

On a new class of weakly Berwald spaces with (α, β) -metric

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Abstract

We have two concepts of Douglas spaces and Landsberg spaces as generalizations of Berwald spaces. S. Bacsó [1] gave the definition of a weakly-Berwald space as another generalization of Berwald spaces. In 1972, M. Matsumoto has introduced the concept of (α, β) -metric, which is a Finsler metric, constructed from a Riemannian metric and a differential 1-form. In this paper, we study an important class of (α, β) -metrics in the form $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$, known as second approximate Matsumoto metric on an n -dimensional manifold and get the conditions for such metrics to be weakly-Berwald metrics, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form. A Finsler space with an (α, β) -metric is a weakly-Berwald space, if and only if $B_m^m = \partial B^m / \partial y^m$ is a 1-form. We show that it becomes a weakly Berwald space under some geometric and algebraic conditions.

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1 Introduction

Let M^n be an n -dimensional differential manifold and let $F^n = (M^n, L)$ be an n -dimensional Finsler space where L is a fundamental function. Let $g_{ij} = \dot{\partial}_i \dot{\partial}_j L^2 / 2$ be the fundamental tensor, where the symbol $\dot{\partial}_i$ means $\frac{\partial}{\partial y^i}$ and we define G_i as

$$G_i = \{y^r (\partial_r \dot{\partial}_i L^2) - \partial_i L^2\} / 4,$$

and $G^i = g^{ij}G_j$ where the symbol ∂_i means $\frac{\partial}{\partial x^i}$ and (g^{ij}) is the inverse matrix of (g_{ij}) . The coefficients (G_{jk}^i, G_j^i) of the Berwald connection $B\Gamma$ are defined as $G_j^i = \dot{\partial}_j G^i$ and $G_{jk}^i = \dot{\partial}_k G_j^i$.

A Berwald space is a Finsler space which satisfies the condition $G_{ijk}^h = 0$, that is to say, whose coefficients G_{ij}^h of the Berwald connection are functions of the position (x^i) alone. Therefore the equations $y_r G_{ijk}^r = 0$ hold, so $2G^i = G_{rs}^i y^r y^s$ are homogeneous polynomials in (y^i) of degree two, so $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three. Then we can consider the notions of Landsberg spaces and Douglas spaces as two generalizations of Berwald spaces. The notion of weakly-Berwald spaces is the third generalization of Berwald spaces. Thus if a Finsler space satisfies the condition $G_{ij} = 0$, we call it a weakly-Berwald space.

The concept of an (α, β) -metric on a Finsler space $F^n = (M^n, L)$ was introduced by M. Matsumoto in [6] and has been studied by many authors [3, 5, 8, 10]. Some important examples of (α, β) -metric are Randers metric ($L = \alpha + \beta$), Kropina metric $\left(L = \frac{\alpha^2}{\beta}\right)$, generalized Kropina metric $\left(L = \frac{\alpha^{m+1}}{\beta^m}\right)$, ($m \neq 0, -1$), and Matsumoto metric $\left(L = \frac{\alpha^2}{\alpha - \beta}\right)$. The study of Finsler spaces with these metrics has greatly contributed to the growth of Finsler geometry and its applications to the theory of Relativity and allied areas.

Definition 1.1. A Finsler metric $L(x, y)$ is called an (α, β) -metric if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M^n .

S. Bacso [1] introduced the notion of weakly-Berwald space as another generalization of Berwald spaces. IL-Yong Lee and Myung-Han Lee [4] have studied weakly Berwald spaces with special (α, β) -metrics. In this paper we extend the study on **weakly** Berwald spaces with second approximate Matsumoto metric.

Let $F^n = (M^n, L)$ be a Finsler space of dimension n , and the domain of the fundamental metric function $L(x, y)$ is the set of $TM \setminus (0)$ of the non-zero tangent vectors. We assume that L is positive and the fundamental metric tensor $g_{ij}(x, y) = \frac{1}{2}L_{(i)(j)}^2$ (where $(i) := \frac{\partial}{\partial y^i}$) is not necessarily positive definite.

The equation of the (canonically parametrized) geodesics of F^n is given by

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, y) = 0, \quad \left(\frac{dx^i}{dt} = y^i\right),$$

where

$$G^i = \frac{1}{2}g^{ir} \left(y^s \frac{\partial L_r^2}{\partial x^s} - \frac{\partial L^2}{\partial x^r} \right).$$

The Berwald connection of the space is defined by its connection coefficients $G_{jk}^i(x, y)$ which can be computed from G^i according to the following **formula**:

$$G_j^i(x, y) = G_{(j)}^i; \quad G_{jk}^i(x, y) = G_{j(k)}^i.$$

Definition 1.2. A Finsler space is called Berwald space, if G_{jk}^i are functions of position alone, i. e., Berwald connection BF is linear.

A Finsler space is called a weakly Berwald space if the (hv) -Ricci curvature tensor $G_{jk} = 0$. The spray function G^i of a Finsler space with an (α, β) -metric are given by $2G^i = \gamma_{00}^i + 2B^i$, where γ_{jk}^i stands for the Christoffel symbols in the associated Riemannian space (M^n, α) . Then we have $G_j^i = \gamma_{0j}^i + B_j^i$ and $G_{jk}^i = \gamma_{jk}^i + B_{jk}^i$, where $\dot{\partial}_j B^i = B_j^i$ and $\dot{\partial}_k B_j^i = B_{jk}^i$. Thus a Finsler space with an (α, β) -metric is a weakly-Berwald space, if and only if $B_m^m = \partial B^m / \partial y^m$ is a one-form.

Recently, S. Bacso and B. Szilagyi [1] gave an example for the weakly Berwald Finsler space, and a sufficient condition for the existence of a weakly Berwald Finsler space of Kropina type was also determined. Recently, S. Bacso and R. Yoshikawa [2] investigated the conditions for Randers and Kropina spaces to be weakly-Berwald spaces. R. Yoshikawa and K. Okubo [11] obtained the conditions for generalized Kropina spaces and Matsumoto spaces to be weakly-Berwald spaces and Berwald spaces. In [9], it has been shown under which condition a Finsler space with first approximate Matsumoto metric becomes a weakly Berwald space. In this paper, first we discuss the conditions for the Finsler space F^n with an (α, β) -metric to be a weakly-Berwald space and then we find the conditions for Finsler space with second approximate Matsumoto metric $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ to be weakly-Berwald space.

2 Weakly-Berwald space with respect to (α, β) -metric

In the present section, we deal with the condition that a Finsler space with an (α, β) -metric be a weakly-Berwald space.

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space equipped with (α, β) -metric $L(\alpha, \beta)$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. In this paper, the symbol $(;)$ stands for h -covariant derivation with respect to the Riemannian connection in the associated Riemannian space (M^n, α) and γ_{jk}^i stands for the Christoffel symbols in the space (M^n, α) .

We use the following notations [2]

$$\begin{aligned} (a) \quad & b^i = a^{ir} b_r, \quad b^2 = a^{rs} b_r b_s, \\ (b) \quad & 2r_{ij} = b_{i;j} + b_{j;i}, \quad 2s_{ij} = b_{i;j} - b_{j;i}, \\ (c) \quad & r_j^i = a^{ir} r_{rj}, \quad s_j^i = a^{ir} s_{rj}, \quad r_i = b_r r_i^r, \quad s_i = b_r s_i^r. \end{aligned}$$

We now consider the function $G^i(x, y)$ of F^n with an (α, β) -metric. According to [7], they are written in the form

$$2G^m = \gamma_{00}^m + 2B^m,$$

$$B^m = \frac{E^*}{\alpha} y^m + \frac{\alpha L_\beta}{L_\alpha} s_0^m - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} C^* \left(\frac{1}{\alpha} y^m - \frac{\alpha}{\beta} b^m \right), \quad (2.1)$$

where we have put

$$E^* = \left(\frac{\beta L_\beta}{L} \right) C^*, C^* = \frac{\alpha \beta (r_{00} L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha})}, \gamma^2 = b^2 \alpha^2 - \beta^2, \quad (2.2)$$

and

$$L_\alpha = \frac{\partial L}{\partial \alpha}, L_\beta = \frac{\partial L}{\partial \beta}, L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2}, L_{\alpha\beta} = \frac{\partial^2 L}{\partial \alpha \partial \beta}, L_{\alpha\alpha\alpha} = \frac{\partial^3 L}{\partial \alpha^3}. \quad (2.3)$$

Since $\gamma_{00}^i = \gamma_{jk}^i(x) y^j y^k$ are homogeneous polynomials in (y^i) of degree two, it is obvious that a Finsler space with an (α, β) -metric is a Berwald space, if and only if B^m are homogeneous polynomials in (y^i) of degree two and Berwald connection $B\Gamma$ is linear.

Differentiating (2.1) with respect to y^n and contracting m and n in the obtained equation, we have

$$\begin{aligned} B_m^m &= \left\{ \dot{\partial}_m \left(\frac{\beta L_\beta}{\alpha L} \right) y^m + \frac{n \beta L_\beta}{\alpha L} - \dot{\partial}_m \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha} \right) \left(\frac{\beta y^m - \alpha^2 b^m}{\alpha \beta} \right) \right\} C^* \\ &\quad - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} \left\{ \dot{\partial}_m \left(\frac{1}{\alpha} \right) y^m + \frac{1}{\alpha} \delta_m^m - \dot{\partial}_m \left(\frac{\alpha}{\beta} \right) b^m \right\} C^* + \dot{\partial}_m \left(\frac{\alpha L_\beta}{L_\alpha} \right) s_0^m \\ &\quad + \left(\frac{\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}}{\alpha L L_\alpha} \right) (\dot{\partial}_m C^*) y^m + \left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} \right) (\dot{\partial}_m C^*) b^m. \end{aligned} \quad (2.4)$$

Since $L = L(\alpha, \beta)$ is a positively homogeneous function of α and β of degree one, we have

$$\begin{aligned} L_\alpha \alpha + L_\beta \beta &= L, L_{\alpha\alpha} \alpha + L_{\alpha\beta} \beta = 0, \\ L_{\beta\alpha} \alpha + L_{\beta\beta} \beta &= 0, L_{\alpha\alpha\alpha} \alpha + L_{\alpha\alpha\beta} \beta = -L_{\alpha\alpha}. \end{aligned}$$

Using the above relations and the homogeneity of (y^i) , we obtain the following equations

$$\dot{\partial}_m \left(\frac{\beta L_\beta}{\alpha L} \right) y^m = -\frac{\beta L_\beta}{\alpha L}, \quad (2.5)$$

$$\dot{\partial}_m \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha} \right) \left(\frac{\beta y^m - \alpha^2 b^m}{\alpha \beta} \right) = \frac{\gamma^2}{(\beta L_\alpha)^2} \{ L_\alpha L_{\alpha\alpha} + \alpha L_\alpha L_{\alpha\alpha\alpha} - \alpha (L_{\alpha\alpha})^2 \}, \quad (2.6)$$

$$\left\{ \dot{\partial}_m \left(\frac{1}{\alpha} \right) y^m + \frac{1}{\alpha} \delta_m^m - \dot{\partial}_m \left(\frac{\alpha}{\beta} \right) b^m \right\} = \frac{1}{\alpha \beta^2} (\gamma^2 + (n-1)\beta^2), \quad (2.7)$$

$$(\dot{\partial}_m C^*) y^m = 2C^*, \quad (2.8)$$

$$\begin{aligned} (\dot{\partial}_m C^*) b^m &= \frac{1}{2\alpha\beta\Omega^2} \left(\Omega \left(\beta(\gamma^2 + 2\beta^2)W + 2\alpha^2\beta^2 L_\alpha r_0 - \right. \right. \\ &\quad \left. \alpha\beta\gamma^2 L_{\alpha\alpha} r_{00} - 2\alpha(\beta^3 L_\beta + \alpha^2\gamma^2 L_{\alpha\alpha}) s_0 \right) - \\ &\quad \left. \alpha^2\beta W (2b^2\beta^2 L_\alpha - \gamma^4 L_{\alpha\alpha\alpha} - b^2\alpha\gamma^2 L_{\alpha\alpha}) \right), \end{aligned} \quad (2.9)$$

$$\dot{\partial}_m \left(\frac{\alpha L_\beta}{L_\alpha} \right) s_0^m = \frac{\alpha^2 L L_{\alpha\alpha} s_0}{(\beta L_\alpha)^2}, \quad (2.10)$$

where

$$W = (r_{00} L_\alpha - 2\alpha s_0 L_\beta), \Omega = (\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), (\Omega \neq 0) \quad (2.11)$$

$$Y_i = a_{ir} y^r, \quad s_{00} = 0, \quad b^r s_r = 0, \quad a^{ij} s_{ij} = 0.$$

Substituting (2.2), (2.3), (2.5), (2.6), (2.7), (2.8), (2.9) and (2.10) into (2.4), we get

$$B_m^m = \frac{1}{2\alpha L(\beta L_\alpha)^2 \Omega^2} \{2\Omega^2 A C^* + 2\alpha L \Omega^2 B s_0 + \alpha^2 L_\alpha L_{\alpha\alpha} (C r_{00} + D s_0 + E r_0)\}, \quad (2.12)$$

where

$$\begin{aligned} A &= (n+1)\beta^2 L_\alpha (\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}) + \alpha\gamma^2 L(\alpha(L_{\alpha\alpha})^2 \\ &\quad - 2L_\alpha L_{\alpha\alpha} - \alpha L_\alpha L_{\alpha\alpha\alpha}), \\ B &= \alpha^2 L L_{\alpha\alpha}, \\ C &= \beta\gamma^2 (-\beta^2(L_\alpha)^2 + 2b^2\alpha^3 L_\alpha L_{\alpha\alpha} - \alpha^2\gamma^2(L_{\alpha\alpha})^2 + \alpha^2\gamma^2 L_\alpha L_{\alpha\alpha\alpha}), \\ D &= 2\alpha(\beta^3(\gamma^2 - \beta^2)L_\alpha L_\beta - \alpha^2\beta^2\gamma^2 L_\alpha L_{\alpha\alpha} \\ &\quad - 2\alpha\beta\gamma^2(\gamma^2 + 2\beta^2)L_\beta L_{\alpha\alpha} - \alpha^3\gamma^4(L_{\alpha\alpha})^2 - \alpha^2\beta\gamma^4 L_\beta L_{\alpha\alpha\alpha}), \\ E &= 2\alpha^2\beta^2 L_\alpha \Omega. \end{aligned} \quad (2.13)$$

Summarizing up the above, we have the following

Theorem 2.1. *The necessary and sufficient condition for a Finsler space F^n with an (α, β) -metric to be a weakly-Berwald space is that $G_m^m = \gamma_{0m}^m + B_m^m$ and B_m^m is a homogeneous polynomial in (y^m) of degree one, where B_m^m is given by (2.12) and (2.13), provided that $\Omega \neq 0$.*

Lemma 2.2. *If $\alpha^2 \equiv 0 \pmod{\beta}$, that is, $a_{ij}(x)y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension n is equal to 2 and b^2 vanishes. In this case we have 1-form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

3 Finsler space with second approximate Matsumoto metric

In the present section, we consider the condition that the Finsler space with second approximate Matsumoto metric be a weakly-Berwald space.

Let us consider a Finsler space $F^n = (M^n, L)$ with second approximate Matsumoto metric

$$L(\alpha, \beta) = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}. \quad (3.1)$$

We now find the conditions for F^n with (3.1) to be a weakly-Berwald space. For F^n with (3.1), we have

$$\begin{aligned} L_\alpha &= \frac{\alpha^3 - \alpha\beta^2 - 2\beta^3}{\alpha^3}, \quad L_\beta = \frac{\alpha^2 + 2\alpha\beta + 3\beta^2}{\alpha^2}, \\ L_{\alpha\alpha} &= \frac{2\beta^2(\alpha + 3\beta)}{\alpha^4}, \quad L_{\alpha\alpha\alpha} = -\frac{6\beta^2(\alpha + 4\beta)}{\alpha^5}. \end{aligned} \quad (3.2)$$

Substituting (3.2) into (2.1), we get

$$\begin{aligned} B^m &= \frac{r_{00}(\alpha^3 - \alpha\beta^2 - 2\beta^3) - 2s_0\alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)}{(\alpha^3 - \alpha\beta^2 - 2\beta^3) + 2(b^2\alpha^2 - \beta^2)(\alpha + 3\beta)} \\ &\quad \left(\left(\frac{(\alpha^2 + 2\alpha\beta + 3\beta^2)}{2(\alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3)} - \frac{\beta(\alpha + 3\beta)}{(\alpha^3 - \alpha\beta^2 - 2\beta^3)} \right) y^m \right. \\ &\quad \left. - \frac{\alpha^2(\alpha + 3\beta)}{(\alpha^3 - \alpha\beta^2 - 2\beta^3)} b^m \right) + \frac{\alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)}{(\alpha^3 - \alpha\beta^2 - 2\beta^3)} s_0^m. \end{aligned} \quad (3.3)$$

Again, substituting (3.2) into (2.2), (2.11), and (2.13) in respective quantities, we obtain

$$\begin{aligned} A &= (n+1) \frac{\beta^3}{\alpha^8} (\alpha^3 - \alpha\beta^2 - 2\beta^3) (\alpha^5 - 6\alpha^3\beta^2 - 12\alpha^2\beta^3 - 15\alpha\beta^4 - 12\beta^5) \\ &\quad + 2 \frac{\beta^2}{\alpha^8} (b^2\alpha^2 - \beta^2) (\alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3) (\alpha^4 + 6\alpha^3\beta + \alpha^2\beta^2 + 4\alpha\beta^3 + 6\beta^4), \\ B &= 2 \frac{\beta^2}{\alpha^4} (\alpha + 3\beta) (\alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3), \\ C &= \frac{\beta^3}{\alpha^6} (b^2\alpha^2 - \beta^2) \left((4b^2 - 1)\alpha^6 - 12b^2\alpha^5\beta - 2(4 - b^2)\alpha^4\beta^2 \right. \\ &\quad \left. - 4(2b^2 + 5)\alpha^3\beta^3 + 3(36b^2 - 1)\alpha^2\beta^4 + 16\alpha\beta^5 - 16\beta^6 \right), \\ D &= 2 \frac{\beta^3}{\alpha^4} \left(b^2(1 + 2b^2)\alpha^7 + 12b^4\alpha^6\beta + 2(3b^4 - 8b^2 - 1)\alpha^5\beta^2 \right. \\ &\quad \left. - 2(33b^2 + 1)\alpha^4\beta^3 + (36 - 41b^2)\alpha^3\beta^4 \right. \\ &\quad \left. + 2(29 - 33b^2)\alpha^2\beta^5 + 82\alpha\beta^6 + 72\beta^7 \right), \\ E &= 2 \frac{\beta^4}{\alpha^4} \left((1 + 2b^2)\alpha^6 + 6b^2\alpha^5\beta - (2b^2 + 4)\alpha^4\beta^2 - 10(b^2 + 1)\alpha^3\beta^3 \right. \\ &\quad \left. - 3(4b^2 - 1)\alpha^2\beta^4 + 14\alpha\beta^5 + 16\beta^6 \right), \end{aligned} \quad (3.4)$$

$$\begin{aligned}
W &= \frac{r_{00}(\alpha^3 - \alpha\beta^2 - 2\beta^3) - 2\alpha^2 s_0(\alpha^2 + 2\alpha\beta + 3\beta^2)}{\alpha^3}, \\
\Omega &= \frac{\beta^2}{\alpha^3} \left(\alpha^3(1 + 2b^2) + 2b^2\alpha^2\beta - 3\alpha\beta^2 - 8\beta^3 \right), \\
C^* &= \frac{\alpha}{2\beta} \left(\frac{r_{00}(\alpha^3 - \alpha\beta^2 - 2\beta^3) - 2s_0\alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)}{(\alpha^3 - \alpha\beta^2 - 2\beta^3) + 2(b^2\alpha^2 - \beta^2)(\alpha + 3\beta)} \right), \\
E^* &= \frac{\alpha(\alpha^2 + 2\alpha\beta + 3\beta^2)}{2(\alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3)} \\
&\quad \left(\frac{r_{00}(\alpha^3 - \alpha\beta^2 - 2\beta^3) - 2s_0\alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)}{(\alpha^3 - \alpha\beta^2 - 2\beta^3) + 2(b^2\alpha^2 - \beta^2)(\alpha + 3\beta)} \right),
\end{aligned}$$

Substituting (3.4) into (2.12), we get

$$\begin{aligned}
&2B_m^m \left((1 + 2b^2)^2 \alpha^{15} \beta + a_1 \alpha^{14} \beta^2 + a_2 \alpha^{13} \beta^3 - a_3 \alpha^{12} \beta^4 - 2a_4 \alpha^{11} \beta^5 \right. \\
&\quad + 2a_5 \alpha^{10} \beta^6 + 2a_6 \alpha^9 \beta^7 + 2a_7 \alpha^8 \beta^8 + a_8 \alpha^7 \beta^9 + a_9 \alpha^6 \beta^{10} + a_{10} \alpha^5 \beta^{11} \\
&\quad + a_{11} \alpha^4 \beta^{12} + 40(19 - 26b^2) \alpha^3 \beta^{13} + 12(83 - 32b^2) \alpha^2 \beta^{14} + 704 \alpha \beta^{15} + 256 \beta^{16} \Big) \\
&\quad - r_{00} \left(12b^4 \alpha^{15} + a_{12} \alpha^{14} \beta + 2b^2 a_{13} \alpha^{13} \beta^2 + a_{14} \alpha^{12} \beta^3 + 2a_{15} \alpha^{11} \beta^4 + 2a_{16} \alpha^{10} \beta^5 \right. \\
&\quad + 2a_{17} \alpha^9 \beta^6 + 2a_{18} \alpha^8 \beta^7 + 4a_{19} \alpha^7 \beta^8 + a_{20} \alpha^6 \beta^9 + 2a_{21} \alpha^5 \beta^{10} + a_{22} \alpha^4 \beta^{11} \\
&\quad + 2a_{23} \alpha^3 \beta^{12} + 2a_{24} \alpha^2 \beta^{13} - 160 \alpha \beta^{14} - 192 \beta^{15} \Big) \\
&\quad - 2s_0 \left(a_{25} \alpha^{15} \beta + 2a_{26} \alpha^{14} \beta^2 + a_{27} \alpha^{13} \beta^3 + 2a_{28} \alpha^{12} \beta^4 + 2a_{29} \alpha^{11} \beta^5 \right. \\
&\quad + 2a_{30} \alpha^{10} \beta^6 + 2a_{31} \alpha^9 \beta^7 + 6a_{32} \alpha^8 \beta^8 + a_{33} \alpha^7 \beta^9 + 2a_{34} \alpha^6 \beta^{10} + a_{35} \alpha^5 \beta^{11} \\
&\quad + 6a_{36} \alpha^4 \beta^{12} - 1384 \alpha^3 \beta^{13} - 864 \alpha^2 \beta^{14} \Big) \\
&\quad - 2r_0 \left(2(1 + 2b^2) \alpha^{15} \beta + 4(7b^2 + 2) \alpha^{14} \beta^2 + 2(28b^2 - 1) \alpha^{13} \beta^3 \right. \\
&\quad - 8(b^2 + 7) \alpha^{12} \beta^4 - 4(44b^2 + 29) \alpha^{11} \beta^5 - 12(32b^2 - 3) \alpha^{10} \beta^6 \\
&\quad + 4(-46b^2 + 63) \alpha^9 \beta^8 + 4(50b^2 + 87) \alpha^8 \beta^8 + 2(334b^2 - 17) \alpha^7 \beta^9 \\
&\quad + 4(133b^2 - 30) \alpha^6 \beta^{10} + 6(64b^2 - 95) \alpha^5 \beta^{11} + 8(18b^2 - 91) \alpha^4 \beta^{12} \\
&\quad \left. - 640 \alpha^3 \beta^{13} - 192 \alpha^2 \beta^{14} \right) = 0, \tag{3.5}
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= (28b^4 + 16b^2 + 1), \quad a_2 = (56b^4 - 4b^2 - 7), \\
a_3 &= (8b^4 + 112b^2 + 27), \quad a_4 = (88b^4 + 116b^2 + 3), \\
a_5 &= (51 - 16b^2 - 152b^4), \quad a_6 = (109 + 252b^2 - 92b^4), \\
a_7 &= (33 + 464b^2 + 60b^4), \quad a_8 = (428b^4 + 628b^2 - 355), \\
a_9 &= (532b^4 - 240b^2 - 699), \quad a_{10} = (384b^4 - 1140b^2 - 515), \\
a_{11} &= (144b^4 - 1156b^2 + 105), \quad a_{12} = (48b^4 + 8b^2 + 2b^2n + n + 1), \\
a_{13} &= (22b^2 - 10 + 3n), \quad a_{14} = (8b^4 - 120b^2 - 16b^2n - 11n - 17), \\
a_{15} &= (78b^4 - 26b^2 - 40b^2n - 8n - 1), \quad a_{16} = (412b^4 - 50b^2n + 11n + 79), \\
a_{17} &= (294b^4 - 17b^2 + 34b^2n + 44n + 168), \\
a_{18} &= (-324b^4 - 250b^2 + 196b^2n + 41n + 95), \\
a_{19} &= (-562b^4 + 87b^2 + 138b^2n - 18n - 91), \\
a_{20} &= (1952b^4 + 1356b^2 + 310b^2n - 247n - 835), \\
a_{21} &= (-1320b^4 + 164b^2 - 237b^2n - 168n - 468), \\
a_{22} &= (-1440b^4 + 2216b^2 - 840b^2n + 201n - 557), \\
a_{23} &= (1064b^2 - 372b^2n + 156n - 33), \\
a_{24} &= (672b^2 - 144b^2n + 36n - 16), \\
a_{25} &= (-4b^2 - 2b^2n - n + 1), \quad a_{26} = (-8b^2 - 5nb^2 - n + 1), \\
a_{27} &= (-24b^4 - 96b^2 - 4b^2n + 7n + 11), \\
a_{28} &= (-12b^4 - 180b^2 + 40b^2n + 17n + 11), \\
a_{29} &= (-555b^4 - 280b^2 + 142b^2n + 23n + 63), \\
a_{30} &= (587b^4b^4 - 387b^2 + 222b^2n - 18n + 298), \\
a_{31} &= (-334b^4 - 854b^2 + 108b^2n - 105n + 467), \\
a_{32} &= (-12b^4 - 79b^2 - 88b^2n - 29n + 193), \\
a_{33} &= (72b^4 - 564b^2 - 1386b^2n - 117n + 13), \\
a_{34} &= (-214b^2 - 849b^2n + 147n - 1903), \\
a_{35} &= (-312b^2 - 1188b^2n + 603n + 631), \quad a_{36} = (-72b^2n + 87n - 304).
\end{aligned}$$

Now we assume that F^n is a weakly Berwald space, then B_m^m is hp(1). Since α is irrational in (y^i) , the equation (3.5) is divided into two equations as follows,

$$\beta^2 B_m^m F_1 + \beta r_{00} G_1 + \alpha^2 \beta^2 s_0 H_1 + \alpha^2 \beta^2 r_0 I_1 = 0, \quad (3.6)$$

$$\alpha \beta B_m^m F_2 + \alpha r_{00} G_2 + \alpha^3 \beta s_0 H_2 + \alpha^3 \beta r_0 I_2 = 0, \quad (3.7)$$

where

$$\begin{aligned}
F_1 &= 2a_1\alpha^{14} - 2a_3\alpha^{12}\beta^2 + 4a_5\alpha^{10}\beta^4 + 4a_7\alpha^8\beta^6 + 2a_9\alpha^6\beta^8 \\
&\quad + 2a_{11}\alpha^4\beta^{10} + 24(83 - 32b^2)\alpha^2\beta^{12} + 512\beta^{14}, \\
F_2 &= 2a_1\alpha^{14} + 2a_2\alpha^{12}\beta^2 - 4a_4\alpha^{10}\beta^4 + 4a_6\alpha^8\beta^6 + 2a_8\alpha^6\beta^8 \\
&\quad + 2a_{10}\alpha^4\beta^{10} + 80(19 - 26b^2)\alpha^2\beta^{12} + 1408b^{14}, \\
G_1 &= -(a_{12}\alpha\alpha^{14} + a_{14}\alpha^{12}\beta^2 + 2a_{16}\alpha^{10}\beta^4 + 2a_{18}\alpha^8\beta^6 + a_{20}\alpha^6\beta^8 + a_{22}\alpha^4\beta^{10} \\
&\quad + 2a_{24}\alpha^2\beta^{12} - 192\beta^{14}), \\
G_2 &= -2(6b^4\alpha^{14} + b^2a_{13}\alpha^{12}\beta^2 + a_{15}\alpha^{10}\beta^4 + a_{17}\alpha^8\beta^6 + 2a_{19}\alpha^6\beta^8 + a_{21}\alpha^4\beta^{10} \\
&\quad + a_{23}\alpha^2\beta^{12} - 80\beta^{14}), \\
H_1 &= -4(a_{26}\alpha^{12} + a_{28}\alpha^{10}\beta^2 + a_{30}\alpha^8\beta^4 + 3a_{32}\alpha^6\beta^6 + a_{34}\alpha^4\beta^8 + 3a_{36}\alpha^2\beta^{10} \\
&\quad - 432\beta^{12}), \\
H_2 &= -2(a_{25}\alpha^{12} + a_{27}\alpha^{10}\beta^2 + 2a_{29}\alpha^8\beta^4 + 2a_{31}\alpha^6\beta^6 + a_{33}\alpha^4\beta^8 + a_{35}\alpha^2\beta^{10} \\
&\quad - 1384\beta^{12}), \\
I_1 &= -8\{(7b^2 + 2)\alpha^{12} - 2(b^2 + 7)\alpha^{10}\beta^2 - 3(32b^2 - 3)\alpha^8\beta^4 \\
&\quad + (50b^2 + 87)\alpha^6\beta^6 + (133b^2 - 30)\alpha^4\beta^8 + 2(18b^2 - 91)\alpha^2\beta^{10} - 48\beta^{12}\}, \\
I_2 &= -4\{(1 + 2b^2)\alpha^{12} + (28b^2 - 1)\alpha^{10}\beta^2 - 2(44b^2 + 29)\alpha^8\beta^4 \\
&\quad + 2(-46b^2 + 63)\alpha^6\beta^6 + (334b^2 - 17)\alpha^4\beta^8 \\
&\quad + 3(64b^2 - 95)\alpha^2\beta^{10} - 320\beta^{12}\}.
\end{aligned}$$

Eliminating B_m^m from these equations, we obtain

$$Rr_{00} + \alpha^2\beta S_0 + \alpha^2\beta Tr_0 = 0, \quad (3.8)$$

where

$$R = \beta^2 F_2 G_1 - F_1 G_2, \quad S = F_2 H_1 - F_1 H_2, \quad T = I_1 F_2 - I_2 F_1.$$

From (3.8), we get

$$(R/\alpha^2\beta)r_{00} + Ss_0 + Tr_0 = 0. \quad (3.9)$$

Since only the term $\epsilon_1\alpha^{26}$ of Ss_0 in (3.9) does not contain β , we must have $hp(26)V_{26}$, such that

$$\alpha^{26}s_0 = \beta V_{26}, \quad (3.10)$$

where

$$\epsilon_1 = -4a_1(a_{26} - a_{25}).$$

First consider that $\alpha^2 \neq 0(mod\beta)$ and $b^2 \neq 0$. (3.10) shows the existence of a function $k(x)$ satisfying $V_{26} = k\alpha^{26}$, and hence $s_0 = k\beta$, (3.9) reduces to

$$(R/\alpha^2\beta)r_{00} + Sk\beta + Tr_0 = 0,$$

which implies that

$$Rr_{00} + Sk\alpha^2\beta^2 + \alpha^2\beta Tr_0 = 0.$$

Only the term $-2a_1\alpha^{28}(a_{12} - 12b^4)r_{00}$ of the above does not contain β . Thus there exist $hp(29)U_{29}$ satisfying $-2a_1\alpha^{28}(a_{12} - 12b^4)r_{00} = \beta U_{29}$. It is a contradiction, which implies $k = 0$. Hence we obtain $s_0 = 0; s_j = 0$. Therefore, (3.8) becomes

$$Rr_{00} + \alpha^2\beta Tr_0 = 0. \quad (3.11)$$

Only the term $188416\beta^{28}r_{00}$ of (34) seemingly does not contain α^2 , and hence we must have $hp(28)V_{28}$ such that $\beta^{28}r_{00} = \alpha^2V_{28}$. From $\alpha^2 \neq 0(mod\beta)$ there exist a function $f(x)$ such that

$$r_{00} = \alpha^2 f(x); \quad r_{ij} = a_{ij}f(x). \quad (3.12)$$

Transvecting above by $b^i y^j$, we have

$$r_0 = \beta f(x); \quad r_j = b_j f(x). \quad (3.13)$$

Substituting (3.12) and (3.13) into (3.11), we have

$$f(x)(R + \beta^2 T) = 0. \quad (3.14)$$

Assume that $f(x) \neq 0$, from (3.14) we get

$$R + \beta^2 T = 0,$$

The term $-2a_1\alpha^{28}(a_{12} - 12b^4)$ of the above does not contain β . Thus there exist $hp(27)V_{27}$ satisfying $-2a_1\alpha^{28}(a_{12} - 12b^4) = \beta V_{27}$, where V_{27} is $hp(27)$ this implies $V_{27} = 0$, provided that $b^2 \neq 0$. Hence $f(x) = 0$ must hold and we obtain

$$r_{00} = 0; \quad r_{ij} = 0 \text{ and } r_0 = 0; \quad r_j = 0.$$

Conversely, substituting $r_{00} = 0, s_0 = 0$ and $r_0 = 0$ into (3.5), we have $B_m^m = 0$, that is, the Finsler space with (3.1) is a weakly-Berwald space.

On the other hand, if we suppose that the Finsler space with (3.1) is a Berwald space, then we have $r_{00} = 0, s_0 = 0$ and $r_0 = 0$, because the space is weakly Berwald space from the above discussion. Hence $s_{ij} = 0$ hold good.

Now consider $\alpha^2 \equiv 0(mod\beta)$, Lemma (2.2) shows that $n = 2, b^2 = 0$ and $\alpha^2 = \beta\delta, \delta = d_i(x)y_i$. From these conditions (3.8) is rewritten in the form

$$R'r_{00} + \beta\delta S' s_0 = 0, \quad (3.15)$$

where

$$\begin{aligned}
R' = & a_1 a_{12} \beta^{14} \delta^{14} + (a_1 a_{14} + a_2 a_{12}) \beta^{15} \delta^{13} + (2a_1 a_{16} + a_2 a_{14} - 2a_4 a_{12} \\
& - 2a_1 a_{15}) \beta^{16} \delta^{12} + 2(a_1 a_{18} + a_2 a_{16} - a_4 a_{14} + a_{12} a_6 - a_1 a_{17} + a_3 a_{15}) \beta^{17} \delta^{11} \\
& + (a_1 a_{20} + 2a_2 a_{18} - 4a_4 a_{16} + 2a_6 a_{14} + a_8 a_{12} - 4a_1 a_{19} + 2a_3 a_{17} \\
& - 4a_5 a_{15}) \beta^{18} \delta^{10} + (a_1 a_{22} + a_2 a_{20} - 4a_4 a_{18} + 4a_6 a_{16} + a_8 a_{14} + a_{10} a_{12} \\
& - 2a_1 a_{21} + 4a_3 a_{19} - 4a_{17} a_5 - 4a_7 a_{15}) \beta^{19} \delta^9 + (2a_1 a_{24} + a_2 a_{22} - 2a_4 a_{20} \\
& + 4a_6 a_{18} + 2a_8 a_{16} + a_{10} a_{14} + 760 - 2a_1 a_{23} + 2a_3 a_{21} - 8a_5 a_{19} - 4a_7 a_{19} \\
& - 2a_9 a_{15}) \beta^{20} \delta^8 + 2(-96 + a_2 a_{24} - a_4 a_{22} + a_6 a_{20} + a_8 a_{18} + a_{10} a_{16} + 380a_{14} \\
& + 352 + 80a_1 + a_3 a_{23} - 2a_5 a_{21} - 4a_7 a_{19} - a_9 a_{17} - a_{11} a_{15}) \beta^{21} \delta^7 + (-192 \\
& - 4a_4 a_{24} + 2a_6 a_{22} + a_8 a_{20} + 2a_{10} a_{18} + 1520a_{16} + 704a_{14} - 160a_3 - 4a_5 a_{23} \\
& - 4a_7 a_{21} - 4a_9 a_{19} - 2a_{11} a_{17} - 1992a_{15}) \beta^{22} \delta^6 + (384a_4 + 4a_6 a_{24} + a_8 a_{22} \\
& + a_{10} a_{20} + 1520a_{18} + 1408a_{16} + 320a_5 - 4a_7 a_{23} - 2a_9 a_{21} - 4a_{11} a_{19} \\
& - 1992a_{17} - 512a_{15}) \beta^{23} \delta^5 + (-384a_6 + 2a_8 a_{24} + a_{10} a_{22} + 760a_{20} \\
& + 1408a_{18} + 320 - 2a_9 a_{23} - 2a_{11} a_{21} - 3984a_{19} - 512a_{17}) \beta^{24} \delta^4 \\
& + (-a_8 + 2a_{10} a_{24} + 760a_{22} + 704a_{20} + 160a_9 - 2a_{11} a_{23} - 1992a_{21} \\
& - 1024a_{19}) \beta^{25} \delta^3 + 8(-24a_{10} + 190a_{24} + 88a_{22} + 20a_{11} - 249a_2 \\
& - 64a_{21}) \beta^{26} \delta^2 + 128(105 + 11a_{24} - 4a_{23}) \beta^{27} \delta - 94208 \beta^{28},
\end{aligned}$$

$$\begin{aligned}
S' = & (2a_1 a_{26} - a_1 a_{25}) \beta^{13} \delta^{13} + (2a_1 a_{28} + 2a_2 a_{26} - a_1 a_{27} - a_3 a_{25}) \beta^{14} \delta^{12} \\
& + (2a_1 a_{30} + 2a_2 a_{28} - 4a_4 a_{26} - 2a_1 a_{29} + a_3 a_{27} - 2a_5 a_{25}) \beta^{15} \delta^{11} \\
& + (6a_1 a_{32} + 2a_2 a_{30} - 4a_4 a_{28} + 4a_6 a_{26} - 2a_1 a_{31} + 2a_3 a_{29} - 2a_5 a_{27} \\
& - 2a_7 a_{25}) \beta^{16} \delta^{10} + (2a_1 a_{34} + 6a_2 a_{32} - 4a_4 a_{30} + 4a_6 a_{28} + 2a_8 a_{26} \\
& - a_1 a_{33} + 2a_3 a_{31} - 4a_5 a_{29} - 2a_7 a_{27} - a_9 a_{25}) \beta^{17} \delta^9 + (6a_1 a_{36} \\
& + 2a_2 a_{34} - 12a_4 a_{32} + 4a_6 a_{30} + 2a_8 a_{28} + 2a_{10} a_{26} - a_1 a_{35} + a_3 a_{33} \\
& - 4a_5 a_{31} - 4a_7 a_{29} - a_9 a_{27} - a_{11} a_{25}) \beta^{18} \delta^8 + (-864a_1 + 6a_2 a_{36} \\
& - 4a_4 a_{34} + 12a_6 a_{32} + 2a_8 a_{30} + 2a_{10} a_{28} + 1520a_{26} + 1384 + a_3 a_{35} \\
& - 2a_5 a_{33} - 4a_7 a_{31} - 2a_9 a_{29} - a_{11} a_{27} - 996a_{25}) \beta^{19} \delta^7 + (-864a_2 \\
& - 12a_3 a_{36} + 4a_4 a_{34} + 6a_6 a_{32} + 2a_{10} a_{30} + 1520a_{28} + 1408a_{26} - 1384 \\
& - 2a_5 a_{35} - 2a_7 a_{33} - 2a_9 a_{31} - 2a_{11} a_{29} - 996 - 256a_{25}) \beta^{20} \delta^6 \\
& + (-4.432 + 12a_6 a_{36} + 2a_8 a_{34} + 6a_{10} a_{32} + 1520a_{30} + 1408a_{28} \\
& + 2.1384 - 2a_7 a_{35} - a_9 a_{33} - 2a_{11} a_{31} - 1992a_{29} - 256a_{27}) \beta^{21} \delta^5 \\
& + (-4.432 + 6a_8 a_{36} + 2a_{10} a_{34} + 3.1520a_{32} + 1408 + 2.1384 - a_9 a_{35}
\end{aligned}$$

$$\begin{aligned}
& -a_{11}a_{33} - 1992a_{31} - 512a_{29})\beta^{22}\delta^4 + (-864a_8 + 6a_{10}a_{36} + 1520a_{34} \\
& + 3.1408 + 1384 - a_{11}a_{35} - 996a_{33} - 512a_{31})\beta^{23}\delta^3 \\
& + (-864a_{10} + 1520.3a_{36} + 1408a_{34} + 1384a_{11} - 996a_{35} - 256a_{33})\beta^{24}\delta^2 \\
& + (-432.1520 + 3.1408a_{36} + 996.1384 - 256a_{35})\beta^{25}\delta \\
& - (608256 - 354304)\beta^{26},
\end{aligned}$$

Since only the term $94208\beta^{28}$ of $R'r_{00} + \beta\delta S's_0$ in (3.15) seemingly does not contain δ , we must have $hp(1)V_1$ such that $r_{00} = \delta V_1$. We have $s_0 = 0$; $s_j = 0$, now (3.15) becomes

$$R'r_{00} = 0, \quad (3.16)$$

which implies

$$r_{00} = 0; \quad r_{ij} = 0 \text{ and } r_0 = 0; \quad r_j = 0.$$

Conversely from $r_{00} = 0$, $r_0 = 0$ and $s_0 = 0$ we have $B_m^m = 0$. Thus the space with (3.1) is weakly-Berwald space. Thus we state that

Theorem 3.1. *A Finsler space with the metric (3.1) is weakly Berwald space if and only if the following conditions holds;*

- (1) $\alpha^2 \neq 0(\text{mod}\beta)$ implies $r_{ij} = 0$ and $s_j = 0$.
- (2) $\alpha^2 \equiv 0(\text{mod}\beta)$ implies $n = 2$, $b^2 = 0$ and $r_{ij} = 0$, $s_j = 0$ are satisfied, where $\alpha^2 = \beta\delta$, $\delta = d_i y^i$.

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