

Oscillatory Blood Flow in Convergent and Divergent Channels,

Part 1: Effects of Pulse Amplitude and Local Constriction Height

ABSTRACT Oscillatory blood flow in convergent and divergent channels is investigated. The problem which involves a set of non-linear differential equations is handled analytically using the method of regular perturbation series solutions. Solutions are obtained for the velocities, pressure and wall shear stress, and are analyzed graphically. It is found that the variations in the pulse amplitude and height of constriction reduce the axial velocity and pressure but increase the radial velocity and wall shear stress. More so, it is observed that flow separation occurs in the radial velocity and pressure structures in the convergent and divergent regions respectively, when the height of the constriction are varied.

Keywords: oscillatory flow, pulse amplitude, constriction height, convergent and divergent channels

1. INTRODUCTION

The oscillatory flow of an incompressible viscous fluid through convergent and divergent channels has applications in engineering and biological systems.

The human cardiovascular system is infested by a number of diseases. Some of which are due to the nature of the food we eat. An example of such is the atherosclerosis of the artery. This involves the hardening of the artery due to deposition of lipids (a generic name for esters) on the arterial internal walls. The calcification of the lipids leads to lose of distensibility at the point of infection. Also, the progressive encroachment of the plaque on the internal walls tends to block the passage of blood to other parts of the body. This imposes extra load on the heart musculature ([1], [2]). To maintain the peripheral flow, the heart enlarges itself, and thus, increasing the amplitude of the pulse or pressure wave, which subsequently leads to abrupt rise in the flow variables ([2], [3]). More so, the encroachment of the plaque leads to distortion of the tapered geometry of the artery to other forms, such as the idealized locally constricted (see ([4], [5]) and peristaltic channels (see [6], [7]). The plaque is found predominantly in the internal carotid artery which supplies blood to the brain; the coronary artery which supplies blood to the cardiac muscles, and the femoral artery which supplies blood to the lower limbs [1].

[8] investigated the fluid mechanical aspect of the pulsatile flow of an incompressible fluid through a tapered tube, locally constricted and peristaltic channels. Their model did not consider the flow behavior in the regions before and after the peak of the stenosis height, which are idealized as the convergent and divergent channels. Therefore, we are motivated to examine the flow behaviour in these channels.

Different aspects of the flow of an incompressible viscous fluid through tubes of varying cross-sections have been reported in literature. For example, [9] examined the characteristics of pulsating flow in a confined channel, and observed that a given rate of mass flow is obtained in oscillating motion under an average pressure gradient as in steady flow. [8] investigated the pulsating flow of an incompressible viscous fluid in tubes of varying cross-sections using the perturbation series expansion solutions which they developed, and observed that flow separation occurs in the wall shear stress structure in all geometries. [10] studied the steady laminar flow of an incompressible fluid in a channel of varying width with permeable boundaries, and observed that the fluid absorbed at the wall has a strong influence in reducing the occurrence of initial flow separation. [11] considered the pulsating flow in stenotic vessels using the Reynolds-averaged Navier-Stokes approach. Their predicted a wall shear stress peak at the throat of the stenosis with maximum values observed distal to the stenosis where the flow separation occurred. [12] examined the influence of heat and mass transfer on the flow of blood (which they considered as a Phan-Tien-Tanner fluid) through convergent tapering, divergent tapering and non-tapered arteries for different parameters of

interest, and noticed among others, that the shearing stress at the stenosis throat increases with an increase in the external parameter, volumetric flow and Weissenberg number.

The purpose of this paper is to address the effects of the amplitude of the pressure wave and height of constriction on the flow velocity, pressure and wall shear stress structure in the regions before and after the peak of the constriction using the perturbation series solutions of the form developed by [9].

This paper is organized in the following manner: section 2 is the methodology; section 3 gives the results and discussion, while section 4 holds the conclusions.

2. METHODOLOGY

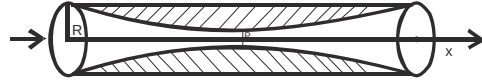


Fig.1 The convergent and divergent models of the sclerotic artery.

We consider the flow of blood in axisymmetric convergent and divergent channels whose radii vary slowly with the cross-sections. The fluid viscosity is a function of temperature. With the loss of distensibility, the channels now have rigid walls. At the entrance of the tube, we assume the flow to be fully developed and pulsating with a prescribed periodic frequency β and time average volume flux Q . If (R, θ, X) and (u, v, w) are the polar cylindrical orthogonal coordinates and vector components, respectively, assuming the flow is symmetrical about the X -axis, then for a two-dimensional flow situation the continuity, momentum and energy equations describing the flow are:

$$\frac{1}{R} \frac{\partial}{\partial R} (Ru) + \frac{\partial u}{\partial X} = 0 \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial R} + w \frac{\partial u}{\partial X} = -\frac{\partial p}{\partial R} + \nu \left(\frac{\partial^2 u}{\partial R^2} + \frac{1}{R} \frac{\partial u}{\partial R} + \frac{u}{R^2} + \frac{\partial^2 u}{\partial X^2} \right) \quad (2)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial R} + w \frac{\partial w}{\partial X} = -\frac{1}{\rho} \frac{\partial p}{\partial X} + \nu \left(\frac{\partial^2 w}{\partial R^2} + \frac{1}{R} \frac{\partial w}{\partial R} + \frac{\partial^2 w}{\partial X^2} \right) \quad (3)$$

where ρ the density, p the pressure, ν the kinematic viscosity of the fluid, and t is the time. $R=0$ is the centre or symmetric axis of the tube; $R = a_o(X, t)$, which is an arbitrary function of X and t is the cross-sectional radius of the tube; a_o is the characteristic radius of the tube, and t is the time.

The boundary conditions are:

$$(i) \quad w = 0, \quad \frac{\partial u}{\partial R} = 0 \quad \text{at } R=0 \quad (4)$$

(ii) there is no tangential motion at the wall of the tube ie:

$$u(a_o, t) = 0 \text{ at } R = a_o(X, t) \quad (5)$$

(iii) the flux across a cross-section of the tube is prescribed as:

$$a_o(X, t) \int_0^{2\pi} dR \int_0^{2\pi} R u d\theta = 2 \pi \psi_o (1 + k e^{i\beta t}) \quad (6)$$

where ψ_o is a constant, k is the amplitude of the pulse which is assumed to be small, and β is the frequency of oscillation.

We eliminate the pressure terms in equations (2) and (3) by taking the $\frac{\partial}{\partial R}$ of equation (2) and $\frac{\partial}{\partial X}$ of equation (3), then subtracting the first result from the second one, we have

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial R} - \frac{\partial w}{\partial X} \right) + \frac{\partial u}{\partial X} \left(\frac{\partial u}{\partial R} - \frac{\partial w}{\partial X} \right) + \frac{\partial w}{\partial R} \left(\frac{\partial u}{\partial R} - \frac{\partial w}{\partial X} \right) = \\ & \nu \left[\frac{\partial^2}{\partial X^2} \left(\frac{\partial u}{\partial R} - \frac{\partial w}{\partial X} \right) + \frac{\partial^2}{\partial R^2} \left(\frac{\partial u}{\partial R} - \frac{\partial w}{\partial X} \right) + \frac{1}{R} \frac{\partial}{\partial R} \left(\frac{\partial u}{\partial R} - \frac{\partial w}{\partial X} \right) - \frac{1}{R^2} \left(\frac{\partial u}{\partial R} - \frac{\partial w}{\partial X} \right) \right] \quad (7) \end{aligned}$$

Also, the pressure expression in the axial direction can be obtained from equation (3), as

$$\frac{\partial p}{\partial X} = \nu \left(\frac{\partial^2 w}{\partial R^2} + \frac{1}{R} \frac{\partial w}{\partial R} + \frac{\partial^2 w}{\partial X^2} \right) - \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial R} + w \frac{\partial w}{\partial X} \right) \quad (8)$$

Introducing the stream function ψ and vortices Ω , respectively i.e.

$$u = -\frac{1}{R} \frac{\partial \psi}{\partial R}, \quad w = \frac{1}{R} \frac{\partial \psi}{\partial X} \quad (9)$$

$$\Omega = \frac{\partial u}{\partial R} - \frac{\partial w}{\partial X} = \frac{1}{R} \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R^2} \frac{\partial \psi}{\partial R} + \frac{1}{R} \frac{\partial^2 \psi}{\partial X^2} \quad (10)$$

into equations (7) and (8), we have

$$\frac{\partial \Omega}{\partial t} - \frac{1}{R} \frac{\partial \Omega}{\partial X} \frac{\partial \psi}{\partial R} + \frac{1}{R} \frac{\partial \Omega}{\partial R} \frac{\partial \psi}{\partial X} + \frac{\Omega}{R^2} \frac{\partial \psi}{\partial X} = \nu \left[\frac{\partial^2 \Omega}{\partial X^2} + \frac{\partial^2 \Omega}{\partial R^2} + \frac{1}{R} \frac{\partial \Omega}{\partial R} - \frac{\Omega}{R^2} \right] \quad (11)$$

$$\frac{\partial p}{\partial X} = \nu \left(-\frac{1}{R} \frac{\partial^3 \psi}{\partial X^2 \partial R} - \frac{1}{R} \frac{\partial^3 \psi}{\partial R^3} - \frac{1}{R^2} \frac{\partial^2 \psi}{\partial R^2} \right) - \frac{1}{R^2} \frac{\partial \psi}{\partial R} \frac{\partial^2 \psi}{\partial X \partial R} + \frac{1}{R^2} \frac{\partial \psi}{\partial X} \frac{\partial^2 \psi}{\partial R^2} + \frac{1}{R} \frac{\partial^2 \psi}{\partial t \partial R} \quad (12)$$

with the boundary conditions as

$$\psi = 0, \quad \frac{1}{R} \frac{\partial \psi}{\partial X} = 0, \quad \frac{1}{R} \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R^2} \frac{\partial \psi}{\partial R} = 0 \quad \text{at } R = 0 \quad (13)$$

$$\frac{\partial \psi}{\partial R} = 0, \quad \psi = \psi_o (1 + k e^{i\beta t}) \quad \text{at } R = a(X, t) \quad (14)$$

Also, we assume that the cross-section of the tube in the model vary in the axial direction, and for which we take $a(X, t) = a_o s(\varepsilon X/a_o, t)$, $r = R_o(1 + \varepsilon f(x, t))$, $0 \leq r \leq 1$, $0 \leq x \leq 1$ where s is an

arbitrary function of X ; a is the variation in the r along the axial direction; $0 < \varepsilon = \frac{a_o}{L} \ll 1$ is a small

dimensionless parameter that characterizes the slow variation in the channel radius; L defines the channel characteristic length; $\varepsilon = 0$ corresponds to a tube with constant radius. As ε increases from zero the variation of ψ in the axial direction depends upon εX instead of X .

Similarly, introducing the following non-dimensionalized variables

$$r = \frac{R}{a_o}, \quad x = \varepsilon \frac{X}{a_o}, \quad T = \beta t, \quad \phi(r, x, T) = \frac{\psi}{\psi_o},$$

$$\omega(r, x, T) = \frac{\Omega a_o^3}{\psi_o}, \quad \text{Re} = \frac{\psi_o}{a_o \nu}, \quad \eta = \frac{\beta}{a_o^2 \nu}, \quad a_o = \frac{a}{s}$$

where Re is the Reynolds number of the flow, η is a dimensionless number for the frequency of oscillation, T is the dimensionless time, ε is the height of constriction, ϕ and ω are the dimensionless stream function and vortices respectively; into equations (8) - (12) respectively, we have

$$u = -\frac{1}{r} \frac{\partial \phi}{\partial r} \quad (15)$$

$$w = -\frac{1}{r} \frac{\partial \phi}{\partial x} \quad (16)$$

$$\omega = \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r^2} \frac{\partial \phi}{\partial r} \quad (17)$$

$$\eta \frac{\partial \omega}{\partial T} + \frac{\text{Re}}{r} \varepsilon \left(\frac{\partial \omega}{\partial r} \frac{\partial \phi}{\partial x} + \frac{\partial \omega}{\partial x} \frac{\partial \phi}{\partial r} + \frac{\omega}{r} \frac{\partial \phi}{\partial x} \right) = \frac{1}{r^2} \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} - \frac{1}{r} \omega \quad (18)$$

$$\frac{\partial p}{\partial X} = -\frac{\text{Re}}{\varepsilon} \left(-\frac{1}{r} \frac{\partial^3 \phi}{\partial r^3} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial r^2} \right) + \frac{\eta}{\varepsilon r} \frac{\partial^2 \phi}{\partial r \partial T} \quad (19)$$

Equation (19) tends to suggest that the pressure field is of order $\frac{1}{\varepsilon}$. Also, the $O(\varepsilon)^2$ terms are assumed to be very small, therefore, they are neglected.

Similarly, the wall shear stress is given as

$$\tau_w = \rho \nu \Omega \quad (\text{see [8]}) \text{ yielding}$$

$$\tau_w = \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \phi}{\partial r} \quad (20)$$

The boundary conditions become:

$$\phi = 0, \quad \frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial r} \right) = 0 \quad \text{at } r = 0 \quad (21)$$

$$\phi = 1 + k e^{iT}, \quad \frac{\partial \phi}{\partial r} = 0 \quad \text{at } r = s(x, T) \quad (22)$$

Now, we shall seek for perturbation series solutions about the small parameter ε . In particular, we shall use the form developed by [9], and which is of the form:

$$f = f^{(0)} + k e^{iT} \bar{f}^{(0)} + \varepsilon \left(f^{(1)} + k e^{iT} \bar{f}^{(1)} \right) + \dots \quad (23)$$

where f_n represents ω and ϕ .

Substituting equation (23) into equations (17), (18), (21) and (22), we have: the zeroth order terms as

$$\omega^{(0)} = \frac{1}{r} \left(\frac{\partial^2 \phi^{(0)}}{\partial r^2} - \frac{1}{r} \frac{\partial \phi^{(0)}}{\partial r} \right) \quad (24)$$

$$\omega^{(0)} = \frac{1}{r} \left(\frac{\partial^2 \bar{\phi}^{(0)}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\phi}^{(0)}}{\partial r} \right) \quad (25)$$

$$\frac{\partial^2 \omega^{(0)}}{\partial r^2} + \frac{1}{r} \frac{\partial \omega^{(0)}}{\partial r} - \frac{1}{r^2} \omega^{(0)} = 0 \quad (26)$$

$$\frac{\partial^2 \bar{\omega}^{(0)}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\omega}^{(0)}}{\partial r} - \left(\lambda^2 + \frac{1}{r} \right) \bar{\omega}^{(0)} = 0 \quad (27)$$

where $\lambda^2 = i\eta$
with the boundary conditions are:

$$\left. \begin{aligned} \phi^{(0)} = \bar{\phi}^{(0)} = 0 \\ \frac{1}{r} \frac{\partial \phi^{(0)}}{\partial x} = \frac{1}{r} \frac{\partial \bar{\phi}^{(0)}}{\partial x} = 0 \\ \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi^{(0)}}{\partial r} \right) = \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \bar{\phi}^{(0)}}{\partial r} \right) = 0 \end{aligned} \right\} \quad \text{at } r = 0 \quad (28)$$

$$\left. \begin{aligned} \frac{\partial \phi^{(0)}}{\partial r} = \frac{\partial \bar{\phi}^{(0)}}{\partial r} = 0 \\ \phi^{(0)} = \bar{\phi}^{(0)} = 1 \end{aligned} \right\} \quad \text{at } r = s \quad (29)$$

while the first order terms are

$$\omega^{(1)} = \frac{1}{r} \left(\frac{\partial^2 \phi^{(1)}}{\partial r^2} - \frac{1}{r} \frac{\partial \phi^{(1)}}{\partial r} \right) \quad (30)$$

$$\bar{\omega}^{(1)} = \frac{1}{r} \left(\frac{\partial^2 \bar{\phi}^{(1)}}{\partial r^2} - \frac{1}{r} \frac{\partial \bar{\phi}^{(1)}}{\partial r} \right) \quad (31)$$

$$\frac{\partial^2 \omega^{(1)}}{\partial r^2} + \frac{1}{r} \frac{\partial \omega^{(1)}}{\partial r} - \frac{1}{r} \omega^{(1)} = \text{Re} \left[\frac{\partial \phi^{(0)}}{\partial r} \frac{\partial \omega^{(0)}}{\partial x} + \frac{\partial \phi^{(0)}}{\partial x} \left(\frac{\omega^{(0)}}{r} - \frac{\partial \omega^{(0)}}{\partial r} \right) \right] \quad (32)$$

$$\frac{\partial^2 \bar{\omega}^{(1)}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\omega}^{(1)}}{\partial r} - \left(\lambda^2 + \frac{1}{r^2} \right) \bar{\omega}^{(1)} = \text{Re} \left[\frac{\partial \phi^{(0)}}{\partial r} \frac{\partial \omega^{(0)}}{\partial x} + \frac{\partial \bar{\phi}^{(0)}}{\partial r} \frac{\partial \omega^{(0)}}{\partial x} - \frac{\partial \bar{\omega}^{(0)}}{\partial x} \left(\frac{\partial \omega^{(0)}}{\partial r} - \frac{\omega^{(0)}}{r} \right) - \frac{\partial \bar{\phi}^{(0)}}{\partial x} \left(\frac{\partial \omega^{(0)}}{\partial r} - \frac{\omega^{(0)}}{r} \right) \right] \quad (33)$$

with boundary conditions

$$\left. \begin{aligned}
\phi^{(1)} = \bar{\phi}^{(1)} &= 0 \\
\frac{1}{r} \frac{\partial \phi^{(1)}}{\partial x} = \frac{1}{r} \frac{\partial \bar{\phi}^{(1)}}{\partial x} &= 0 \\
\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi^{(1)}}{\partial r} \right) = \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \bar{\phi}^{(1)}}{\partial r} \right) &= 0
\end{aligned} \right\} \quad \text{at } r = 0 \quad (34)$$

$$\left. \begin{aligned}
\frac{\partial \phi^{(1)}}{\partial r} = \frac{\partial \bar{\phi}^{(1)}}{\partial r} &= 0 \\
\phi^{(1)} = \bar{\phi}^{(1)} &= 0
\end{aligned} \right\} \quad \text{at } r = s \quad (35)$$

164
165 **More so**, by equation (22), equations (15), (16), (19) and (20) become

$$166 \quad u = -\frac{1}{r} \frac{\partial \phi}{\partial r} = -\frac{1}{r} \frac{\partial}{\partial r} \left(\phi^{(o)} + ke^{iT} \phi^{(o)} + \varepsilon \phi^{(1)} + \varepsilon ke^{It} \phi^{(1)} \right) \quad (36)$$

$$167 \quad w = \frac{1}{r} \frac{\partial \phi}{\partial x} = \frac{1}{r} \frac{\partial s}{\partial x} \cdot \frac{\partial \phi}{\partial s} = \frac{1}{r} \frac{\partial s}{\partial x} \cdot \frac{\partial}{\partial s} \left(\phi^{(o)} + ke^{iT} \phi^{(o)} + \varepsilon \phi^{(1)} + \varepsilon ke^{It} \phi^{(1)} \right) \quad (37)$$

$$168 \quad \frac{\partial p}{\partial x} = -\frac{\text{Re}}{\varepsilon} \left[-\frac{1}{r} \frac{\partial^3 \phi^{(o)}}{\partial r^3} + \frac{1}{r^2} \frac{\partial^2 \phi^{(o)}}{\partial r^2} + ke^{iT} \left(-\frac{1}{r} \frac{\partial^3 \bar{\phi}^{(o)}}{\partial r^3} + \frac{1}{r^2} \frac{\partial^2 \bar{\phi}^{(o)}}{\partial r^2} \right) \right] \quad (38)$$

$$169 \quad \tau_w = -\left(\frac{1}{r} \frac{\partial^2 \phi^{(o)}}{\partial r^2} + \frac{1}{r^2} \frac{\partial \phi^{(o)}}{\partial r} \right) - ke^{iT} \left(\frac{1}{r} \frac{\partial^2 \bar{\phi}^{(o)}}{\partial r^2} + \frac{1}{r^2} \frac{\partial \bar{\phi}^{(o)}}{\partial r} \right) - \varepsilon \left(\frac{1}{r} \frac{\partial^2 \phi^{(1)}}{\partial r^2} + \frac{1}{r^2} \frac{\partial \phi^{(1)}}{\partial r} \right) \\ 170 \quad -\varepsilon ke^{iT} \left(\frac{1}{r} \frac{\partial^2 \bar{\phi}^{(1)}}{\partial r^2} + \frac{1}{r^2} \frac{\partial \bar{\phi}^{(1)}}{\partial r} \right) \quad (39)$$

171 The analyses of the solutions show that the non-steady part is very insignificant.

172
173 By exhaustive algebraic operations, the solutions of the zeroth order equations (23) - (26), (29) - (32) and
174 (35) - (38), are:

$$175 \quad \phi^{(o)} = 2 \left(\frac{r}{s} \right)^2 - \left(\frac{r}{s} \right)^4 \quad (40)$$

$$176 \quad \bar{\phi}^{(o)} = \frac{r^2}{\lambda s^2 I_2(\lambda s)} [I_o(\lambda s) - I_o(\lambda r) + I_2(\lambda r)] \quad (41)$$

$$177 \quad \omega^o = -\frac{8}{s^3} \left(\frac{r}{s} \right) \quad (42)$$

$$178 \quad \bar{\omega}^{(o)} = -2 \frac{\lambda I_1(\lambda r)}{s^2 I_2(\lambda s)} \quad (43)$$

$$\phi^{(1)} = -\frac{\text{Re}}{s} \frac{\partial s \varepsilon}{\partial x} \left[\frac{1}{9} \left(\frac{r}{s} \right)^8 - \frac{2}{3} \left(\frac{r}{s} \right)^6 + \left(\frac{r}{s} \right)^4 - \frac{4}{9} \left(\frac{r}{s} \right)^2 \right] \quad (44)$$

$$\begin{aligned} \bar{\phi}^{(1)} = & \frac{4 \text{Re}}{\lambda^4 s^4} \frac{\partial s}{\partial x} \left\{ \frac{I_1(\lambda r)}{s \lambda^2 I_2(\lambda s)} \left[\left(\frac{2}{3} s^4 \lambda^4 - 5 s^2 \lambda^2 - 32 \right) \lambda I_1^2(\lambda s) \right. \right. \\ & + \left(\frac{1}{2} s^2 \lambda^2 + 16 \right) \lambda^2 s I_o(\lambda s) I_1(\lambda s) + \left(\frac{2}{3} \lambda^4 s^4 + 12 \lambda^2 s^2 - 128 \right) \frac{I_1(\lambda s)}{s} \\ & \left. \left. + \left(64 - 6 \lambda^2 s^2 - \frac{1}{4} \lambda^4 s^4 \right) \lambda I_o(\lambda s) \right] \right\} \quad (45) \end{aligned}$$

$$\omega^{(1)} = \frac{8 \text{Re}}{s^4} \frac{\partial s}{\partial x} \left[\frac{2}{3} \left(\frac{r}{s} \right)^5 - 2 \left(\frac{r}{s} \right)^3 + \left(\frac{r}{s} \right) \right] \quad (46)$$

$$\begin{aligned} \bar{\omega}^{(1)} = & 4 \frac{\text{Re}}{\lambda^4 s^4} \frac{\partial s}{\partial x} \left\{ \frac{1}{\lambda^2 I^2(\lambda s)} \left[\left(\frac{2 s^4 \lambda^4}{3} - 5 s^2 \lambda^2 - 32 \right) \lambda I_1^2(\lambda s) \right] \right. \\ & \left. + \left[I_o(\lambda r) \left(\lambda s^2 r - \frac{\lambda r^2}{3} \right) I_1(\lambda s) - \frac{r^3}{s} + 2 s r \right] \right\} \quad (47) \end{aligned}$$

$$u = -\frac{4}{s^2} + \frac{4 r^2}{s^4} - \frac{k e^{iT}}{\lambda s^2 I_2(\lambda s)} 2(I_o(\lambda s) - I_o(\lambda r) + \lambda r I_1(\lambda r))$$

$$+ \frac{\varepsilon \text{Re}}{s^3} \frac{\partial s}{\partial x} \left[\frac{8}{9} \left(\frac{r}{s} \right)^6 - 4 \left(\frac{r}{s} \right)^4 + 4 \left(\frac{r}{s} \right)^2 - 18 \right]$$

$$\begin{aligned} & + \frac{\varepsilon k e^{iT}}{r \lambda^4 s^6 I_2(\lambda s)} 4 \text{Re} \frac{\partial s}{\partial x} \left\{ \frac{I_o(\lambda r)}{s \lambda I_2(\lambda s)} \left[\left(\frac{2}{3} s^4 \lambda^4 - 5 s^2 \lambda^2 - 32 \right) \lambda I_1^2(\lambda s) \right. \right. \\ & \quad + \left(\frac{1}{2} s^2 \lambda^2 + 16 \right) \lambda^2 s I_o(\lambda s) \\ & \quad + \left(\frac{3}{2} s^4 \lambda^4 - 12 s^2 \lambda^2 - 128 \right) \frac{I_1(\lambda s)}{s} \\ & \quad \left. \left. + \left(64 - 6 s^2 \lambda^2 - \frac{1}{4} s^4 \lambda^4 \right) I_o(\lambda s) \right] \right. \\ & \quad \left. + \lambda^4 r^2 \left(\frac{3}{2} s^2 - \frac{8}{\lambda^2} \right) I_1(\lambda r) I_1(\lambda s) - \frac{4}{3} \lambda^3 r^2 I_o(\lambda r) I_1(\lambda s) + \dots \right\} \quad (48) \end{aligned}$$

$$w = \left(4 \frac{r^3}{s^5} - 2 \frac{r}{s^3} \right) \frac{\partial s}{\partial x} - \frac{k e^{iT} r}{s I_2^2(\lambda s)} \frac{\partial s}{\partial x} [I_o(\lambda s) I_1(\lambda s) - I_1(\lambda s) I_o(\lambda r) + I_1(\lambda s) I_2(\lambda r)]$$

$$\begin{aligned}
& + \varepsilon \frac{\text{Re}}{s^3} \left(\frac{\partial s}{\partial x} \right)^2 \left[\left(\frac{r}{s} \right)^7 - \frac{14}{3} \left(\frac{r}{s} \right)^5 + 5 \left(\frac{r}{s} \right)^3 - 27 \left(\frac{r}{s} \right) \right] \\
& + \varepsilon k e^{iT} \left\{ \begin{aligned} & \left[\frac{2}{3} \lambda^6 \left(2s^3 I_o(\lambda s) I_1(\lambda s) I_2^2(\lambda s) + \frac{s^2}{\lambda} (I_1^2(\lambda s) I_2^2(\lambda s)) \right) \right. \\ & \frac{4 \text{Re}}{r \lambda^6} I_2(\lambda r) \frac{\partial s}{\partial x} - 5 \lambda^4 \left(2s^5 I_o(\lambda s) I_1(\lambda s) I_2^2(\lambda s) - \frac{s^4}{\lambda} I_1^2(\lambda s) I_2^2(\lambda s) \right) \\ & \left. - 32 \lambda^2 \left(2s^7 I_o(\lambda s) I_1(\lambda s) I_2^2(\lambda s) - 3 \frac{s^6}{\lambda} I_1^2(\lambda s) I_2^2(\lambda s) \right) \right] \\ & + \frac{4 \text{Re} r}{s 4 I_2^2(\lambda s)} I_2(\lambda r) \frac{\partial s}{\partial x} \left[\begin{aligned} & I_o(\lambda s) - \frac{4}{3s^2} \left(I_o(\lambda s) - \frac{2 I_1(\lambda s)}{\lambda s} \right) \\ & - \frac{8}{\lambda^2 s^2} \left(I_o(\lambda s) - \frac{2 I_1(\lambda s)}{\lambda s} \right) \end{aligned} \right] \end{aligned} \right\}
\end{aligned}
\tag{49}$$

$$p = \frac{1}{\varepsilon} \left[\frac{36x}{s^4} - \frac{k e^{iT} x}{s^2 I_2(\lambda r)} \left[\frac{10}{r} I_1(\lambda r) + 7 \lambda r I_o(\lambda r) - (\lambda^2 r + \lambda^2 r^2) I_1(\lambda r) \right] \right] \quad 0 \leq x \leq 1 \tag{50}$$

$$\begin{aligned}
& \tau_w = \frac{8}{s^2} - \frac{k e^{iT} x}{s^2 I_2(\lambda s)} (\lambda s^2 I_o(\lambda s) + 4s I_1(\lambda s)) + \varepsilon \frac{8}{3s^3} \text{Re} \frac{\partial s}{\partial x} + \\
& \frac{\varepsilon k e^{iT}}{\lambda^4 s^6 I_2(\lambda r)} \text{Re} \frac{\partial s}{\partial x} \left\{ \begin{aligned} & \left[\begin{aligned} & \left(\frac{2}{3} s^4 \lambda^4 - s^2 \lambda^2 - 32 \right) I_1^2(\lambda s) \\ & + \left(\frac{1}{2} s^2 \lambda^2 + 16 \right) \lambda^2 s I_o(\lambda s) I_1(\lambda s) \end{aligned} \right. \\ & \frac{I_1(\lambda s)}{s^2 I_2(\lambda s)} \left(\frac{3}{2} s^4 \lambda^4 - 12s^2 \lambda^2 - 128 \right) \frac{I_1(\lambda s)}{s} \\ & \left. + \left(64 - 6s^2 \lambda^2 - \frac{1}{4} s^4 \lambda^4 \right) I_o(\lambda s) \right] \\ & + \lambda^4 \left(\frac{4s}{3} - 6s^3 + \frac{32s}{\lambda^2} \right) I_1^2(\lambda s) \end{aligned} \right\}
\end{aligned}
\tag{51}$$

where $I_o(\lambda r)$, $I_1(\lambda r)$, $I_2(\lambda r)$, $I_o(\lambda s)$, $I_1(\lambda s)$ and $I_2(\lambda s)$ are the modified Bessel function of order zero, one, two, respectively. It is worthy to note that the solutions to equations (31) and (33) as expressed in equations (45) and (47) are culled and developed from [8], p. 206.

Furthermore, the geometries under consideration are: the convergent channel $s = e^{-x/2}$, and divergent channel $s = e^{x/2}$

204

205 3 RESULTS AND DISCUSSION

206 We considered the flow of blood through a stenosis artery with emphasis on the effects of pressure wave
207 amplitude and height of constriction on the overall flow structure in the regions before and after the peak
208 of the constriction. These two regions under consideration approximate the convergent and divergent
209 channels, as shown in figure (1). The analyses of the results show that variations in the amplitude and
210 height of the constriction have tremendous effects on the flow structure. To this end, using Maple 12
211 computational software and the following physically realistic parameters: $T=\pi/4$; $\lambda=2$; $k = 0.01, 0.03, 0.05,$
212 0.07 ; $\varepsilon = 0.01, 0.03, 0.05, 0.07$; $Re= 10, 100, 500, 1000$ we obtained the results shown in Fig. 2 – Fig. 17.
213 These figures show the profiles of the computational results for the velocities pressure and wall shear
214 stress. The profiles show that increase in the amplitude decreases the axial velocity and axial pressure
215 (see Fig. 2, Fig. 3, Fig. 6 and Fig. 7), but increases the radial velocity and wall shear stress (see Fig. 4,
216 Fig. 5, Fig. 8 and Fig. 9). Blood flow in a normal artery is laminar and Poiseuille. But due to the stenosis,
217 the arterial channel is blocked moderately or severely. For this, the heart enlarges itself to maintain the
218 peripheral flow. The enlargement of the heart leads to increase in the amplitude of the pulse or pressure
219 wave. And, the increase in the amplitude energizes the flow variables in the direction of the increase,
220 which in this case, the radial velocity and wall shear stress. This accounts for what is seen in Fig. 4, Fig.
221 5, Fig. 8 and Fig. 9. On the other hand, the increase in the amplitude reduces the axial velocity and axial
222 pressure as seen in Fig. 2, Fig. 3, Fig. 6 and Fig. 7. This could be due to the increase in the radial
223 velocity.

224

225 Furthermore, the profiles show that the height of the constriction decreases the axial velocity (see Fig. 10
226 and Fig. 11), while it increases the axial pressure and wall shear stress (see Fig. 14, Fig. 16 and Fig. 17).
227 As said above, blood flow in the artery is laminar, Poiseuille and fully developed. It moves with the
228 velocity with which it is released from the heart. As it gets to the region of constriction, the flow pattern
229 changes. The blockage caused by the stenosis reduces the axial pressure as shown in Fig. 11 and Fig. 11.

230

231 **More so**, the changes in the geometrical configuration **affect** the flow. In particular, for the convergent
232 channel, as the geometry thins down exponentially, the pressure factor increases, and subsequently, the
233 velocities and wall shear stress increase. It is also seen that in the divergent channel whose geometry
234 increases exponentially, the radial velocity, axial pressure and wall shear stress are increased. The
235 increase could be due to the influence of the flow situations in the convergent channel. All these account
236 for the results in Fig. 13, Fig. 14, Fig. 16 and Fig. 17.

237

238 Similarly, a special feature called the flow separation occurs in the radial and pressure structures at $r \leq$
239 0.3 and $r \leq 0.46$ **for $T=\pi/4$ as shown in our data**, respectively (see Fig. 12 and Fig. 15). Initially, the
240 radial velocity and pressure increase with the height of constriction drop at these points, and then the flow
241 pattern changes such that these variables decrease as the height of constriction increases. The
242 occurrence of the flow separation (which is in agreement with [8]) is due to the adverse flow conditions at
243 such points. [8] adduced that it is due to viscous effects.

244

245 The overall analyses show that the axial velocity drops as the pulse amplitude and height of constriction
246 increase. This has some health implications on the stenotic patient. The drop in the axial velocity leads
247 subsequently to a drop in the rate at which the rejuvenated blood which bears oxygen and food nutrients
248 is transported to other parts of the body, thus causing cells starvation.

249

**OSCILLATORY BLOOD FLOW IN CONVERGENT AND DIVERGENT CHANNELS:
PART 1-EFFECTS OF PULSE AMPLITUDE AND HEIGHT OF CONSTRICTION
CONVERGENT**

EFFECTS OF AMPLITUDE

$Re=700$; $\varepsilon = 0.01$; $k = 0.0, 0.01, 0.03, 0.05$; $H(x)=1$; $Pr=0.5$; $\mu=0.01$; $T=\pi/4$; $\lambda=2$

AXIAL VELOCITY

r	K=0.0	K=0.01	K=0.03	K=0.05
0	59.9014	-110.135	-448.207	-786.28
0.2	71.9161	-268.722	-949.999	-1631.28
0.4	87.91997	-602.861	-1984.18	-3365.5
0.6	107.181	-1301.68	-4119.41	-6937.14
0.8	130.827	-2759.0	-8538.65	-14318.3
1.0	159.668	-5794.86	-17703	-29612.9

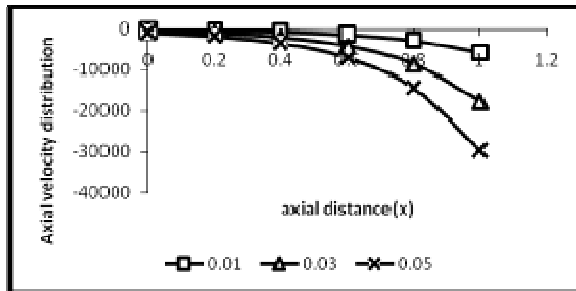


**Fig. 2 Axial velocity-amplitude (k) profiles in a convergent channel
DIV**

EFFECTS OF ARBITRARY AMPLITUDE

$Re=700$; $\varepsilon = 0.01$; $k = 0.0, 0.01, 0.03, 0.05$; $H(x)=1$; $Pr=0.5$; $\mu=0.01$; $T=\pi/4$; $\lambda=2$

AXIAL VELOCITY

r	K=0.0	K=0.01	K=0.03	K=0.05
0	58.9014	-110.135	-448.811	-786.28
0.2	48.2388	-36.111	-204.811	-373.51
0.4	39.5044	-2.76464	-87.3027	-171.841
0.6	32.35	11.1313	-31.3062	-73.7437
0.8	26.4903	15.8644	-5.38756	-26.6395
1.0	21.6914	16.4197	5.87637	-5.667

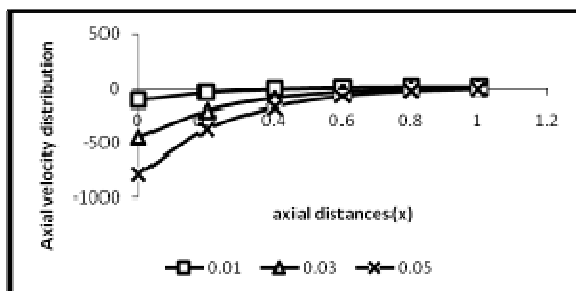


Fig. 3 Axial velocity-amplitude (k) profiles in a divergent channel

RADIAL VELOCITY

r	K=0.0	K=0.01	K=0.03	K=0.05
0	-4.6827	4.98359	24.1874	43.3913
0.2	--5.03907	-3.00333	-2.26814	7.53961
0.4	-6.88494	-6.07377	-4.45143	-2.82909
0.6	-8.40538	-8.05767	-7.36236	-6.66685
0.8	-10.2606	-9.95964	-9.35774	-8.75583
1.0	-12.5238	-12.0795	-11.191	-10.3025

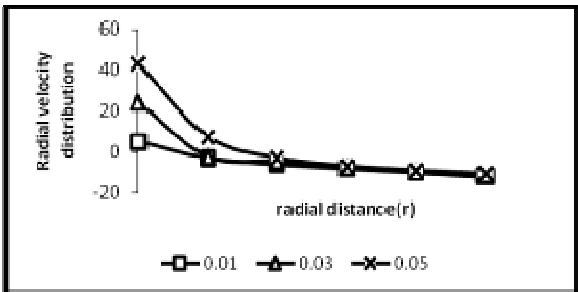


Fig. 4 Radial velocity-amplitude (k) profiles in a convergent channel

DIVERGENT CHANNEL

RADIAL VELOCITY

r	K=0.0	K=0.01	K=0.03	K=0.05
0	-4.61833	4.98359	24.1874	43.3913
0.2	-3.78215	33.8746	109.188	184.502
0.4	-3.09722	156.023	474.263	792.504
0.6	-2.53623	727.55	2184.72	3647.89
0.8	-2.07679	3669.78	1103.5	18375.2
1.0	-1.70053	20460.0	61383.3	10237.0

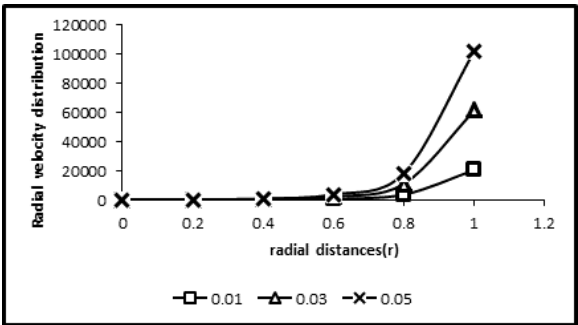
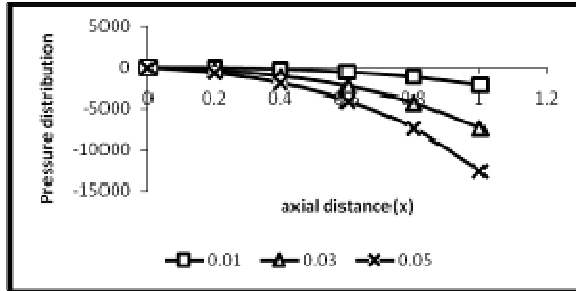


Fig. 5 Radial velocity-amplitude (k) profiles in a divergent channel



CONVERGENT

EFFECTS OF ARBITRARY AMPLITUDE

$Re=700$; $\varepsilon = 0.01$; $k = 0.0, 0.01, 0.03, 0.05$; $H(x)=1$; $Pr=0.5$; $\mu=0.01$; $T=\pi/4$; $\lambda=2$

PRESSURE

R	K=0.0	K=0.01	K=0.03	K=0.05
0	0	0	0	0
0.2	154.539	-12.5345	-346.681	-680.828
0.4	262.118	-180.833	-1066.73	-1952.64
0.6	337.192	-546.33	-2313.37	-4080.42
0.8	389.25	-1181.57	-4323.21	-7464.85
1.0	424.656	-2199.94	-7449.14	-12698.3

Fig. 6 Pressure-amplitude (k) profiles in a convergent channel

DIVERGENT

EFFECTS OF AMPLITUDE

$Re=700$; $\varepsilon = 0.01$; $k = 0.0, 0.01, 0.03, 0.05$; $H(x)=1$; $Pr=0.5$; $\mu=0.01$; $T=\pi/4$; $\lambda=2$

PRESSURE

r	K=0.0	K=0.01	K=0.03	K=0.05
0	0	0	0	0
0.2	224.003	127.895	-64.3195	-256.534
0.4	553.295	406.622	113.276	-180.07
0.6	1046.28	877.653	540.394	203.135
0.8	1800.64	1627.51	1281.24	934.971
1.0	2984.33	2816.88	2481.99	2147.1

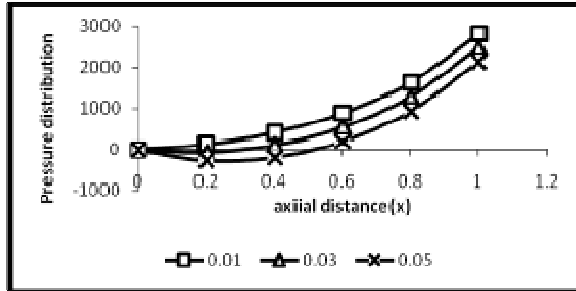


Fig. 7 Pressure-amplitude (k) profiles in a divergent channel

CONVERGENT CHANNELS

EFFECTS OF ARBITRARY AMPLITUDE ON THE FLOW

$Re=700$; $\varepsilon = 0.01$; $k = 0.0, 0.01, 0.03, 0.05$; $H(x)=1$; $Pr=0.5$; $\mu=0.01$; $T=\pi/4$; $\lambda=2$

Wall shear stress

X	K=0.0	K=0.01	K=0.03	K=0.05
0	17.333	20.474	26.7553	33.067
0.2	21.171	28.4077	42.8812	57.3547
0.4	25.8583	41.7788	73.6199	105.461
0.6	31.5834	65.6969	133.924	202.151
0.8	38.576	110.66	254.829	398.998
1.0	47.1169	198.499	501.264	804.029

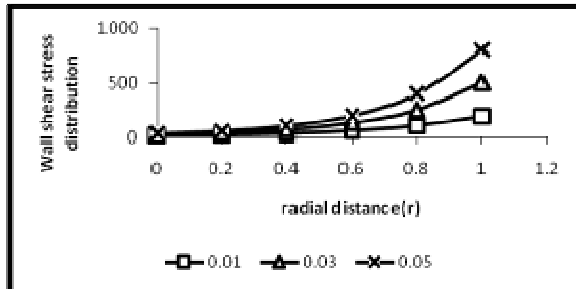


Fig. 8 Wall shear stress-amplitude (k) profiles in a convergent channel

DIVERGENT CHANNEL

EFFECTS OF VARIATION IN AMPLITUDE

$Re=700$; $\varepsilon = 0.01$; $k = 0.0, 0.01, 0.03, 0.05$; $H(x)=1$; $Pr=0.5$; $\mu=0.01$; $T=\pi/4$; $\lambda=2$

WALL SHEAR STRESS

r	K=0.0	K=0.01	K=0.03	K=0.05
0	17.3333	20.474	26.2553	33.0367
0.2	14.1913	15.4531	17.9766	20.5
0.4	11.6189	12.0729	12.9809	13.8888
0.6	9.51274	9.67623	10.0032	10.3302
0.8	7.7837	7.91309	8.16252	8.41195
1.0	6.37658	6.59099	7.01981	7.44863

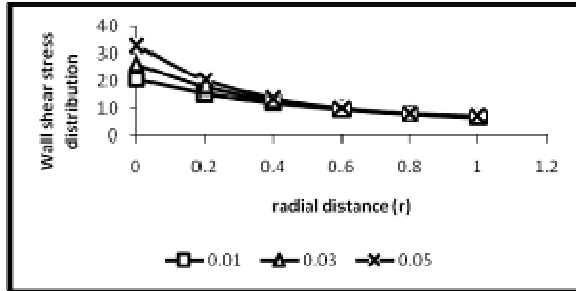


Fig. 9 Wall shear stress-amplitude (k) profiles in a divergent channel
NO Nu here

CONV VELO

EFFECTS OF ARBITRARY RADIUS

Re=700; $\varepsilon = 0.0, 0.01, 0.03, 0.05$; $k = 0.01$; $H(x)=1$; $Pr=0.5$; $\mu=0.01$; $T=\pi/4$; $\lambda=2$

AXIAL VELOCITY

r	EP=0.0	EP=0.01	EP=0.03	EP=0.05
0	-3.9639	-786.28	-2350.91	-3915.54
0.2	-4.8232	-1631.28	-4884.16	-8137.05
0.4	-5.8806	-3365.5	-10184.7	-16804.0
0.6	-7.1708	-6939.14	-20792.1	-34657.0
0.8	-8.7272	-14318.3	-42937.5	-71556.6
1.0	-10.6133	-29612.9	-88817.6	-148022.0

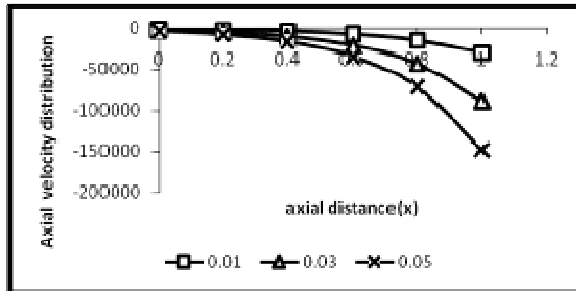


Fig. 10 Axial velocity-height of constriction (ε) profiles in a convergent channel
DIVERGENT CHANNEL

EFFECTS OF ARBITRARY RADIUS

Re=700; $\varepsilon = 0.0, 0.01, 0.03, 0.05$; $k = 0.01$; $H(x)=1$; $Pr=0.5$; $\mu=0.01$; $T=\pi/4$; $\lambda=2$

AXIAL VELOCITY

R	EP=0.0	EP=0.01	EP=0.03	EP=0.05
0	-3.96397	-786.28	-2350.91	-3915.54
0.2	-3.25059	-543.355	-1114.03	-1854.55
0.4	-2.66484	-171.841	-510.192	-848.544

0.6	-2.18413	-73.7439	-216.863	-359.982
0.8	-1.7898	-26.6395	-76.3389	-126.038
1.0	-1.46643	4-.667	-11.0682	-17.4693

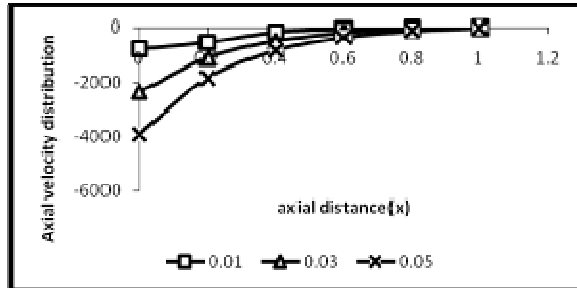


Fig. 11 Axial velocity-height of constriction (ϵ) profiles in a divergent channel

RADIAL VELOCITY

r	EP=0.0	EP=0.01	EP=0.03	EP=0.05
0	0.124396	43.3913	120.925	216.459
0.2	0.156226	7.53961	22.3064	37.0731
0.4	0.196513	-2.82909	-8.88029	-14.9315
0.6	0.247615	-6.66685	-20.4958	-34.3247
0.8	0.312587	-8.75583	-26.8927	-45.0295
1.0	0.395403	-10.3025	-31.883	-53.0941

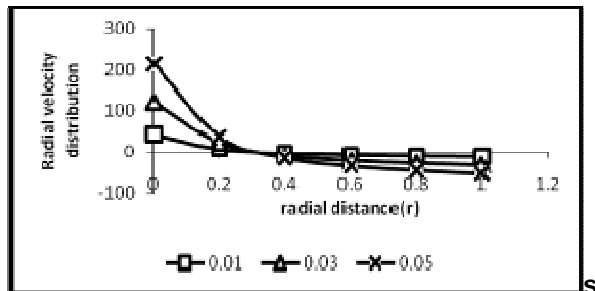


Fig. 12 Radial velocity-height of constriction (ϵ) profiles in a convergent channel

RADIAL VELOCITY

R	EP=0.0	EP=0.01	EP=0.03	EP=0.05
0	-.124396	43.3913	129.925	216.459
0.2	0.099199	90.0479	553.306	922.111
0.4	0.0792177	792.504	2377.35	3962.2
0.6	0.063349	3647.89	10943.6	18239.2
0.8	0.0507304	18357.2	55071.5	91785.9
1.0	0.0406858	192307.0	306920.0	511533.0

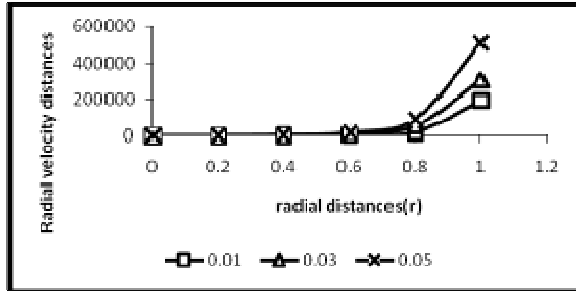


Fig. 13 Radial velocity-height of constriction (ϵ) profile in a divergent channel

CONV

EFFECTS OF ARBITRARY RADIUS

$Re=700$; $\epsilon = 0.0, 0.01, 0.03, 0.05$; $k = 0.01$; $H(x)=1$; $Pr=0.5$; $\mu=0.01$; $T=\pi/4$; $\lambda=2$

PRESSURE

r	EP=0.0	EP=0.01	EP=0.03	EP=0.05
0	-	0	0	0
0.2	-	-680.828	-226.943	-136.166
0.4	-	-1952.63	-650.878	-390.527
0.6	-	-4080.42	-1360.14	-816.083
0.8	-	-7464.85	-3188.28	-2492.97
1.0	-	-12698.3	-5232.78	-4739.67

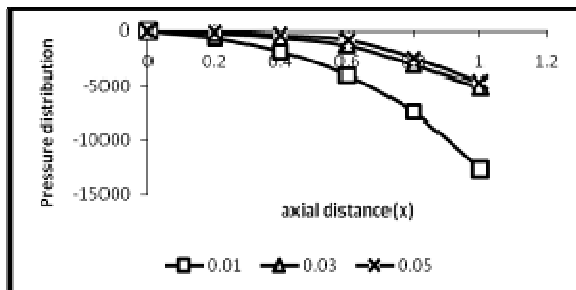


Fig. 14 Pressure-height of constriction (ϵ) profiles in a convergent channel

DIV

EFFECTS OF ARBITRARY RADIUS

$Re=700$; $\epsilon = 0.0, 0.01, 0.03, 0.05$; $k = 0.01$; $H(x)=1$; $Pr=0.5$; $\mu=0.01$; $T=\pi/4$; $\lambda=2$

PRESSURE

r	EP=0.0	EP=0.01	EP=0.03	EP=0.05
0	-	0	0	0
0.2	-	-174.012	-85.5114	-51.3069
0.4	-	-180.07	-60.0234	-36.014
0.6	-	203.134	67.7118	40.6271
0.8	-	934.971	311.657	186.994

1.0	-	2147.1	715.701	429.42
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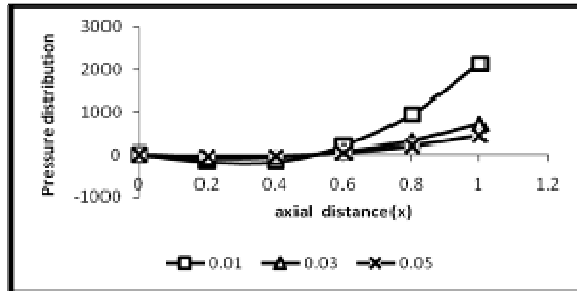


Fig. 15 Pressure-height of constriction (ϵ) profiles in a divergent channel
CONV

EFFECTS OF VARIATION IN RADIUS

$Re=700$; $\epsilon = 0.0, 0.01, 0.03, 0.05$; $k = 0.01$; $H(x)=1$; $Pr=0.5$; $\mu=0.01$; $T=\pi/4$; $\lambda=2$

WALL STRESS

X	EPSILON =0.0	EPSILON=0.01	EPSILON =0.03	EPSILON =0.05
0	5.84539	33.069	87.4196	141.802
0.2	7.25223	57.3547	157.559	257.764
0.4	8.97177	105.461	298.439	491.462
0.6	11.0731	202.451	584.306	966.462
0.8	18.6405	398.98	1169.71	1940.43
1.0	26.7771	804.029	2378.58	3953.04

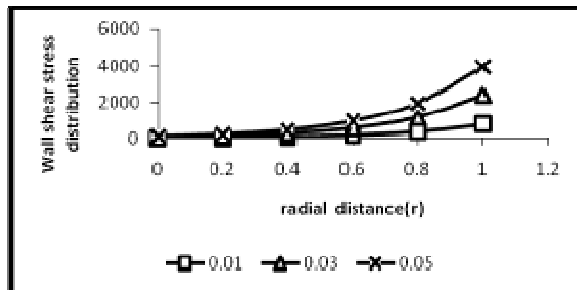


Fig. 16 Wall shear stress-height of constriction (ϵ) profiles in a convergent channel
DIV

EFFECTS OF VARIATION IN RADIUS

$Re=700$; $\epsilon = 0.0, 0.01, 0.03, 0.05$; $k = 0.01$; $H(x)=1$; $Pr=0.5$; $\mu=0.01$; $T=\pi/4$; $\lambda=2$

WALL SHEAR STRESS

r	EP=0.0	EP=0.01	EP=0.03	EP=0.05
0	5.84539	33.0367	87.4195	141.802
0.2	4.69471	25.7577	52.1107	83.7213
0.4	3.7539	13.8888	34.1587	54.4286
0.6	2.98508	10.3302	25.0204	39.7106

0.8	2.35723	8.41195	20.5214	32.6308
1.0	1.84493	7.44863	18.656	29.8634

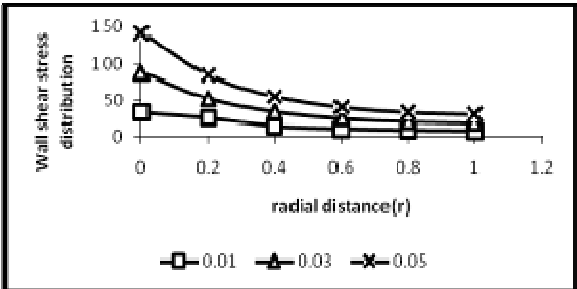


Fig. 17 Wall shear stress-height of constriction (ϵ) profiles in a divergent channel

4 CONCLUSIONS

The analyses of the results indicate that the pulse amplitude and height of constriction reduce the axial velocity and pressure but increase the radial velocity and wall shear stress. The occurrence of flow separation in the radial velocity and pressure structures are due to some adverse flow situations at such points. Furthermore, the drop in the axial velocity has some physiological implications on the stenotic patient.

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