any idempotent of R and x any element of R . Then,
181 $a = ex - exe \in N$.
182 Suppose that $a \notin C$, then $1 + a \notin C$, and hence using (3),
183 $1 + a = e_1 \cdot e_2 \dots e_n$, where e_i is idempotent. So
184 $e_1(1 + a) = e_1 \cdot e_2 \dots e_n = 1 + a$. This implies that $e_1 = 1$ since $1 + a$ is a unit. Similarly,
185 we get $e_2 = e_3 = \dots = e_n = 1$. So $a = 0$ and $ex = exe$. Similarly we can show that
186 $xe = exe$, and hence $e \in C$.
187 On the other hand, if $a \in C$, then it is easy to show that also the idempotents are
188 central.
189 So, in all cases, the idempotents are central which implies using (3) that R is
190 commutative.
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Theorem 4: Let R be a ring such that

(4) The set $R \setminus J$ is multiplicatively generated by idempotents and nilpotents.

If R satisfies the polynomial identity $x^m = x^{m+1}f(x)$, where f is a polynomial with integer coefficients, and if the set N of nilpotent elements is commutative, then N is an ideal of R and R/N is Boolean.

Proof.

A simple induction shows that $x^m = x^{2m}f(x)^m$. This implies that $e = x^m f(x)^m$ is idempotent for each x in R .

Let $x \in J$. Since J is an ideal then $e = x^m f(x)^m$ is an idempotent element in J . But J contains no nonzero idempotent elements and hence $e = 0$.

So, $x^m f(x)^m = 0$, and hence $x^m = x^{2m}f(x)^m = 0$.

So every element in J is nilpotent. This implies, using (4), that every element in R is multiplicatively generated by idempotents and nilpotents.

Now, using Theorem C, N is an ideal and R/N is Boolean.

Corollary: Let R be a finite ring such that

(5) The set $R \setminus (NUJ)$ is multiplicatively generated by idempotents.

Then N is an ideal of R and R/N is Boolean.

Proof.

Since R is Artinian (being finite), then the Jacobson radical J is nil and hence $NUJ = N$. This implies by (5) that $R \setminus N$ is multiplicatively generated by idempotents. Now, using Theorem 1, N is an ideal of R and R/N is Boolean.

The following example shows the essentiality that R is finite in Theorems 1 and 2, and the essentiality that R has identity in Lemmas 3 and 5.

Example Let R_0 be a ring with identity, and let R be the ring of all $\infty \times \infty$ matrices over R_0 in which at most a finite number of entries are non zero. For every $X \in R$, there exists a finite $n \times n$ matrix A over R_0 such that

$$X = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where } 0\text{'s are zero matrices.}$$

Let L , M and N be matrices in R defined by

$$L = \begin{pmatrix} I_n & I_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0_n & 0_n & 0 \\ A & I_n & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and } N = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$$

It is readily verified that L , M , and N are idempotents.

Clearly for every X in R , $X = LMN$. Thus, every X in R can be expressed as a product of idempotents. However, N is not an ideal, since

235 $S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R$, and $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in N$, but $ST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is not in N .

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245 References

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247 [1] H. Abu-Khuzam and A.Yaqub, "*Structure of certain periodic rings*", Canadian
248 Mathematical Bulletin, Vol. 28, (1985), 120-123.

249

250 [2] H. Abu-Khuzam, *A note on rings which are multiplicatively generated by*
251 *idempotents and nilpotents*, International J. Math & Math Sci. Vol 11, No 1
252 (1988), 5-8.

253

254 [3] H. Abu-Khuzam, "*Commutativity results for periodic rings*", *Acta Math. Hungar.*,
255 58(1991), 273-277.

256

257 [4] H. Abu-Khuzam, H. Bell, and A. Yaqub, "*Commutativity of rings satisfying*
258 *certain polynomial identities*", Bull. Austral. Math. Soc., 44(1991), 273-277.

259

260 [5] H. Abu-Khuzam and Howard Bell, "*On rings with a condition on*
261 *nonperiodic elements*", Result. Math. 43 (2003), 198-201.

262

263 [6] H. Abu-Khuzam and A. Yaqub, *Generalized Boolean and Boolean-Like Rings*,
264 International Journal of Algebra, Vol. 7, no. 9 (2013) 429 – 438.

265

266 [7] H. E. Bell, "*A commutativity study for periodic rings*", *Pacific J. Math.*,
267 70(1977), 29-36.

268

269 [8] I. N. Herstein, *A generalization of a theorem of Jacobson III*, Amer. J. Math.
270 75(1953),105-111.

271

272 [9] I. N. Herstein, *Noncommutative Rings*, The Carus Math. Monographs, John Wiley
273 and Sons, Inc., 1968.

274

275 [10] I. N. Herstein, *Rings With Involution*, University of Chicago Press, Chicago,
276 1976.

277

278 [11] T. Hungerford, *Algebra*, Graduate Texts in Mathemactics. Volume 73, New York:
279 Springer-Verlag, (1980).

280

281 [12] N. Jacobson, *Structure of rings*, Amer. Math. Soc.Colloq. Publications, 37,
282 Providence, RI, 1964.

283

- 284 [13] M.S. Putcha and A. Yaqub, A, *Rings which are Multiplicatively Generated by*
285 *Idempotents*, Communications in Algebra 8(1980), 153-159.
286
287 [14] A. Yaqub, *A generalization of Boolean rings*, International Journal of Algebra,
288 1(2007), 353-362.
289
290
291
292