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# Computing the Determinant and Inverse of the Complex Fibonacci Hermitian Toeplitz Matrix


## Abstract

In this paper, we consider the determinant and the inverse of the complex Fibonacci Hermitian Toeplitz matrix, firstly, give the definition of the complex Fibonacci Hermitian Toeplitz matrix. Secondly, we computing the determinant and inverse of the complex Fibonacci Hermitian Toeplitz matrix by constructing the transformation matrices.


*Keywords: Hermitian Toeplitz matrix, determinant, inverse, complex Fibonacci number.*

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## 1 Introduction

Studies show that there has been an increasing interest on Fibonacci sequence and their generalizations. One of them is the concept of complex Fibonacci numbers $F_n^*$, which was defined by Horadam [1]. The $n$-th complex Fibonacci number is given by the equality $F_n^* = F_n + iF_{n+1}$, where $i$ is the imaginary unit which satisfies $i^2 = -1$. Complex Fibonacci numbers satisfy the same recurrence relation $F_n^* = F_{n-1}^* + F_{n-2}^* (n \geqslant 2)$ of the classical Fibonacci numbers with different initial conditions, i.e., $F_0^* = i, F_1^* = 1 + i$.

On the other hand, Hermitian Toeplitz matrices have important applications in various disciplines including image processing, signal processing, and solving least squares problems [2, 3, 4].

It is an ideal research area and hot topic for the inverses of the special matrices with famous numbers. Some scholars showed the explicit determinants and inverses of the special matrices involving famous numbers. Lin showed the determinant of the Fibonacci-Lucas quasi-cyclic matrices in [5]. Circulant matrices with Fibonacci and Lucas numbers are discussed and their explicit determinants and inverses are proposed in [6]. The authors provided determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [7]. In [8], circulant type matrices with the $k$-Fibonacci and $k$-Lucas numbers are considered and the explicit determinants and inverse matrices are presented by constructing the transformation matrices. The explicit determinants of

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circulant and left circulant matrices including Tribonacci numbers and generalized Lucas numbers are shown based on Tribonacci numbers and generalized Lucas numbers in [9]. And for the RSFPLR circulant matrices and the RSLPFL circulant matrices involving Padovan, Perrin, Tribonacci and the generalized Lucas numbers by the inverse factorization of polynomial in [10]. Jiang et al. [11] gave the invertibility of circulant type matrices with the sum and product of Fibonacci and Lucas numbers and provided the determinants and the inverses of these matrices. It should be noted that Jiang and Zhou [12] obtained the explicit formula for spectral norm of an $r$-circulant matrix whose entries in the first row are alternately positive and negative, and the authors [13] investigated explicit formulas of spectral norms for $g$-circulant matrices with Fibonacci and Lucas numbers. Furthermore, in [14] the determinants and inverses are discussed and evaluated for Tribonacci skew circulant type matrices. The authors [15] proposed the invertibility criterium of the generalized Lucas skew circulant type matrices and provided their determinants and the inverse matrices. The determinants and inverses of Tribonacci circulant type matrices are discussed in [16]. Determinants and inverses of Fibonacci and Lucas skew symmetric Toeplitz matrices are given by constructing the special transformation matrices in [17].

The purpose of this paper is to obtain the better results for the determinants and inverses of complex Fibonacci Hermitian Toeplitz matrix. In this paper we adopt the following two conventions $0^0 = 1$, $i^2 = -1$, ~~then~~ we define a kind of special matrix ~~defined as follow.~~

**Definition 1.1.** A complex Fibonacci Hermitian Toeplitz matrix is ~~meant~~ a square matrix of the form

$$\mathbf{T}_{F_n^*} = \begin{pmatrix} 0 & F_0^* & F_1^* & \cdots & F_{n-4}^* & F_{n-3}^* & F_{n-2}^* \\ \bar{F}_0^* & 0 & F_0^* & \cdots & F_{n-5}^* & F_{n-4}^* & F_{n-3}^* \\ \bar{F}_1^* & \bar{F}_0^* & 0 & \cdots & F_{n-6}^* & F_{n-5}^* & F_{n-4}^* \\ & \cdots & & \cdots & & \cdots & \\ \bar{F}_{n-4}^* & \bar{F}_{n-5}^* & \bar{F}_{n-6}^* & \cdots & 0 & F_0^* & F_1^* \\ \bar{F}_{n-3}^* & \bar{F}_{n-4}^* & \bar{F}_{n-5}^* & \cdots & \bar{F}_0^* & 0 & F_0^* \\ \bar{F}_{n-2}^* & \bar{F}_{n-3}^* & \bar{F}_{n-4}^* & \cdots & \bar{F}_1^* & \bar{F}_0^* & 0 \end{pmatrix}_{n\times n}, \tag{1.1}$$

where $F_0^*,\ F_1^*, \cdots,\ F_{n-2}^*$ are the complex Fibonacci numbers.

It is evidently determined by its first row and first column, and $\mathbf{T}_{F_n^*} = \mathbf{T}_{F_n^*}^*$.

**Lemma 1.1.** *Let $p_i$ is a Hessenberg matrix,*

$$p_i = \begin{pmatrix} \kappa_2 & \kappa_1 & 0 & \cdots & \cdots & \cdots & 0 \\ \kappa_3 & \kappa_2 & \kappa_1 & \ddots & & & \vdots \\ \kappa_4 & \kappa_3 & \kappa_2 & \kappa_1 & \ddots & & \vdots \\ \kappa_5 & \kappa_4 & \kappa_3 & \kappa_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \kappa_1 \\ \kappa_{i+1} & \cdots & \cdots & \kappa_5 & \kappa_4 & \kappa_3 & \kappa_2 \end{pmatrix}_{i\times i},$$

*then the determinant of $p_i$ can be expressed as*

$$\det p_i = \sum_{j=1}^{i} (-1)^{i+j} \kappa_{i-j+2} \kappa_1^{i-j} \det p_{j-1},\ \ i \geqslant 2,$$

*~~with~~ the initial values of $\det p_i$ are $\det p_0 = 1$, $\det p_1 = \kappa_2$.*

where ^and

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*Proof.* Do the Laplace expansion of matrix $p_i$ along the last row, it is easy to check that

$$\begin{aligned}\det p_i &= (-1)^{i+1}\kappa_{i+1}\kappa_1^{i-1}\det p_0 + (-1)^{i+2}\kappa_i\kappa_1^{i-2}\det p_1 \\ &+ (-1)^{i+3}\kappa_{i-1}\kappa_1^{i-3}\det p_2 + \cdots + (-1)^{i+i-1}\kappa_3\kappa_1\det p_{i-2} \\ &+ (-1)^{i+i}\kappa_2\kappa_1^0\det p_{i-1} \\ &= \sum_{j=1}^{i}(-1)^{i+j}\kappa_{i-j+2}\kappa_1^{i-j}\det p_{j-1},\end{aligned}$$

for $i \geqslant 2$. The initial values of $\det p_i$ are $\det p_0 = 1,\ \det p_1 = \kappa_2$. □

**Lemma 1.2.** *Define an $i \times i$ Toeplitz-like matrix by*

$$\nabla_i([\mu_k]_{k=1}^i, \alpha_1, \alpha_2, \cdots, \alpha_i) = \begin{pmatrix} \mu_i & \mu_{i-1} & \mu_{i-2} & \mu_{i-3} & \mu_{i-4} & \cdots & \mu_2 & \mu_1 \\ \alpha_2 & \alpha_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & \ddots & & & & \vdots \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \ddots & & & \vdots \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 0 \\ \alpha_i & \cdots & \cdots & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{pmatrix}_{i\times i},$$

*we have*

$$\det \nabla_i([\mu_k]_{k=1}^i, \alpha_1, \alpha_2, \cdots, \alpha_i) = \sum_{j=1}^{i}(-1)^{1+j}\mu_{i+1-j}\alpha_1^{i-j}\det p_{j-1},$$

*where* $\det p_j (0 \leqslant j \leqslant i-1)$ *is the determinant of Hessenberg matrix, and can be calculated by Lemma 1.1.*

*Proof.* Do the Laplace expansion of matrix $\nabla_i([\mu_k]_{k=1}^i, \alpha_1, \alpha_2, \cdots, \alpha_i)$ along the first row, it is easy to check that

$$\begin{aligned}\det \nabla_i([\mu_k]_{k=1}^i, \alpha_1, \alpha_2, \cdots, \alpha_i) &= (-1)^{1+1}\mu_i\alpha_1^{i-1}\det p_0 + (-1)^{1+2}\mu_{i-1}\alpha_1^{i-2}\det p_1 + (-1)^{1+3}\mu_{i-2}\alpha_1^{i-3}\det p_2 \\ &+ \cdots + (-1)^{1+i-1}\mu_2\alpha_1\det p_{i-2} + (-1)^{1+i}\mu_1\alpha_1^0\det p_{i-1} \\ &= \sum_{j=1}^{i}(-1)^{1+j}\mu_{i-j+1}\alpha_1^{i-j}\det p_{j-1},\end{aligned}$$

where $p_j (0 \leqslant j \leqslant i-1)$ is the determinant of Hessenberg matrix, and can be calculated by Lemma 1.1. □

## 2 Determinant and Inverse of the Complex Fibonacci Hermitian Toeplitz Matrix

Let $\mathbf{T}_{F_n^*}$ be a complex Fibonacci Hermitian Toeplitz matrix. We give the determinant and inverse of the matrix $\mathbf{T}_{F_n^*}$ in this section, at first we give the determinant of the matrix $\mathbf{T}_{F_n^*}$, and then we find the inverse of the matrix $\mathbf{T}_{F_n^*}$ supposed that it is invertible.

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**Theorem 2.1.** *Let $\mathbf{T}_{F_n^*}$ be a complex Fibonacci Hermitian Toeplitz matrix as the form of (1.1), then we have* . Then

$$\det \mathbf{T}_{F_1^*} = 0,\ \ \det \mathbf{T}_{F_2^*} = -1,\ \ \det \mathbf{T}_{F_3^*} = -2,\ \ \det \mathbf{T}_{F_4^*} = 6,\ \ \det \mathbf{T}_{F_5^*} = -2$$

*and*

$$\begin{aligned}\det \mathbf{T}_{F_n^*} = &-\bar{F}_0^*[F_0^*(K_3 \det \nabla_{n-3}([\eta_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})\\ &- K_4 \det \nabla_{n-3}([\xi_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}))\\ &- \bar{F}_{n-3}^*(K_1 \det \nabla_{n-3}([\eta_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})\\ &- K_4 \det \nabla_{n-3}([F_k^*]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}))\\ &+ \bar{F}_{n-4}^*(K_1 \det \nabla_{n-3}([\xi_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})\\ &- K_3 \det \nabla_{n-3}([F_k^*]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}))],\ \ n > 5,\end{aligned} \tag{2.1}$$

*where*

$$K_1 = \sum_{k=1}^{n-2} F_k^* \Delta_{n-2-k},\ \ K_3 = \sum_{k=1}^{n-2} \xi_k \Delta_{n-2-k},\ \ K_4 = \sum_{k=1}^{n-2} \eta_k \Delta_{n-2-k},$$

$\Delta_0 = 1,\ \Delta_1$ *is one root of the characteristic equation* $-x^2 + \alpha x + t_0 = 0,$

$$\Delta_2 = \Delta_1^2,\ \ \Delta_i = \alpha \Delta_{i-1} + \sum_{k=0}^{i-2} t_k \Delta_{i-2-k},\ \ (3 \leqslant i \leqslant n-3)$$

$$\xi_i = xF_{i-1}^* + \bar{F}_{n-i-3}^*,\ \ (1 \leqslant i \leqslant n-3),\ \ \xi_{n-2} = xF_{n-3}^*,$$

$$\eta_i = yF_{i-1}^* + \bar{F}_{n-4-i},\ \ (1 \leqslant i \leqslant n-4),\ \ \eta_{n-3} = yF_{n-4}^*,\ \ \eta_{n-2} = yF_{n-3}^* + F_0^*,$$

$$\alpha = 2i,\ \ t_0 = F_2^*,\ \ t_i = 2F_i^*,\ \ (1 \leqslant i \leqslant n-5),\ \ x = -\frac{\bar{F}_{n-2}^*}{\bar{F}_0^*},\ \ y = -\frac{\bar{F}_{n-3}^*}{\bar{F}_0^*},$$

$\det \nabla_{n-3}([\eta_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}),\ \det \nabla_{n-3}([\xi_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})$ *and* $\det \nabla_{n-3}([F_k^*]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})$ *can be calculated by Lemma 1.2.*

*Proof.* Let $\mathbf{T}_{F_n^*}$ be a complex Fibonacci Hermitian Toeplitz matrix, and we can easily get the following conclusions:

$$\det \mathbf{T}_{F_1^*} = 0,\ \ \det \mathbf{T}_{F_2^*} = -1,\ \ \det \mathbf{T}_{F_3^*} = -2,\ \ \det \mathbf{T}_{F_4^*} = 6,\ \ \det \mathbf{T}_{F_5^*} = -2.$$

Introduce the following two transformation matrices when $n > 5$,

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & x & \vdots & & & & & \iddots & 1 \\ \vdots & y & \vdots & & & & \iddots & 1 & 0 \\ \vdots & 0 & \vdots & & & \iddots & 1 & 1 & -1 \\ \vdots & \vdots & \vdots & & \iddots & 1 & 1 & -1 & 0 \\ \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & \vdots & 0 & 1 & 1 & -1 & \iddots & & \vdots \\ 0 & 0 & 1 & 1 & -1 & 0 & \cdots & \cdots & 0 \end{pmatrix}_{n\times n},$$

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and

$$\mathcal{N}_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \Delta_{n-3} & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ \vdots & \vdots & \Delta_{n-4} & \vdots & & & \iddots & 1 & 0 \\ \vdots & \vdots & \Delta_{n-5} & \vdots & & \iddots & 1 & \iddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \vdots & \Delta_2 & 0 & \iddots & \iddots & & & \vdots \\ \vdots & \vdots & \Delta_1 & 1 & \iddots & & & & \vdots \\ 0 & 0 & \Delta_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n\times n},$$

where

$$x = -\frac{\bar{F}^*_{n-2}}{\bar{F}^*_0},\ y = -\frac{\bar{F}^*_{n-3}}{\bar{F}^*_0},\ \Delta_0 = 1,$$

$\Delta_1$ is one root of the characteristic equation $-x^2 + \alpha x + t_0 = 0$,

$$\Delta_2 = \Delta_1^2,\ \Delta_i = \alpha\Delta_{i-1} + \sum_{k=0}^{i-2} t_k \Delta_{i-2-k},\ (3 \leqslant i \leqslant n-3),$$

$$\alpha = 2i,\ t_0 = F^*_2,\ t_i = 2F^*_i,\ (1 \leqslant i \leqslant n-5).$$

By using $\mathcal{M}_1$, $\mathcal{N}_1$ and the recurrence relations of the complex Fibonacci sequences, the matrix $\mathbf{T}_{F^*_n}$ is changed into the following form,

$$\mathcal{M}_1\mathbf{T}_{F^*_n}\mathcal{N}_1 = \begin{pmatrix} 0 & F^*_0 & K_1 & F^*_{n-3} & F^*_{n-4} & F^*_{n-5} & F^*_{n-6} & \cdots & F^*_2 & F^*_1 \\ \bar{F}^*_0 & 0 & K_2 & F^*_{n-4} & F^*_{n-5} & F^*_{n-6} & F^*_{n-7} & \cdots & F^*_1 & F^*_0 \\ 0 & \bar{F}^*_{n-3} & K_3 & \xi_{n-3} & \xi_{n-4} & \xi_{n-5} & \xi_{n-6} & \cdots & \xi_2 & \xi_1 \\ \vdots & \bar{F}^*_{n-4} & K_4 & \eta_{n-3} & \eta_{n-4} & \eta_{n-5} & \eta_{n-6} & \cdots & \eta_2 & \eta_1 \\ \vdots & 0 & 0 & \alpha & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & t_0 & \alpha & -1 & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & t_1 & t_0 & \alpha & -1 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & t_{n-6} & \cdots & \cdots & t_1 & t_0 & \alpha & -1 \end{pmatrix}_{n\times n},$$

where

$$\xi_i = xF^*_{i-1} + \bar{F}^*_{n-i-3},\ (1 \leqslant i \leqslant n-3),\ \xi_{n-2} = xF^*_{n-3},$$
$$\eta_i = yF^*_{i-1} + \bar{F}_{n-4-i},\ (1 \leqslant i \leqslant n-4),\ \eta_{n-3} = yF^*_{n-4},\ \eta_{n-2} = yF^*_{n-3} + F^*_0,$$
$$\alpha = 2i,\ t_0 = F^*_2,\ t_i = 2F^*_i,\ (1 \leqslant i \leqslant n-5),$$
$$K_1 = \sum_{k=1}^{n-2} F^*_k \Delta_{n-2-k},\ K_2 = \sum_{k=0}^{n-3} F^*_k \Delta_{n-3-k},\ K_3 = \sum_{k=1}^{n-2} \xi_k \Delta_{n-2-k},\ K_4 = \sum_{k=1}^{n-2} \eta_k \Delta_{n-2-k}.$$

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Do the Laplace expansion of matrix $\mathcal{M}_1\mathbf{T}_{F_n^*}\mathcal{N}_1$ along the first column, we can get that

$$\begin{aligned}\det(\mathcal{M}_1\mathbf{T}_{F_n^*}\mathcal{N}_1) = &-\bar{F}_0^*[F_0^*(K_3\det\nabla_{n-3}([\eta_k]_{k=1}^{n-3},-1,\alpha,t_0,t_1,\cdots,t_{n-6})\\ &-K_4\det\nabla_{n-3}([\xi_k]_{k=1}^{n-3},-1,\alpha,t_0,t_1,\cdots,t_{n-6}))\\ &-\bar{F}_{n-3}^*(K_1\det\nabla_{n-3}([\eta_k]_{k=1}^{n-3},-1,\alpha,t_0,t_1,\cdots,t_{n-6})\\ &-K_4\det\nabla_{n-3}([F_k^*]_{k=1}^{n-3},-1,\alpha,t_0,t_1,\cdots,t_{n-6}))\\ &+\bar{F}_{n-4}^*(K_1\det\nabla_{n-3}([\xi_k]_{k=1}^{n-3},-1,\alpha,t_0,t_1,\cdots,t_{n-6})\\ &-K_3\det\nabla_{n-3}([F_k^*]_{k=1}^{n-3},-1,\alpha,t_0,t_1,\cdots,t_{n-6}))],\end{aligned}$$

where

$$\begin{aligned}&\nabla_{n-3}([\eta_k]_{k=1}^{n-3},-1,\alpha,t_0,t_1,\cdots,t_{n-6})\\ &=\begin{pmatrix}\eta_{n-3} & \eta_{n-4} & \eta_{n-5} & \eta_{n-6} & \cdots & \eta_4 & \eta_3 & \eta_2 & \eta_1\\ \alpha & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ t_0 & \alpha & -1 & \ddots & & & & & \vdots\\ t_1 & t_0 & \alpha & -1 & \ddots & & & & \vdots\\ t_2 & t_1 & t_0 & \alpha & \ddots & \ddots & & & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots\\ \vdots & \vdots & & \ddots & \ddots & \alpha & -1 & \ddots & \vdots\\ \vdots & \vdots & & & \ddots & t_0 & \alpha & -1 & 0\\ t_{n-6} & t_{n-7} & \cdots & \cdots & \cdots & t_1 & t_0 & \alpha & -1\end{pmatrix}_{(n-3)\times(n-3)},\end{aligned}$$

$$\begin{aligned}&\nabla_{n-3}([\xi_k]_{k=1}^{n-3},-1,\alpha,t_0,t_1,\cdots,t_{n-6})\\ &=\begin{pmatrix}\xi_{n-3} & \xi_{n-4} & \xi_{n-5} & \xi_{n-6} & \cdots & \xi_4 & \xi_3 & \xi_2 & \xi_1\\ \alpha & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ t_0 & \alpha & -1 & \ddots & & & & & \vdots\\ t_1 & t_0 & \alpha & -1 & \ddots & & & & \vdots\\ t_2 & t_1 & t_0 & \alpha & \ddots & \ddots & & & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots\\ \vdots & \vdots & & \ddots & \ddots & \alpha & -1 & \ddots & \vdots\\ \vdots & \vdots & & & \ddots & t_0 & \alpha & -1 & 0\\ t_{n-6} & t_{n-7} & \cdots & \cdots & \cdots & t_1 & t_0 & \alpha & -1\end{pmatrix}_{(n-3)\times(n-3)},\end{aligned}$$

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$$\nabla_{n-3}([F_k^*]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})$$
$$= \begin{pmatrix} F_{n-3}^* & F_{n-4}^* & F_{n-5}^* & F_{n-6}^* & \cdots & F_4^* & F_3^* & F_2^* & F_1^* \\ \alpha & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ t_0 & \alpha & -1 & \ddots & & & & & \vdots \\ t_1 & t_0 & \alpha & -1 & \ddots & & & & \vdots \\ t_2 & t_1 & t_0 & \alpha & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \alpha & -1 & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & t_0 & \alpha & -1 & 0 \\ t_{n-6} & t_{n-7} & \cdots & \cdots & \cdots & t_1 & t_0 & \alpha & -1 \end{pmatrix}_{(n-3)\times(n-3)}$$

and the determinant of them can be calculated by Lemma 1.2.

While
$$\det \mathcal{M}_1 = \det \mathcal{N}_1 = (-1)^{\frac{(n+1)(n-2)}{2}},$$
we can obtain $\det \mathbf{T}_{F_n^*}$ as (2.1), which completes the proof. □

**Theorem 2.2.** *Let $\mathbf{T}_{F_n^*}$ be an invertible complex Fibonacci Hermitian Toeplitz matrix and $n > 6$. Then we have*

$$\mathbf{T}_{F_n^*}^{-1} = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \psi_{1,3} & \cdots & \psi_{1,n-2} & \psi_{1,n-1} & \psi_{1,n} \\ \bar{\psi}_{1,2} & \psi_{2,2} & \psi_{2,3} & \cdots & \psi_{2,n-2} & \psi_{2,n-1} & \psi_{1,n-1} \\ \bar{\psi}_{1,3} & \bar{\psi}_{2,3} & \psi_{3,3} & \cdots & \psi_{3,n-3} & \psi_{2,n-2} & \psi_{1,n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \bar{\psi}_{1,n-2} & \bar{\psi}_{2,n-2} & \bar{\psi}_{3,n-3} & \cdots & \psi_{3,3} & \psi_{2,3} & \psi_{1,3} \\ \bar{\psi}_{1,n-1} & \bar{\psi}_{2,n-1} & \bar{\psi}_{2,n-2} & \cdots & \bar{\psi}_{2,3} & \psi_{2,2} & \psi_{1,2} \\ \bar{\psi}_{1,n} & \bar{\psi}_{1,n-1} & \bar{\psi}_{1,n-2} & \cdots & \bar{\psi}_{1,3} & \bar{\psi}_{1,2} & \psi_{1,1} \end{pmatrix}, \tag{2.2}$$

*where*

$$\psi_{1,1} = \frac{K_2}{\bar{F}_0^*}\left(\frac{\bar{F}_{n-3}^*}{F_0^*}a_{1,1} + \frac{\bar{F}_{n-4}^*}{F_0^*}a_{1,2}\right) + \sum_{k=0}^{n-4}\frac{F_k^*}{\bar{F}_0^*}\left(\frac{\bar{F}_{n-3}^*}{F_0^*}a_{n-2-k,1} + \frac{\bar{F}_{n-4}^*}{F_0^*}a_{n-2-k,2}\right),$$

$$\psi_{1,2} = \frac{1}{\bar{F}_0^*} - \frac{K_2}{\bar{F}_0^*}\left(xa_{1,1} + ya_{1,2}\right) - \sum_{k=0}^{n-4}\frac{F_k^*}{\bar{F}_0^*}\left(xa_{n-2-k,1} + ya_{n-2-k,2}\right),$$

$$\psi_{1,3} = -\frac{K_2}{\bar{F}_0^*}a_{1,n-2} - \sum_{k=0}^{n-4}\frac{F_k^*}{\bar{F}_0^*}a_{n-2-k,n-2},$$

$$\psi_{1,4} = -\frac{K_2}{\bar{F}_0^*}(a_{1,n-3} + a_{1,n-2}) - \sum_{k=0}^{n-4}\frac{F_k^*}{\bar{F}_0^*}(a_{n-2-k,n-3} + a_{n-2-k,n-2}),$$

$$\psi_{1,j} = -\frac{K_2}{\bar{F}_0^*}(a_{1,n+1-j} + a_{1,n+2-j} - a_{1,n+3-j})$$
$$- \sum_{k=0}^{n-4}\frac{F_k^*}{\bar{F}_0^*}(a_{n-2-k,n+1-j} + a_{n-2-k,n+2-j} - a_{n-2-k,n+3-j}),\ (5 \leqslant j \leqslant n-1),$$

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$$
\begin{aligned}
\psi_{1,n} &= -\frac{K_2}{F_0^*}(a_{1,1}-a_{1,3}) - \sum_{k=0}^{n-4}\frac{F_k^*}{F_0^*}(a_{n-2-k,1}-a_{n-2-k,3}),\\
\psi_{2,2} &= -\frac{K_1}{F_0^*}\left(xa_{1,1}+ya_{1,2}\right) - \sum_{k=1}^{n-3}\frac{F_k^*}{F_0^*}\left(xa_{n-1-k,1}+ya_{n-1-k,2}\right),\\
\psi_{2,3} &= -\frac{K_1}{F_0^*}a_{1,n-2} - \sum_{k=1}^{n-3}\frac{F_k^*}{F_0^*}a_{n-1-k,n-2},\\
\psi_{2,4} &= -\frac{K_1}{F_0^*}(a_{1,n-3}+a_{1,n-2}) - \sum_{k=1}^{n-3}\frac{F_k^*}{F_0^*}(a_{n-1-k,n-3}+a_{n-1-k,n-2}),\\
\psi_{2,j} &= -\frac{K_1}{F_0^*}(a_{1,n+1-j}+a_{1,n+2-j}-a_{1,n+3-j})\\
&\quad - \sum_{k=1}^{n-3}\frac{F_k^*}{F_0^*}(a_{n-1-k,n+1-j}+a_{n-1-k,n+2-j}-a_{n-1-k,n+3-j}),\ (5\leqslant j\leqslant n-1),\\
\psi_{3,3} &= \Delta_{n-3}a_{1,n-2}+a_{n-2,n-2},\\
\psi_{i,4} &= \Delta_{n-i}(a_{1,n-3}+a_{1,n-2})+(a_{n-i+1,n-3}+a_{n-i+1,n-2}),\ (3\leqslant i\leqslant 4),\\
\psi_{i,j} &= \Delta_{n-i}(a_{1,n+1-j}+a_{1,n+2-j}-a_{1,n+3-j})+(a_{n-i+1,n+1-j}+a_{n-i+1,n+2-j}-a_{n-i+1,n+3-j}),\\
&\qquad (3\leqslant i\leqslant [\frac{n+1}{2}];\ i\leqslant j\leqslant n+1-i;\ \textit{except } \psi_{3,3},\ \psi_{3,4} \textit{ and } \psi_{4,4}),
\end{aligned}
$$

*in which $a_{i,j}$ is defined by*

$$
\begin{aligned}
a_{1,1} &= \frac{1}{s}\left(\omega_{n-3}-\omega_{n-4}b_1\alpha-\sum_{k=1}^{n-5}\omega_k(b_{n-3-k}\alpha+\sum_{j=1}^{n-4-k}b_jt_{n-4-k-j})\right),\\
a_{1,2} &= -\frac{1}{s}\left(\beta_{n-3}-\beta_{n-4}b_1\alpha-\sum_{k=1}^{n-5}\beta_k(b_{n-3-k}\alpha+\sum_{j=1}^{n-4-k}b_jt_{n-4-k-j})\right), a_{2,1}=-\frac{d}{s},\ a_{2,2}=\frac{c}{s},\\
a_{i,j} &= \begin{cases}
-\sum_{k=1}^{n-j-1}b_k(a_{i,1}\beta_{n-j-k}+a_{i,2}\omega_{n-j-k}),\ (1\leqslant i\leqslant 2,\ 3\leqslant j\leqslant n-2),\\
-b_1\alpha a_{2,j},\ (i=3,\ 1\leqslant j\leqslant 2),\\
-a_{2,j}(b_{i-2}\alpha+\sum_{k=1}^{i-3}b_kt_{i-3-k}),\ (4\leqslant i\leqslant n-2,\ 1\leqslant j\leqslant 2),\\
b_1\alpha\left(\sum_{k=1}^{n-4}b_k(a_{2,1}\beta_{n-3-k}+a_{2,2}\omega_{n-3-k})\right)+b_1,\ (i=3,\ j=3),\\
b_1\alpha\left(\sum_{k=1}^{n-1-j}b_k(a_{2,1}\beta_{n-j-k}+a_{2,2}\omega_{n-j-k})\right),\ (i=3,\ 4\leqslant j\leqslant n-2),\\
(b_{i-2}\alpha+\sum_{k=1}^{i-3}b_kt_{i-3-k})\left(\sum_{k=1}^{n-1-j}b_k(a_{2,1}\beta_{n-j-k}+a_{2,2}\omega_{n-j-k})\right)+b_{i-j+1},\\
\qquad (4\leqslant i\leqslant n-2,\ 3\leqslant j\leqslant n-2,\ i\geqslant j),\\
(b_{i-2}\alpha+\sum_{k=1}^{i-3}b_kt_{i-3-k})\left(\sum_{k=1}^{n-1-j}b_k(a_{2,1}\beta_{n-j-k}+a_{2,2}\omega_{n-j-k})\right),\\
\qquad (4\leqslant i\leqslant n-2,\ 3\leqslant j\leqslant n-2,\ i<j).
\end{cases}
\end{aligned}
$$

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*with*

$$
\begin{aligned}
s &= c\left(\omega_{n-3} - \omega_{n-4}b_1\alpha - \sum_{k=1}^{n-5}\omega_k(b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j})\right) \\
&\quad - d\left(\beta_{n-3} - \beta_{n-4}b_1\alpha - \sum_{k=1}^{n-5}\beta_k(b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j})\right), \\
b_i &= -\det p_{i-1},\ (1 \leqslant i \leqslant n-4),\ c = -\frac{\bar{F}^*_{n-3}}{F^*_0}K_1 + K_3,\ d = -\frac{\bar{F}^*_{n-4}}{F^*_0}K_1 + K_4, \\
\beta_i &= -\frac{\bar{F}^*_{n-3}}{F^*_0}F^*_i + \xi_i,\ \omega_i = -\frac{\bar{F}^*_{n-4}}{F^*_0}F^*_i + \eta_i,\ (1 \leqslant i \leqslant n-3),
\end{aligned}
$$

*$K_1$, $K_2$, $K_3$, $K_4$, $x$, $y$, $\alpha$, $\Delta_i(0 \leqslant i \leqslant n-3)$, $\xi_i(1 \leqslant i \leqslant n-3)$, $\eta_i(1 \leqslant i \leqslant n-3)$ and $t_i(0 \leqslant i \leqslant n-6)$ are as in Theorem 2.1, $\det p_i(0 \leqslant i \leqslant n-5)$ is the determinant of Hessenberg matrix, and can be calculated by Lemma 1.1, $[\cdot]$ is the integer-valued function.*

observe

*we can ~~observed~~ that $\mathbf{T}^{-1}_{F^*_n}$ is not only a Hermitian matrix, but also a symmetric matrix along its second diagonal, i.e., sub-symmetric matrix.*

*Proof.* Introduce the following two transformation matrices when $n > 6$,

$$
\mathcal{M}_2 = \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 1 & \ddots & & & & & & \vdots \\
-\frac{\bar{F}^*_{n-3}}{F^*_0} & 0 & 1 & \ddots & & & & & \vdots \\
-\frac{\bar{F}^*_{n-4}}{F^*_0} & \vdots & \ddots & 1 & \ddots & & & & \vdots \\
0 & \vdots & & \ddots & 1 & \ddots & & & \vdots \\
\vdots & \vdots & & & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & & & & \ddots & 1 & \ddots & \vdots \\
\vdots & \vdots & & & & & \ddots & 1 & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{pmatrix},
$$

$$
\mathcal{N}_2 = \begin{pmatrix}
1 & 0 & -\frac{K_2}{\bar{F}^*_0} & -\frac{F^*_{n-4}}{\bar{F}^*_0} & -\frac{F^*_{n-5}}{\bar{F}^*_0} & \cdots & -\frac{F^*_1}{\bar{F}^*_0} & -\frac{F^*_0}{\bar{F}^*_0} \\
0 & 1 & -\frac{K_1}{F^*_0} & -\frac{F^*_{n-3}}{F^*_0} & -\frac{F^*_{n-4}}{F^*_0} & \cdots & -\frac{F^*_2}{F^*_0} & -\frac{F^*_1}{F^*_0} \\
\vdots & \ddots & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & & \ddots & 1 & \ddots & & & \vdots \\
\vdots & & & \ddots & 1 & \ddots & & \vdots \\
\vdots & & & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & & & \ddots & 1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{pmatrix}.
$$

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Multiply $\mathcal{M}_1\mathbf{T}_{F_n^*}\mathcal{N}_1$ by $\mathcal{M}_2$ and $\mathcal{N}_2$, $\mathcal{M}_1$ and $\mathcal{N}_1$ are as in the proof of Theorem 2.1, we obtain

$$\mathcal{M}_2\mathcal{M}_1\mathbf{T}_{F_n^*}\mathcal{N}_1\mathcal{N}_2 = \begin{pmatrix} 0 & F_0^* & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \bar{F}_0^* & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & c & \beta_{n-3} & \beta_{n-4} & \beta_{n-5} & \cdots & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\ \vdots & \vdots & d & \omega_{n-3} & \omega_{n-4} & \omega_{n-5} & \cdots & \omega_4 & \omega_3 & \omega_2 & \omega_1 \\ \vdots & \vdots & 0 & \alpha & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & t_0 & \alpha & -1 & \ddots & & & & \vdots \\ \vdots & \vdots & \vdots & t_1 & t_0 & \alpha & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & t_{n-6} & \cdots & \cdots & \cdots & t_1 & t_0 & \alpha & -1 \end{pmatrix},$$

with

$$c = -\frac{\bar{F}_{n-3}^*}{F_0^*}K_1 + K_3,\ \beta_i = -\frac{\bar{F}_{n-3}^*}{F_0^*}F_i^* + \xi_i,\ (1 \leqslant i \leqslant n-3),$$
$$d = -\frac{\bar{F}_{n-4}^*}{F_0^*}K_1 + K_4,\ \omega_i = -\frac{\bar{F}_{n-4}^*}{F_0^*}F_i^* + \eta_i,\ (1 \leqslant i \leqslant n-3).$$

We have

$$\mathcal{M}_2\mathcal{M}_1\mathbf{T}_{F_n^*}\mathcal{N}_1\mathcal{N}_2 = \mathbf{\Phi} \oplus \mathbf{\Lambda},$$

where $\mathbf{\Phi} = \begin{pmatrix} 0 & F_0^* \\ \bar{F}_0^* & 0 \end{pmatrix}_{2\times 2}$ is a diagonal matrix, and $\mathbf{\Lambda}$ is a Toeplitz-like matrix

$$\mathbf{\Lambda} = \begin{pmatrix} c & \beta_{n-3} & \beta_{n-4} & \beta_{n-5} & \cdots & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\ d & \omega_{n-3} & \omega_{n-4} & \omega_{n-5} & \cdots & \omega_4 & \omega_3 & \omega_2 & \omega_1 \\ 0 & \alpha & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & t_0 & \alpha & -1 & \ddots & & & & 0 \\ \vdots & t_1 & t_0 & \alpha & \ddots & \ddots & & & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & t_{n-6} & \cdots & \cdots & \cdots & t_1 & t_0 & \alpha & -1 \end{pmatrix}_{(n-2)\times(n-2)}.$$

$\mathbf{\Phi} \oplus \mathbf{\Lambda}$ is the direct sum of $\mathbf{\Phi}$ and $\mathbf{\Lambda}$. Let $\mathcal{M} = \mathcal{M}_2\mathcal{M}_1$, $\mathcal{N} = \mathcal{N}_1\mathcal{N}_2$, then we obtain

$$\mathbf{T}_{F_n^*}^{-1} = \mathcal{N}(\mathbf{\Phi}^{-1} \oplus \mathbf{\Lambda}^{-1})\mathcal{M},$$

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$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \vdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -\frac{\bar{F}^*_{n-3}}{F^*_0} & x & \vdots & \vdots & & & & \iddots & 1 \\ -\frac{\bar{F}^*_{n-4}}{F^*_0} & y & \vdots & \vdots & & & \iddots & 1 & 0 \\ 0 & 0 & \vdots & \vdots & & \iddots & 1 & 1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & 1 & -1 & \iddots & \vdots \\ \vdots & \vdots & 0 & 1 & 1 & -1 & \iddots & & \vdots \\ 0 & 0 & 1 & 1 & -1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

$$\mathcal{N} = \begin{pmatrix} 1 & 0 & -\frac{K_2}{\bar{F}^*_0} & -\frac{F^*_{n-4}}{\bar{F}^*_0} & -\frac{F^*_{n-5}}{\bar{F}^*_0} & \cdots & -\frac{F^*_2}{\bar{F}^*_0} & -\frac{F^*_1}{\bar{F}^*_0} & -\frac{F^*_0}{\bar{F}^*_0} \\ 0 & 1 & -\frac{K_1}{F^*_0} & -\frac{F^*_{n-3}}{F^*_0} & -\frac{F^*_{n-4}}{F^*_0} & \cdots & -\frac{F^*_3}{F^*_0} & -\frac{F^*_2}{F^*_0} & -\frac{F^*_1}{F^*_0} \\ \vdots & 0 & \Delta_{n-3} & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ \vdots & \vdots & \Delta_{n-4} & \vdots & & & \iddots & 1 & 0 \\ \vdots & \vdots & \Delta_{n-5} & \vdots & & \iddots & 1 & \iddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \vdots & \Delta_2 & 0 & 1 & \iddots & & & \vdots \\ \vdots & \vdots & \Delta_1 & 1 & \iddots & & & & \vdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

We can ~~observed~~ that the inverse matrix of $\mathbf{\Phi}$ is of the form as,

observe

$$\mathbf{\Phi}^{-1} = \begin{pmatrix} 0 & \frac{1}{\bar{F}^*_0} \\ \frac{1}{F^*_0} & 0 \end{pmatrix}.$$

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Let $\mathbf{\Lambda}$ be partitioned as

$$\mathbf{\Lambda} = \left(\begin{array}{c:c} \mathcal{J} & V \\ \hdashline U & \mathcal{B} \end{array}\right)$$

$$= \left(\begin{array}{cc:cccccccc} c & \beta_{n-3} & \beta_{n-4} & \beta_{n-5} & \cdots & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\ d & \omega_{n-3} & \omega_{n-4} & \omega_{n-5} & \cdots & \omega_4 & \omega_3 & \omega_2 & \omega_1 \\ \hdashline 0 & \alpha & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & t_0 & \alpha & -1 & \ddots & & & & \vdots \\ \vdots & t_1 & t_0 & \alpha & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & t_1 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \alpha & -1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & t_0 & \alpha & -1 & 0 \\ 0 & t_{n-6} & t_{n-7} & \cdots & \cdots & t_1 & t_0 & \alpha & -1 \end{array}\right) \begin{array}{l} \\ \\ \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} n-4 \\ \\ \end{array}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{n-4}$$

According to Lemma 1.1 in [17], we have

$$\mathcal{B}^{-1} = \begin{pmatrix} b_1 & 0 & \cdots & \cdots & \cdots & 0 \\ b_2 & b_1 & \ddots & & & \vdots \\ b_3 & b_2 & b_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ b_{n-4} & \cdots & \cdots & b_3 & b_2 & b_1 \end{pmatrix},$$

where $b_i = -\det p_{i-1} (1 \leqslant i \leqslant n-4)$, $\det p_i (0 \leqslant i \leqslant n-5)$ is the determinant of Hessenberg matrix, and can be calculated by Lemma 1.1.

Suppose and so is

~~Supposed~~ that $\mathbf{T}_{F_n^*}$ is invertible, ~~so as~~ $\mathbf{\Lambda}$. Since $\mathcal{B}$ is invertible, ~~then~~ the Schur complement of $\mathcal{B}$, ~~noted as~~ $\mathcal{S}$, is also invertible, and we have

denoted by

$$\mathcal{S} = \mathcal{J} - V\mathcal{B}^{-1}U$$

$$= \begin{pmatrix} c & \beta_{n-3} \\ d & \omega_{n-3} \end{pmatrix} - \begin{pmatrix} \beta_{n-4} & \beta_{n-5} & \cdots & \beta_1 \\ \omega_{n-4} & \omega_{n-5} & \cdots & \omega_1 \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & \cdots & 0 \\ b_2 & b_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ b_{n-4} & \cdots & \cdots & b_2 & b_1 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ 0 & t_0 \\ \vdots & t_1 \\ \vdots & t_2 \\ \vdots & \vdots \\ 0 & t_{n-6} \end{pmatrix}$$

$$= \begin{pmatrix} c & \beta_{n-3} - \beta_{n-4} b_1 \alpha - \sum_{k=1}^{n-5} \beta_k (b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j}) \\ d & \omega_{n-3} - \omega_{n-4} b_1 \alpha - \sum_{k=1}^{n-5} \omega_k (b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j}) \end{pmatrix},$$

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with the determinant :

$$s = \det \mathcal{S} = c\left(\omega_{n-3} - \omega_{n-4}b_1\alpha - \sum_{k=1}^{n-5}\omega_k(b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j})\right)$$
$$- d\left(\beta_{n-3} - \beta_{n-4}b_1\alpha - \sum_{k=1}^{n-5}\beta_k(b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j})\right).$$

Hence, the inverse of $\mathbf{\Lambda}$ is of the following form:

$$\mathbf{\Lambda}^{-1} = [a_{i,j}]_{1\leqslant i,j\leqslant n-2} = \begin{pmatrix} \mathcal{S}^{-1} & -\mathcal{S}^{-1}V\mathcal{B}^{-1} \\ -\mathcal{B}^{-1}U\mathcal{S}^{-1} & \mathcal{B}^{-1}U\mathcal{S}^{-1}V\mathcal{B}^{-1} + \mathcal{B}^{-1} \end{pmatrix},$$

where $[a_{i,j}]_{1\leqslant i,j\leqslant n-2}$ are the same as in Theorem 2.2.

We obtain that

$$\mathbf{\Phi}^{-1} \oplus \mathbf{\Lambda}^{-1} = \begin{pmatrix} 0 & \frac{1}{\bar{F}_0^*} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{1}{F_0^*} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-3} & a_{1,n-2} \\ \vdots & \vdots & a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-3} & a_{2,n-2} \\ \vdots & \vdots & a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n-3} & a_{3,n-2} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & a_{n-3,1} & a_{n-3,2} & a_{n-3,3} & \cdots & a_{n-3,n-3} & a_{n-3,n-2} \\ 0 & 0 & a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-3} & a_{n-2,n-2} \end{pmatrix},$$

~~Then~~ we have
and

$$\mathbf{T}_{F_n^*}^{-1} = \mathcal{N}(\mathbf{\Phi}^{-1} \oplus \mathbf{\Lambda}^{-1})\mathcal{M} = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \psi_{1,3} & \cdots & \psi_{1,n-2} & \psi_{1,n-1} & \psi_{1,n} \\ \bar{\psi}_{1,2} & \psi_{2,2} & \psi_{2,3} & \cdots & \psi_{2,n-2} & \psi_{2,n-1} & \psi_{1,n-1} \\ \bar{\psi}_{1,3} & \bar{\psi}_{2,3} & \psi_{3,3} & \cdots & \psi_{3,n-3} & \psi_{2,n-2} & \psi_{1,n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \bar{\psi}_{1,n-2} & \bar{\psi}_{2,n-2} & \bar{\psi}_{3,n-3} & \cdots & \psi_{3,3} & \psi_{2,3} & \psi_{1,3} \\ \bar{\psi}_{1,n-1} & \bar{\psi}_{2,n-1} & \bar{\psi}_{2,n-2} & \cdots & \bar{\psi}_{2,3} & \psi_{2,2} & \psi_{1,2} \\ \bar{\psi}_{1,n} & \bar{\psi}_{1,n-1} & \bar{\psi}_{1,n-2} & \cdots & \bar{\psi}_{1,3} & \bar{\psi}_{1,2} & \psi_{1,1} \end{pmatrix},$$

where $[\psi_{i,j}](1 \leqslant i \leqslant [\frac{n+1}{2}];\ i \leqslant j \leqslant n+1-j)$ are the same as in Theorem 2.2.

Which completes the proof. □

# 3 Numerical Example

In this section, an example demonstrates the method introduced above for the calculation of determinant and inverse of the complex Fibonacci Hermitian Toeplitz matrix. Here we consider a

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$7 \times 7$ matrix:

$$\mathbf{T}_{F_7^*} = \begin{pmatrix} 0 & i & 1+i & 1+2i & 2+3i & 3+5i & 5+8i \\ -i & 0 & i & 1+i & 1+2i & 2+3i & 3+5i \\ 1-i & -i & 0 & i & 1+i & 1+2i & 2+3i \\ 1-2i & 1-i & -i & 0 & i & 1+i & 1+2i \\ 2-3i & 1-2i & 1-i & -i & 0 & i & 1+i \\ 3-5i & 2-3i & 1-2i & 1-i & -i & 0 & i \\ 5-8i & 3-5i & 2-3i & 1-2i & 1-i & -i & 0 \end{pmatrix}_{7\times 7}.$$

~~Use~~ Using the corresponding formulas in Theorem 2.1, we get

$$\begin{aligned} &x = -8-5i,\ y = -5-3i,\ \Delta_1 = 1+2i,\ \alpha = 2i,\ t_0 = 1+2i,\ t_1 = 2+2i, \\ &K_1 = -30-21i,\ K_3 = -20+172i,\ K_4 = -11+108i, \end{aligned}$$

and from Lemma 1.1 we can obtain

$$\begin{aligned} &\det \nabla_4([\eta_k]_{k=1}^4, -1, \alpha, t_0, t_1) = -10-40i, \\ &\det \nabla_4([\xi_k]_{k=1}^4, -1, \alpha, t_0, t_1) = -15-64i, \\ &\det \nabla_4([F_k^*]_{k=1}^4, -1, \alpha, t_0, t_1) = 14+3 \end{aligned}$$

~~from~~ From (2.1), we obtain

$$\begin{aligned} \det \mathbf{T}_{F_7^*} &= -\bar{F}_0^*[F_0^*(K_3 \det \nabla_4([\eta_k]_{k=1}^4, -1, \alpha, t_0, t_1) - K_4 \det \nabla_4([\xi_k]_{k=1}^4, -1, \alpha, t_0, t_1)) \\ &\quad - \bar{F}_4^*(K_1 \det \nabla_4([\eta_k]_{k=1}^4, -1, \alpha, t_0, t_1) - K_4 \det \nabla_4([F_k^*]_{k=1}^4, -1, \alpha, t_0, t_1)) \\ &\quad + \bar{F}_3^*(K_1 \det \nabla_4([\xi_k]_{k=1}^4, -1, \alpha, t_0, t_1) - K_3 \det \nabla_4([F_k^*]_{k=1}^4, -1, \alpha, t_0, t_1))] \\ &= 32. \end{aligned}$$

As the inverse calculation, ~~use~~ using the corresponding formulas in Theorem 2.2, we get

$$\psi_{1,1} = -\frac{13}{32},\ \psi_{1,2} = \frac{1}{8} - \frac{3}{32}i,\ \psi_{1,3} = \frac{25}{32} - \frac{9}{16}i,\ \psi_{1,4} = \frac{5}{8} + \frac{29}{32}i,$$

$$\psi_{1,5} = -\frac{9}{32} + \frac{11}{16}i,\ \psi_{1,6} = -\frac{7}{8} - \frac{3}{32}i,\ \psi_{1,7} = \frac{13}{32} - \frac{1}{4}i,\ \psi_{2,2} = \frac{3}{32},$$

$$\psi_{2,3} = -\frac{17}{16} + \frac{17}{32}i,\ \psi_{2,4} = \frac{3}{32} - \frac{3}{2}i,\ \psi_{2,5} = \frac{15}{16} - \frac{17}{32}i,\ \psi_{2,6} = \frac{35}{32} + i,$$

$$\psi_{3,3} = -\frac{9}{32},\ \psi_{3,4} = -\frac{9}{16} - \frac{17}{32}i,\ \psi_{3,5} = \frac{1}{32} - \frac{5}{8}i,\ \psi_{4,4} = -\frac{61}{32},$$

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so we can get

$$\mathbf{T}_{F_7^*}^{-1} = \begin{pmatrix} -\frac{13}{32} & \frac{1}{8}-\frac{3}{32}i & \frac{25}{32}-\frac{9}{16}i & \frac{5}{8}+\frac{29}{32}i & -\frac{9}{32}+\frac{11}{16}i & -\frac{7}{8}-\frac{3}{32}i & \frac{13}{32}-\frac{1}{4}i \\ \frac{1}{8}+\frac{3}{32}i & \frac{3}{32} & -\frac{17}{16}+\frac{17}{32}i & \frac{3}{32}-\frac{3}{2}i & \frac{15}{16}-\frac{17}{32}i & \frac{35}{32}+i & -\frac{7}{8}-\frac{3}{32}i \\ \frac{25}{32}+\frac{9}{16}i & -\frac{17}{16}-\frac{17}{32}i & -\frac{9}{32} & -\frac{9}{16}-\frac{17}{32}i & \frac{1}{32}-\frac{5}{8}i & \frac{15}{16}-\frac{17}{32}i & -\frac{9}{32}+\frac{11}{16}i \\ \frac{5}{8}-\frac{29}{32}i & \frac{3}{32}+\frac{3}{2}i & -\frac{9}{16}+\frac{17}{32}i & -\frac{61}{32} & -\frac{9}{16}-\frac{17}{32}i & \frac{3}{32}-\frac{3}{2}i & \frac{5}{8}+\frac{29}{32}i \\ -\frac{9}{32}-\frac{11}{16}i & \frac{15}{16}+\frac{17}{32}i & \frac{1}{32}+\frac{5}{8}i & -\frac{9}{16}+\frac{17}{32}i & -\frac{9}{32} & -\frac{17}{16}+\frac{17}{32}i & \frac{25}{32}-\frac{9}{16}i \\ -\frac{7}{8}+\frac{3}{32}i & \frac{35}{32}-i & \frac{15}{16}+\frac{17}{32}i & \frac{3}{32}+\frac{3}{2}i & -\frac{17}{16}-\frac{17}{32}i & \frac{3}{32} & \frac{1}{8}-\frac{3}{32}i \\ \frac{13}{32}+\frac{1}{4}i & -\frac{7}{8}+\frac{3}{32}i & -\frac{9}{32}-\frac{11}{16}i & \frac{5}{8}-\frac{29}{32}i & \frac{25}{32}+\frac{9}{16}i & \frac{1}{8}+\frac{3}{32}i & -\frac{13}{32} \end{pmatrix}_{7\times 7}.$$

# 4 Conclusion

In this paper, by constructing the special transformation matrices we give the determinant and inverse of the complex Fibonacci Hermitian Toeplitz matrix in section 2.

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