

Original Research Article

The n^{th} Power Transformation of the Error Component of the Multiplicative Time Series Model

1.0 Abstract

In this paper, the probability density function (pdf), the mean and the variance of the n^{th} power transformation of the error component of the multiplicative time series model were established as a general rule to any power transformation. Some of the published transformations like the square root and the inverse were used to validate the results obtained. The results showed that they conformed to the general rule.

2.0 Introduction

The pdf of the normal distribution is given in Uche[1] as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-1}{\sigma}\right)^2}, x \geq 0, \sigma^2 > 0 \quad (1)$$

The error component e_t of the multiplicative time series model has a pdf $N(1, \sigma^2)$ where $e_t > 0$, Iwueze[2] established the distribution of the left-truncated normal distribution and is given by

$$f(x) = \frac{e^{-\frac{1}{2}\left(\frac{x-1}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}\left[1 - \Phi\left(\frac{-1}{\sigma}\right)\right]}, x \geq 0, \sigma^2 > 0 \quad (2)$$

With mean $E(X)$ and variance $Var(X)$ given by

$$E(X) = 1 + \frac{\sigma e^{-\frac{1}{2}\sigma^2}}{\sigma\sqrt{2\pi}\left[1 - \Phi\left(\frac{-1}{\sigma}\right)\right]}, x \geq 0, \sigma^2 > 0 \quad (3)$$

and

$$Var(X) = \frac{\sigma^2}{2\left(1 - \Phi\left(\frac{-1}{\sigma}\right)\right)} \left(\left[1 + P_r\left(\chi_{(1)}^2 \leq \frac{1}{\sigma^2}\right)\right] - \frac{\sigma e^{-\frac{1}{2}\sigma^2}}{\sqrt{2\pi}\left[1 - \Phi\left(\frac{-1}{\sigma}\right)\right]} \left[\frac{\sigma e^{-\frac{1}{2}\sigma^2}}{\sqrt{2\pi}\left[1 - \Phi\left(\frac{-1}{\sigma}\right)\right]} \right]^2 \right) \quad (4)$$

Iwueze [2] examined some implications of truncating the $N(1, \sigma^2)$ to the left. He ^{has} briefly summarise them ^{respectively}

The truncated normal distribution ^{has} gained much acceptance in various fields of human endeavours, these include inventory management, regression analysis, operation management, time series analysis and so on. Johnson and Thomopoulos [3] considered the use of the left truncated distribution for improving achieved service levels. They presented the table of the cumulative distribution function of the left truncated normal distribution and

derived the characteristic parameters of the distribution, and also presented the table of the partial expectation of the left truncated normal distribution. A time series is a collection of ordered observation made sequentially in time. Examples abound in Sciences, Engineering, Economics etc and methods of analysing time series constitute a veritable area in the field of Statistics.

According to Spiegel and Stephens [4] the general time series model is always considered as a mixture of four major components, namely the Trend, Seasonal movements, cyclical movements and irregular or Random Movements. Hence classifications of the time series model are

Multiplicative model: $X_t = T_t S_t C_t e_t$ (5)

Additive model: $X_t = T_t + S_t + C_t + e_t$ (6)

Mixed model: $X_t = T_t S_t C_t + e_t$ (7)

In short term series the trend and cyclical components are merged to give the trend-cycle component; hence equation (5) through (7) can be rewritten as

$X_t = M_t S_t C_t e_t \Rightarrow \ln X_t = \ln M_t + \ln S_t + \ln C_t + \ln e_t$ (8)

$X_t = M_t + S_t + C_t + e_t$ (9)

$X_t = M_t S_t C_t + e_t$ (10)

where M_t is the trend cycle component and e_t is independent identically distributed (iid)

normal errors with mean 1 and variance $\sigma^2 > 0$ [$e_t \sim N(1, \sigma^2)$]

Any justification for the error terms to follow a truncated normal distribution

2.0 Data Transformation

In data analysis transformation is the replacement of a variable by a function of that variable, for example, replacing a variable x by the square root of x or the logarithm of x .

In a stronger sense, a transformation is a replacement that changes the shape of a distribution or relationship. Reasons for transformation include stabilizing variance, normalizing, reducing the effect of outliers, making a measurement scale more meaningful, and to linearize a relationship. For more references see Bartlett [5] Box and Cox [6], etc.

Many time series analysis assume normality and it is well known that variance stabilization implies normality of the series. The most popular and common transformation are the logarithm transformation and the power transformations (square, square root, inverse, inverse square, and inverse square root). It is important to note that, if we apply the n^{th} power transformation on model (8), we still obtain a multiplicative time series model given by

$$Y_t^n = M_t^n S_t^n e_t^n = M_t^* S_t^* e_t^* \quad (11)$$

Where $M_t^n = M_t^*$, $S_t^n = S_t^*$, $e_t^n = e_t^*$

Several studies abound in statistical literature on effects of power transformations on the error component of a multiplicative time series model whose error component is classified under the characteristics given in (3). The sole aim of such studies is to establish the conditions for successful transformation. A successful transformation is achieved when the derivable statistical properties of a data set remain unchanged after transformation, there basic properties or assumptions of interest for the studies are (i) unit mean and (ii) constant variance. Also Nwosu et al [7] studied the effects of inverse and square root transformation respectively on the error component of the same model and discovered that the inverse

transform $Y = \frac{1}{e_t}$ can be assumed to be normally distributed with mean, one and the same variance provided $\sigma < 0.07$. Similarly Otuonye et al [8] discovered that the square root transform; $\sqrt{e_t}$ can be assumed to be normally distributed with unit mean and variance $4\sigma^2$, for $\sigma_1 \leq 0.3$ where σ_1^2 is the variance of the original error component before transformation.

Ibeh et al [9] studied the inverse square transformation of error the component of the multiplicative time series model, the results of the research showed that the basic assumptions of the error term of the multiplicative model which is normally distributed with mean 1 and finite variance can only be maintained if the standard deviation of the untransformed error term is less than or equal to 0.07 ($\sigma \leq 0.07$). The study also revealed that the variance of the transformed of the error term is 4 times the variance of the untransformed for $\sigma \leq 0.07$.

Ajibade et al [10] studied the distribution of the inverse square root transformed error component of the multiplication time series model and found out that the means are the same and variance $Var(e_t^*) \approx \frac{1}{4} Var(e_t)$ for $\sigma \leq 0$.

In this paper a general rule expression for the pdf, mean and variance of n^{th} power distribution was used to verify some published power transformations, example, the square root and the inverse transformation.

Derivation of the Probability Density Function (pdf) of n^{th} Power Transformation

Given

$$y = x^n \quad \text{where } X = [e_t \sim N(1, \sigma^2)] \quad (12)$$

$$\Rightarrow X = Y^{\frac{1}{n}} \quad \Rightarrow \frac{dx}{dy} = \frac{1}{n} Y^{\frac{1}{n}-1}$$

$$\text{So } f(y) = f(x) \left| \frac{dx}{dy} \right|$$

$\&$ $x = y^{\frac{1}{n}}$ mixing rvs with their relationships.

$$f(x) = \frac{e^{-\frac{1}{2}\left(\frac{x-1}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}, x \geq 0, \sigma^2 > 0 \quad (12)$$

$$f(y) = \frac{e^{-\frac{1}{2}\left(\frac{\frac{1}{y^n}-1}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]} \cdot \left|\frac{1}{n} \cdot y^{\frac{1}{n}-1}\right|$$

$$f(y) = \frac{\frac{1}{n} \cdot y^{\frac{1}{n}-1} e^{-\frac{1}{2}\left(\frac{\frac{1}{y^n}-1}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]} \quad (13)$$

We now show that it is a proper pdf

$$i.e. \int_0^{\infty} f(y) dy = 1$$

(14)

$$let u = \frac{\frac{1}{y^n}-1}{\sigma}, \quad \frac{du}{dy} = \frac{1}{\sigma} \cdot \frac{1}{n} y^{\frac{1}{n}-1} \text{ and } dy = \frac{n\sigma}{y^{\frac{1}{n}-1}} du, \quad -\frac{1}{\sigma} < u < \infty$$

$$\int_0^{\infty} f(y) dy = \int_{-\frac{1}{\sigma}}^{\infty} \frac{\frac{1}{n} \cdot y^{\frac{1}{n}-1} e^{-\frac{1}{2}\left(\frac{\frac{1}{y^n}-1}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]} \frac{n\sigma}{y^{\frac{1}{n}-1}} du \quad (15)$$

$$= \int_{-\frac{1}{\sigma}}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{\frac{1}{y^n}-1}{\sigma}\right)^2}}{\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]} du$$

$$= \frac{1}{1-\Phi\left(-\frac{1}{\sigma}\right)} \int_{-\frac{1}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = \frac{1}{1-\Phi\left(-\frac{1}{\sigma}\right)} P_r\left(u > -\frac{1}{\sigma^2}\right)$$

$$But P_r\left(u > -\frac{1}{\sigma^2}\right) = 1 - \Phi\left(-\frac{1}{\sigma}\right)$$

$$\therefore \int_0^{\infty} f(y) dy = \frac{1 - \Phi\left(-\frac{1}{\sigma}\right)}{1 - \Phi\left(-\frac{1}{\sigma}\right)} = 1 \quad (16)$$

Therefore (15) is a proper pdf.