

A SIXTH-ORDER SELF-STARTING ALGORITHMS FOR SECOND ORDER INITIAL VALUE PROBLEMS OF ODEs

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ABSTRACT

This paper focuses on derivation of three-step block method through collocation and interpolation technique with Chebyshev polynomial as basis function. The procedure yields uniform order finite difference formulae for the solution of second order Initial Value Problems in Ordinary Differential Equations. The analysis of the self-starting method shows that the method is efficient, consistent, zero-stable and, hence convergent. The method is more accurate when compared with existing methods.

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Keyword: Hybrid, Collocation, Interpolation, Block Method.

1. INTRODUCTION

Differential equations play an important role in the modelling of physical problems arising from almost every discipline of study. Owing to this, many researchers have paid attention to study the solutions of these problems using various methods and different approach. Common among these problems in mathematical physics are initial value problems and boundary value problems. The continuous integration algorithms for the numerical solution of the initial value problems (IVPs) have been discussed in the literature. The works of [1] and [2] focused on the construction of continuous multistep methods by employing the multistep collocation approach. The use of finite difference to construct continuous implicit schemes through which the block formulae are derived has been discussed extensively by some scholars such as [3], [4], [5] to mention a few.

The traditional multistep methods including the hybrid ones can be made continuous through the idea of multistep collocation, [1] and [2]. To obtain multiple discrete hybrid method, the continuous implicit hybrid method is evaluated at some selected points involving grid and off-grid points along with its first derivative. For the derivation of the block

methods, the multiple discrete hybrid formulae obtained are solved simultaneously and the resulting equations constituted a block from which a number of explicit methods will be obtained. We have reported recently two-step IVPs solvers, see [6], but in this paper, we extend the scope to three-step.

In what follows, we shall discuss the procedure for the derivation of the three-step block method with its analysis in section three and use it to solve two test problems in section four. Thereafter, conclusions are made.

2. MATERIALS AND METHODS

In this section, we consider the development of three-step hybrid block method for the solution of the form

$$y'' = f(x, y, y'), y'(a) = z_0, y(a) = y_0 \quad (1)$$

where f is a continuous function.

Consider the Chebyshev polynomial in the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j T_j(x) \quad (2)$$

on the partition $a = x_0 < x_1 < \dots < x_n < x_{n+1} < \dots < x_N = b$ of the integration interval $[a, b]$, with a constant step size h , given by $h = x_{n+1} - x_n$; $n = 0, 1, \dots, N - 1$. The second derivative of (2) is given by

$$y''(x) = \sum_{j=0}^{r+s-1} a_j T_j''(x) \quad (3)$$

where $x \in [a, b]$, the a_j 's are real unknown parameters to be determined and $r + s$ is the sum of the number of collocation and interpolation points. The trial functions $T_n(x)$, $n = 0(1)7$ are clearly stated below.

$$T_0(x) = 1$$

$$T_1(x) = t$$

$$T_2(x) = 2t^2 - 1$$

$$T_3(x) = 4t^3 - 3t$$

$$T_4(x) = 8t^4 - 8t^2 + 1$$

$$T_5(x) = 16t^5 - 20t^3 + 5t$$

$$T_6(x) = 32t^6 - 48t^4 + 18t^2 - 1$$

$$T_7(x) = 64t^7 - 112t^5 + 56t^3 - 7t$$

$$\text{where } t = \frac{2x - 2x_n - h}{h}.$$

In what immediately follows, we shall consider three-step hybrid methods.

2.1 THREE-STEP HYBRID METHOD

To derive this method, we shall interpolate (2) at two points and collocate (3) at six points. Interpolating (2) at $x_{n+\frac{1}{2}}$ & $x_{n+\frac{3}{2}}$ and collocating (2) at $x_n, x_{n+\frac{1}{2}}, x_{n+1}, x_{n+\frac{3}{2}}, x_{n+2}$ & x_{n+3} yield

$$y_{n+\frac{1}{2}} = a_0 - a_2 + a_4 - a_6$$

$$y_{n+\frac{3}{2}} = a_0 + 2a_1 + 7a_2 + 26a_3 + 97a_4 + 362a_5 + 1351a_6 + 5042a_7$$

$$f_n = y_n'' = (16a_2 - 96a_3 + 320a_4 - 800a_5 + 1680a_6 - 3136a_7) / h^2$$

$$f_{n+\frac{1}{2}} = y_{n+\frac{1}{2}}'' = (16a_2 - 64a_4 + 144a_6) / h^2$$

$$f_{n+1} = y_{n+1}'' = (16a_2 + 96a_3 + 320a_4 + 800a_5 + 1680a_6 + 3136a_7) / h^2$$

$$f_{n+\frac{3}{2}} = y_{n+\frac{3}{2}}'' = (16a_2 + 192a_3 + 1472a_4 + 9280a_5 + 52368a_6 + 275072a_7) / h^2$$

$$f_{n+2} = y_{n+2}'' = (16a_2 + 288a_3 + 3392a_4 + 33120a_5 + 290448a_6 + 2374848a_7) / h^2$$

$$f_{n+3} = y_{n+3}'' = (16a_2 + 480a_3 + 9536a_4 + 157600a_5 + 2342544a_6 + 32486720a_7) / h^2$$

which we write as a system of eight equations in matrix form

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 7 & 26 & 97 & 362 & 1351 & 5042 \\ 0 & 0 & 16 & -96 & 320 & -800 & 1680 & -3136 \\ 0 & 0 & 16 & 0 & -64 & 0 & 144 & 0 \\ 0 & 0 & 16 & 96 & 320 & 800 & 1680 & 3136 \\ 0 & 0 & 16 & 192 & 1472 & 9280 & 52368 & 275072 \\ 0 & 0 & 16 & 288 & 3392 & 33120 & 290448 & 2374484 \\ 0 & 0 & 16 & 480 & 9536 & 157600 & 2342544 & 32486720 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{2}} \\ h^2 f_n \\ h^2 f_{n+\frac{1}{2}} \\ h^2 f_{n+1} \\ h^2 f_{n+\frac{3}{2}} \\ h^2 f_{n+2} \\ h^2 f_{n+3} \end{bmatrix} \quad (4)$$

Solving for a_j 's and substituting the resulting equations into (2), we have

$$y(x) = \alpha_1(t) y_{n+\frac{1}{2}} + \alpha_3(t) y_{n+\frac{3}{2}} + h^2 (\beta_0(t) f_n + \beta_1(t) f_{n+\frac{1}{2}} + \beta_1(t) f_{n+1} + \beta_3(t) f_{n+\frac{3}{2}} + \beta_2(t) f_{n+2} + \beta_3(t) f_{n+3}) \quad (5)$$

Evaluating (5) at x_n, x_{n+1}, x_{n+2} and x_{n+3} , we have

$$y_n = \frac{1}{2} y_{n+\frac{1}{2}} - \frac{1}{2} y_{n+\frac{3}{2}} + \frac{h^2}{5760} (106 f_n + 1320 f_{n+\frac{1}{2}} + 621 f_{n+1} + 136 f_{n+\frac{3}{2}} - 24 f_{n+2} + f_{n+3}) \quad (6)$$

$$y_{n+1} = \frac{1}{2} y_{n+\frac{1}{2}} + \frac{1}{2} y_{n+\frac{3}{2}} + \frac{h^2}{1920} (f_n - 24 f_{n+\frac{1}{2}} - 194 f_{n+1} - 24 f_{n+\frac{3}{2}} + f_{n+2}) \quad (7)$$

$$y_{n+2} = \frac{3}{2} y_{n+\frac{3}{2}} - \frac{1}{2} y_{n+\frac{1}{2}} + \frac{h^2}{5760} (-4 f_n + 72 f_{n+\frac{1}{2}} + 711 f_{n+1} + 1256 f_{n+\frac{3}{2}} + 126 f_{n+2} - f_{n+3}) \quad (8)$$

$$y_{n+3} = \frac{5}{2} y_{n+\frac{3}{2}} - \frac{3}{2} y_{n+\frac{1}{2}} + \frac{h^2}{1152} (49 f_n - 264 f_{n+\frac{1}{2}} + 1188 f_{n+1} - 200 f_{n+\frac{3}{2}} + 1317 f_{n+2} + 70 f_{n+3}) \quad (9)$$

To formulate the block schemes, we find the derivative of (5) as

$$y'(x) = \alpha'_{\frac{1}{2}}(t)y_{n+\frac{1}{2}} + \alpha'_{\frac{3}{2}}(t)y_{n+\frac{3}{2}} + \beta'_0(t)f_n + \beta'_{\frac{1}{2}}(t)f_{n+\frac{1}{2}} + \beta'_{\frac{1}{2}}(t)f_{n+1} + \beta'_{\frac{3}{2}}(t)f_{n+\frac{3}{2}} + \beta'_{\frac{3}{2}}(t)f_{n+2} + \beta'_{\frac{3}{2}}(t)f_{n+3} \quad (10)$$

Evaluating (10) at $x_n, x_{n+\frac{1}{2}}, x_{n+1}, x_{n+\frac{3}{2}}, x_{n+2}$ & x_{n+3} results into discrete schemes which are combined with equations (6-9)

and, when solved simultaneously yield

$$\left. \begin{aligned} y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}y'_n + \frac{355h^2}{5772}f_n + \frac{263h^2}{2520}f_{n+\frac{1}{2}} - \frac{297h^2}{4313}f_{n+1} + \frac{143h^2}{3780}f_{n+\frac{3}{2}} - \frac{84h^2}{8171}f_{n+2} + \frac{15h^2}{33914}f_{n+3} \\ y_{n+1} &= y_n + y'_n + \frac{219h^2}{1543}f_n + \frac{134h^2}{315}f_{n+\frac{1}{2}} - \frac{11h^2}{84}f_{n+1} + \frac{82h^2}{945}f_{n+\frac{3}{2}} - \frac{61h^2}{9520}f_{n+2} + \frac{h^2}{945}f_{n+3} \\ y_{n+\frac{3}{2}} &= y_n + \frac{3}{2}y'_n + \frac{326h^2}{1473}f_n + \frac{27h^2}{35}f_{n+\frac{1}{2}} + \frac{81h^2}{8960}f_{n+1} + \frac{9h^2}{56}f_{n+\frac{3}{2}} - \frac{351h^2}{8960}f_{n+2} + \frac{3h^2}{1792}f_{n+3} \\ y_{n+2} &= y_n + 2y'_n + \frac{h^2}{945}(284f_n + 1056f_{n+\frac{1}{2}} + 162f_{n+1} + 416f_{n+\frac{3}{2}} - 30f_{n+2} + 2f_{n+3}) \\ y_{n+3} &= y_n + 3y'_n + \frac{h^2}{280}(141f_n + 432f_{n+\frac{1}{2}} + 324f_{n+1} + 48f_{n+\frac{3}{2}} + 297f_{n+2} + 18f_{n+3}) \\ y'_{n+\frac{1}{2}} &= y'_n + \frac{h}{5760}(959f_n + 2800f_{n+\frac{1}{2}} - 1461f_{n+1} + 784f_{n+\frac{3}{2}} - 211f_{n+2} + 9f_{n+3}) \\ y'_{n+1} &= y'_n + \frac{h}{1080}(169f_n + 768f_{n+\frac{1}{2}} + 99f_{n+1} + 64f_{n+\frac{3}{2}} - 21f_{n+2} + f_{n+3}) \\ y'_{n+\frac{3}{2}} &= y'_n + \frac{h}{640}(103f_n + 432f_{n+\frac{1}{2}} + 243f_{n+1} + 208f_{n+\frac{3}{2}} - 27f_{n+2} + f_{n+3}) \\ y'_{n+2} &= y'_n + \frac{h}{45}(7f_n + 32f_{n+\frac{1}{2}} + 12f_{n+1} + 32f_{n+\frac{3}{2}} + 7f_{n+2}) \\ y'_{n+3} &= y'_n + \frac{h}{40}(11f_n + 81f_{n+1} - 64f_{n+\frac{3}{2}} + 81f_{n+2} + 11f_{n+3}) \end{aligned} \right\} \quad (11)$$

3.

ANALYSIS OF THE METHOD

Hereunder, the basic properties of the derived schemes are discussed.

The implicit schemes (6-9) derived are discrete schemes belonging to the class of LMM of the form

$$\sum_{j=0}^k \alpha_j y_{n+j} = h^2 \sum_{j=0}^k \beta_j f_{n+j} \quad (12)$$

Associated with (12) is the linear differential operator L defined by

$$L[y(x); h] = \sum_{j=0}^k [\alpha_j y(x+jh) - h^2 \beta_j y''(x+jh)] \quad (13)$$

Expanding (13) by Taylor series, we have

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + \dots + C_q h^q y^{(q)}(x) + \dots \quad (14)$$

where

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_k$$

$$C_1 = \alpha_1 + 2\alpha_2 + \dots + k\alpha_k$$

$$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + \dots + k^2\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \dots + \beta_k)$$

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$$C_p = \frac{1}{p!}(\alpha_1 + 2^p\alpha_2 + \dots + k^p\alpha_k) - \frac{1}{(q-2)!}(\beta_1 + 2^{q-2}\beta_2 + \dots + k^{q-2}\beta_k), q \geq 3$$

3.1 ORDER AND ERROR CONSTANT

Definition 1: The LMM (12) is said to be of order p if $C_0 = C_1 = C_2 = \dots = C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$ is the error constant, see [7]. According to this definition, the discrete schemes (6-9) have order $p = (6, 6, 6, 6)^T$ with error constants

$$\left(\frac{-11}{688128}, \frac{-31}{30965760}, \frac{59}{10321920}, \frac{-475}{688128} \right)^T.$$

3.2 CONSISTENCY

Definition 2: The LMM (12) is said to be consistent if it is of order $p \geq 1$ and its first and second characteristic polynomials

defined as $\rho(z) = \sum_{j=0}^k \alpha_j z^j$ and $\sigma(z) = \sum_{j=0}^k \beta_j z^j$ where z satisfies

$$(i) \sum_{j=0}^k \alpha_j = 0, (ii) \rho(1) = \rho'(1) = 0, (iii) \rho''(1) = 2!\sigma(1), \text{ see Lambert (1973)}.$$

The discrete schemes derived are all of order greater than one and satisfy the conditions (i) - (iii).

3.3 ZERO STABILITY

Definition 3: The LMM (12) is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root of modulus one has multiplicity not greater than two.

All the roots of the derived schemes have been verified to be less than or equal to 1 and $|z| = 1$, simple.

4. NUMERICAL EXAMPLES

Here, we consider the application of the derived schemes to three test problems for the efficiency and accuracy of the method implemented as block method.

Problem 1

$$y'' = y + xe^{3x}, y(0) = \frac{-3}{32}, y'(0) = \frac{-5}{32}, h = 0.0025$$

$$\text{Exact Solution : } y(x) = \frac{4x-3}{32e^{-3x}}.$$

Source : Adesanya et al (2009).

Problem 2

$$y'' + \frac{6}{x}y' + \frac{4}{x^2}y = 0, y(1) = 1, y'(1) = 1, h = \frac{0.1}{32}$$

$$\text{Exact Solution : } y(x) = \frac{5}{3x} - \frac{2}{3x^4}.$$

Source : Yahaya and Badmus (2009)

Problem 3

$$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}, h = 0.0025$$

$$\text{Exact Solution : } y(x) = 1 + \frac{1}{2} \left(\frac{2+x}{2-x} \right).$$

Source : Areo and Adeniyi (2013)

TABLE OF RESULTS

Table 1: Comparison of Errors for Example 1

X	Computed	Exact	Error	Error in [5]
0.0025	-0.094140925713285	-0.094140915761849	5.022371407648052e-14	7.020D-14
0.0050	-0.094532404142349	-0.094532404142339	0	1.217D-13
0.0075	-0.094924451608460	-0.094924451608388	1.387778780781446e-17	3.396D-12
0.0100	-0.095317044390747	-0.095317044390700	4.711508960753008e-14	8.122D-12
0.0125	-0.095710168480981	-0.095710168480981	3.608224830031759e-16	1.453D-11
0.0150	-0.096103809629114	-0.096103809629113	3.747002708109903e-16	2.233D-11
0.0175	-0.096497953340360	-0.096497953340316	4.370115380680772e-14	3.156D-11
0.0200	-0.096892584872265	-0.096892584872264	6.245004513516506e-16	4.220D-11
0.0225	-0.097287689232185	-0.097287689232184	5.967448757360216e-16	5.421D-11
0.0250	-0.098475685533470	-0.097683251173920	4.000966224992908e-14	6.754D-11

Table 2: Comparison of Errors for Example 2

X	Exact	Computed	Error	Error in [8]
0.003125	1.003076525857696	1.003076525849324	8.371747739488455e-12	1.40E-09
0.00625	1.006057503083516	1.006057503083517	2.220446049250313e-16	4.09E-09
0.009375	1.008944995088837	1.008944995088838	8.881784197001252e-16	NA
0.0125	1.011741018167989	1.011741018160148	7.840394999902856e-12	3.63E-08
0.015625	1.014447542686414	1.014447542686415	1.110223024625157e-15	8.18E-08
0.01875	1.017066494235673	1.017066494235674	1.332267629550188e-15	NA
0.021875	1.019599754756288	1.019599754748944	7.344125307895411e-12	NA
0.025	1.022049163629432	1.022049163629434	1.998401444325282e-15	1.52E-07
0.028125	1.024416518738403	1.024416518738405	2.442490654175344e-15	NA
0.03125	1.026703577500806	1.026703577493925	6.880718217416870e-12	NA

Table 3: Comparison of Errors for Example 3

X	Exact	Computed	Error	Error in [10]
0.0025	1.001250000651042277	1.001250000651042277	0	1.60316e-13
0.0050	1.0025000052083528647	1.0025000052083528647	0	4.296563e-13
0.0075	1.0037500175782733169	1.0037500175782733169	0	2.575717e-14
0.0100	1.0050000416672916778	1.0050000416672926824	1.0046e-15	1.424638e-12
0.0125	1.0062500813821157352	1.0062500813821157352	0	3.539835e-12
0.0150	1.0075001406297462845	1.0075001406297462844	1.00e-19	6.332490e-12
0.0175	1.0087502233175504064	1.0087502233175524158	2.0094e-15	9.675149e-12
0.0200	1.010000333353334762	1.010000333353334762	0	1.475506e-11
0.0225	1.0112504746454189079	1.0112504746454189078	1.00e-19	2.297873e-11
0.0250	1.0125006511027086357	1.0125006511027116502	3.0145e-15	3.483280e-11

5. CONCLUSION

The derivation of convergent three-step hybrid method has been considered. The method which is efficient and accurate compared favourable well with the existing methods. Tables 1, 2 and 3 present the numerical results of the derived schemes which have demonstrated an apparent agreement between the exact solutions and the solutions obtained through the methods. In Table 2, comparison would have been made with the work of [9] but we discovered that the method of [8] was much more efficient and accurate.

Competing Interests

Authors hereby declare that there is no existence of competing interests.

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