

Original Research Article

Dynamics of Water Hyacinth in a Fishery System with Delay Effect

Abstract

This paper analyses the dynamics of a fishery system in an aquatic environment that consists of two zones: water hyacinth zone and free zone. Fish harvesting is allowed in both the zones and fish migration is allowed from water hyacinth zone to free zone and not back. This paper presents dynamics of the stability when discrete time delay is incorporated in the fish death rate due to oxygen depletion and water pollution caused by water hyacinth. It is shown that the time delay can cause a switch from stable state to unstable state and there by Hopf-bifurcation occurs. Numerical simulations are carried out to validate the analytical findings.

Key words: Water hyacinth, Fishery, Stability, Time delay, Hopf-bifurcation.

1 Introduction

Water hyacinth (*Eichhornia crassipes*) is one of the aquatic plants in the nature though it is native to Brazil; it is a globally wide spread ~~on a global level~~ and creates a nuisance in the aquatic environment. Once these plants are introduced into the natural environment they rapidly increase in coverage, ~~as hyacinth has~~ because of the highest growth rate in any saltwater or fresh water. It ~~is doubling~~ doubles in number in a times range between 6-18 days, for these reasons hyacinth has earned such nicknames such as “the weed from hell” and “the beautiful devil”(3) .

Fish is one of the important species in the aquatic system. Quantitative models of fish population include simple models that consider only the biomass of the population processes, such as growth and recruitment. These are often adjusted to incorporate the impact of external effects, such as predation, competition for food supply and other environmental factors. To fisheries biologists, lumping together a number of species of different sizes and age structure into a uniform stock is implausible. Further, the logistic growth curve has been found in appropriate for many fish species.

The water hyacinth effect greatly influence the fish industry, its pollution appears in different levels ranging from low to moderate and high. The hyacinth mats impacts significantly on fishing activities due to increase in time to access the fishing ground. The mats also had detrimental effects by blocking light, severely reducing oxygen levels and allowing poisonous gases such as hydrogen sulfide and ammonia. This result in loss of aquatic biodiversity (10). Already many researchers (2, 4, 5, 6, and 7) links water hyacinth pollution to fish production. In this they were discussed the effect of water hyacinth in fish growth. |

To protect the fishery population, having much threat from water hyacinth, adopting two zonal systems is better idea. A zone having no influence of water hyacinth, called hyacinth free zone, is formed. In this present paper it is assumed that the density of fish is directly proportional to the abundance of fish at time (t) and the catch ability coefficient decreases as the abundance level of water hyacinth increases. Further it was assumed that water hyacinth abundance leads to biological effects such as fish death and fish migration. The oxygen depletion and the environmental pollution cause by the growth of water hyacinth will not have immediate impact on the death of the fish population, but it will have some time lag. The effect of this time lag on the dynamics of this system is presented in this population.

2 Mathematical Model

To formulate a mathematical representation of water hyacinth model the following notations are being used. Let $W(t)$ and $F(t)$ represents the biomass densities of fish population in water hyacinth zone and free zone of water hyacinth respectively at any time 't'. ρ denote the rate at which water hyacinth reduces fish catch; ψ denote fish migration rate from water hyacinth zone to free zone of water hyacinth; r and s denotes fish death rates due to oxygen depletion and water pollution due to water hyacinth; c and e denote the catchability coefficient and harvesting efforts in both the zones; α and β denote the intrinsic growth rates of fish population in both the zones respectively; m and n denote carrying capacity of fish population in both the zones. Further both the variables W and F are non-negative and the parameters are assumed to be non-negative. The mathematical formulation of water hyacinth model is represented by the following system of first order non linear ordinary differential equation:

$$\begin{aligned} \frac{dW}{dt} &= \alpha W \left(1 - \frac{W}{m}\right) - (1 - \rho) ceW - \psi W - rW - sW \\ \frac{dF}{dt} &= \beta F \left(1 - \frac{F}{n}\right) + \psi W - ceF \end{aligned} \quad (2.1)$$

Now incorporating time delay (τ) in fish death rate due to oxygen depletion and water pollution caused by water hyacinth, the equation (2.1) becomes

$$\begin{aligned} \frac{dW}{dt} &= \alpha W \left(1 - \frac{W}{m}\right) - (1 - \rho) ceW - \psi W - rW(t - \tau) - sW(t - \tau) \\ \frac{dF}{dt} &= \beta F \left(1 - \frac{F}{n}\right) + \psi W - ceF \end{aligned} \quad (2.2)$$

3 Equilibrium Analysis

The equilibrium points of the system (2.1) and (2.2) are the solutions of the steady state equations.

$$\begin{aligned} \alpha W \left(1 - \frac{W}{m}\right) - (1 - \rho) ceW - \psi W - rW - sW &= 0 \\ \beta F \left(1 - \frac{F}{n}\right) + \psi W - ceF &= 0 \end{aligned}$$

The possible equilibrium points are

- $E_1 (0, 0)$ (In the absence of both the zones)
- $E_2 (\gamma, 0)$ (In the presence of water hyacinth zone)
- $E_3 (0, \delta)$ (In the presence of free zone of water hyacinth)
- $E_4 (\gamma^*, \delta^*)$ (In the presence of both the zones i.e., the interior equilibrium)

Case (i): The population is extinct and this trivial steady state always exists.

64 **Case (ii):** if γ is positive solution of $\frac{dW}{dt} = 0$ then

$$65 \quad \gamma = \frac{m}{\alpha} [\rho ce + \alpha - ce - \psi - r - s]$$

66 This positive steady state exists only when $\rho ce + \alpha > ce + \psi + r + s$ (3.1)

67 **Case (iii):** if δ is the positive solution of $\frac{dF}{dt} = 0$ then

$$68 \quad \delta = \frac{n}{\beta} [\beta - ce]$$

69 This positive steady state exists only when

$$70 \quad \beta > ce \quad (3.2)$$

71 **Case (iv):** if (γ^*, δ^*) are the positive solutions of $\frac{dW}{dt} = 0$ and $\frac{dF}{dt} = 0$ then

$$72 \quad \gamma^* = \frac{m}{\alpha} [\rho ce + \alpha - ce - \psi - r - s]$$

$$73 \quad \delta^{*2} + G\delta^* + H = 0 \quad (3.3)$$

$$74 \quad \text{Where } G = -\frac{n}{\beta} [\beta - ce]$$

$$75 \quad H = -\frac{\psi mn}{\alpha \beta} [\alpha + \rho ce - ce + \psi + r + s]$$

76 From the biological point of view we only interested on the interior equilibrium $E_4 (\gamma^*, \delta^*)$.

77 Let $W = \gamma - \gamma^*$, $F = \delta - \delta^*$ be the perturbed variables

78 After removing the non-linear terms we obtain the linearized system corresponding to (2.2) is

$$79 \quad \frac{d\gamma}{dt} = \alpha - \frac{2\alpha\gamma^*}{m} - (1 - \rho) ce - \psi - (r + s) e^{-\lambda\tau} \gamma$$

$$\frac{d\delta}{dt} = \psi\gamma + \beta - \frac{2\beta\delta^*}{n} - ce \delta \quad (3.4)$$

80 The characteristic equation of the linear system is given by

$$81 \quad (\lambda, \tau) = \lambda^2 + P\lambda + Q + e^{-\lambda\tau} (R\lambda + T) = 0 \quad (3.5)$$

82 Where $P = -A-D$; $Q = AD$;

83 $R = r + s$; $T = -D(r + s)$

$$84 \quad \text{And } A = \alpha - \frac{2\alpha\gamma^*}{m} - (1 - \rho) ce - \psi \quad B = \psi \quad D = \beta - \frac{2\beta\delta^*}{n} - ce$$

85 Now we discuss the stability of the interior equilibrium point in the presence and absence of delay.

4 Stability Analysis

4.1 Stability analysis in the absence of delay

The characteristic equation of the model (2.2) is $\lambda^2 + X\lambda + Y = 0$ (4.1)

Where $X = P + R$

$$Y = Q + T$$

Here $X > 0$ and $Y > 0$

So the Eigen values of the characteristic equation are either real and negative or complex conjugate with negative real parts.

Hence the system (2.1) is locally asymptotically stable.

THEOREM 1: The system (2.1) is locally asymptotically stable at $E_4(\gamma^*, \delta^*)$ if the equation (4.1) has both the roots with negative real parts.

Numerical analysis of the model in the absence of delay

Example 1: Consider the following parameters for the model (2.1)

$m = 600; n = 300; e = 10; c = 0.004; \alpha = 1.5; \beta = 0.6; \rho = 0.32; \psi = 0.19; r = 0.45; s = 0.36;$

$W(0) = 30; F(0) = 40$

Equilibrium points for the system (2.1) - (2.2) are (189.125, 333.80)

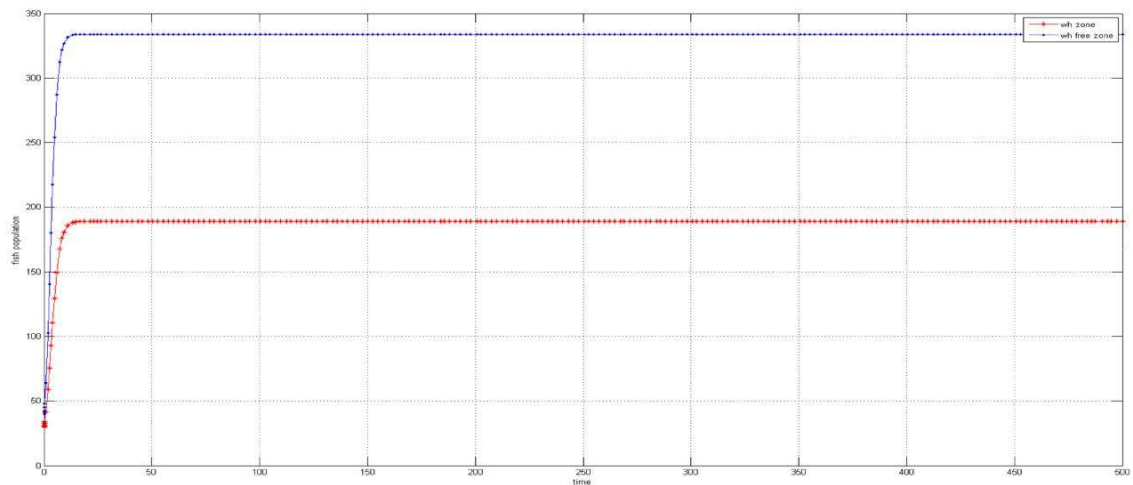


Fig. 1

Fig. 1 shows the time series evolution of fish population in the two zones showing stable oscillation of the population

towards $E_4(\gamma^*, \delta^*)$

4.2 Stability analysis in the presence of delay

Let $\lambda(\tau) = \theta(\tau) + i\eta(\tau)$ be a root of the characteristic equation (3.5).

110 Let T be a particular value of the delay such that $\theta(\tau) = 0, \eta(\tau) > 0$

111 Substituting $\lambda = i\eta$ in (3.5) we get,

112

$$E_4(\gamma^*, \delta^*) \tau \geq 0 \quad P^2 - 2Q - R^2 - T^2 \quad \eta \lambda = i\eta \quad -\eta^2 + i\eta P + Q + [Ri\eta + T] e^{-i\eta\tau} = 0$$

113 $-\eta^2 + i\eta P + Q + [Ri\eta + T][\cos \eta\tau + i \sin \eta\tau] = 0$

114 Separating the real and imaginary parts we get,

115 $\eta^2 - Q = T \cos \eta\tau + R\eta \sin \eta\tau$ (4.2)

116 $P\eta = -R\eta \cos \eta\tau + T \sin \eta\tau$ (4.3)

117 Squaring and adding (4.2)-(4.3), we get

118 $\eta^4 + \eta^2 (P^2 - 2Q - R^2) + (Q^2 - T^2) = 0$

119 The above equation can be written in the form of

120 $\eta^4 + K\eta^2 + L = 0$ (4.4)

121 Where $K = P^2 - 2Q - R^2$; $L = Q^2 - T^2$

122 **Case 1:** If $K = P^2 - 2Q - R^2 > 0$ then the equation (4.4) does not have any real solutions.

123 **THEOREM 2:** If $K > 0, L > 0$ then the equilibrium point $E_4(\gamma^*, \delta^*)$ is locally asymptotically stable for all $T \in \mathbb{R}^+$.

124 **PROOF:** For $T = 0$, $E_4(\gamma^*, \delta^*)$ is locally asymptotically stable from THEOREM 1.

125 When $T > 0$, by case 1 the equation (4.4) does not have any real solution

126 i.e., there exist no real η as a solution for the equation (4.4).

127 Hence no real $\lambda = i\eta$ (η is real) will be a solution to the equation (3.5).

128 It is obvious that the equilibrium point $E_4(\gamma^*, \delta^*)$ is locally asymptotically stable for all $T \in \mathbb{R}^+$.

129 **Case 2:** If $K > 0, L < 0$ then the equation (4.4) have a unique positive root, it is η_0^2 and let the corresponding T be T_0 .

130 **Case 3:** if $K < 0, L > 0$ and $K^2 - 4L > 0$ then the equation (4.4) have two positive roots. Let them be $\eta_{\pm}^2$ and the

131 corresponding T is $T^{\pm}$.

132 Eliminating $\sin \theta\tau$ from (4.2)-(4.3)

133
$$\cos \eta\tau = \frac{\eta^2 (T - PR) - QT}{T^2 + \eta^2 R^2}$$

134
$$\tau_z = \frac{1}{\eta} \arccos \frac{\eta^2 (T - PR) - QT}{T^2 + \eta^2 R^2} + \frac{2\pi z}{\eta}$$
 (4.5)

135 Where $z = 0, 1, 2, \dots$, etc.

136

137 5. Hopf-Bifurcation Analysis

138 In this section, investigate the effect of the time delay on bifurcations of the system.

139 Now differentiating equation (3.5) with respect to T ,

$$140 \quad 2\lambda + P + RE^{-\lambda T} - \tau(R\lambda + T)e^{-\lambda T} \frac{d\lambda}{dT} = \lambda e^{-\lambda T} (R\lambda + T)$$

$$141 \quad \frac{d\lambda}{dT}^{-1} = \frac{2\lambda + P}{-\lambda(\lambda^2 + p\lambda + Q)} + \frac{\frac{d\tau}{dR}}{\lambda(R\lambda + T)} - \frac{\tau}{\lambda} \quad (5.1)$$

$$142 \quad \frac{d\lambda}{dT}^{-1} = RE \frac{d\lambda}{dT}^{-1} = \frac{2(\eta^2 - Q) + P}{4(\frac{\eta^2}{2} - Q)^2} - \frac{R^2}{22\frac{\eta^2}{2}} \quad (5.2)$$

$$143 \quad \frac{d\lambda}{dT}^{-1} = \frac{\frac{\eta^2 + P - 2Q}{2\eta^2 + P^2 - 2Q - R^2} + \frac{QT + \eta}{R^2}}{T^2 + \eta^2 R^2}$$

$$144 \quad \text{Thus } \text{sign} \frac{d}{dT} (\text{RE } \lambda)_{\lambda=i\eta}^{-1} = \text{sign} \text{RE} \frac{d\lambda}{dT}^{-1} = \text{sign} \frac{2\eta^2 + (P^2 - 2Q - R^2)}{T^2 + \eta^2 R^2} \quad (5.3)$$

145 Thus η may be η_0 or $\eta_{\pm}$.

146 The above said positive roots, either from case 2 or from case 3, satisfy all the equations from (4.2)-(4.5).

147 **THEOREM 3:** The system (2.2) is locally asymptotically stable at $E_4(Y^*, \delta^*)$ if

148 $K > 0$ and $L < 0$ for all $T < T_0$ and is unstable for all $T > T_0$ and hopf-bifurcation occurs at $T = T_0$.

149 **PROOF:** From equation (5.3) we have

$$150 \quad \text{sign} \frac{d}{dT} (\text{RE } \lambda)_{\lambda=i\eta}^{-1} = \text{sign} \text{RE} \frac{d\lambda}{dT}^{-1} \quad (5.4)$$

$$151 \quad = \text{sign} \frac{2\eta^2 + (P^2 - 2Q - R^2)}{T^2 + \eta^2 R^2}$$

152 It is clear that

$$153 \quad \text{sign} \frac{d}{dT} (\text{RE } \lambda)_{\lambda=i\eta}^{-1} = \text{sign} \text{RE} \frac{d\lambda}{dT}^{-1} > 0 \quad (5.4)$$

$$154 \quad \eta = \eta_0, T = T_0$$

154 This signifies that there exists Eigen value with negative real part for $T < T_0$ and there exist Eigen value with positive

155 real part for $T > T_0$. More over the condition of hopf-bifurcation is then satisfied yielding the required periodic solution.

156 **THEOREM4:** The system (2.2) is locally asymptotically stable at

157 $E_4(\gamma^*, \delta^*)$ when $\tau \in$
 $(0, \tau_0^+) \cup (\tau_0^-, \tau_1^+) \cup \dots \cup (\tau_{i-1}^-, \tau_i^+)$ if $K < 0, L > 0$ and $K^2 - 4L > 0$ and it is unstable

158 when $\tau \in (\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup \dots \cup (\tau_{i-1}^+, \tau_{i-1}^-)$, for some positive integer i . Therefore there are bifurcations at

159 the equilibrium point $E_4(\gamma^*, \delta^*)$ when $\tau = \tau_z^\pm$, $z=0,1,2,\dots$

160 **PROOF:** From equation (5.3) we have

$$161 \quad \text{sign} \frac{d}{d\tau} (\text{RE } \lambda)_{\lambda=i\eta} = \text{sign} + \frac{\sqrt{K^2 - 4L}}{\left\{ \frac{(-\eta_+ + Q)^2}{2} + P\eta_+ \{T + R\eta_+\} \right\}}$$

$$162 \quad \text{Therefore, } \frac{d}{d\tau} (\text{RE } \lambda)_{\eta=\eta_+, \tau=\tau_n^+} > 0.$$

$$163 \quad \text{Again, } \text{sign} \frac{d}{d\tau} (\text{RE } \lambda)_{\lambda=i\eta} = \text{sign} - \frac{\sqrt{K^2 - 4L}}{\left\{ \frac{(-\eta_- + Q)^2}{2} + P\eta_- \{T + R\eta_-\} \right\}}$$

$$164 \quad \text{Therefore, } \frac{d}{d\tau} (\text{RE } \lambda)_{\eta=\eta_-, \tau=\tau_n^-} < 0 < \frac{d}{d\tau} (\text{RE } \lambda)_{\eta=\eta_-, \tau=\tau_n^-} < 0$$

165 Hence the transversality conditions are satisfied.

166 This completes the proof.

167 **Numerical analysis of the model in the presence of delay:**

168 **Example 2:** Consider the following parameters for the model (2.2)

$m = 600; n = 300; e = 10; c = 0.004; \alpha = 1.5; \beta = 0.6; \rho = 0.32; \psi = 0.19; r = 0.45; s = 0.36;$

$W(0) = 180; F(0) = 330;$

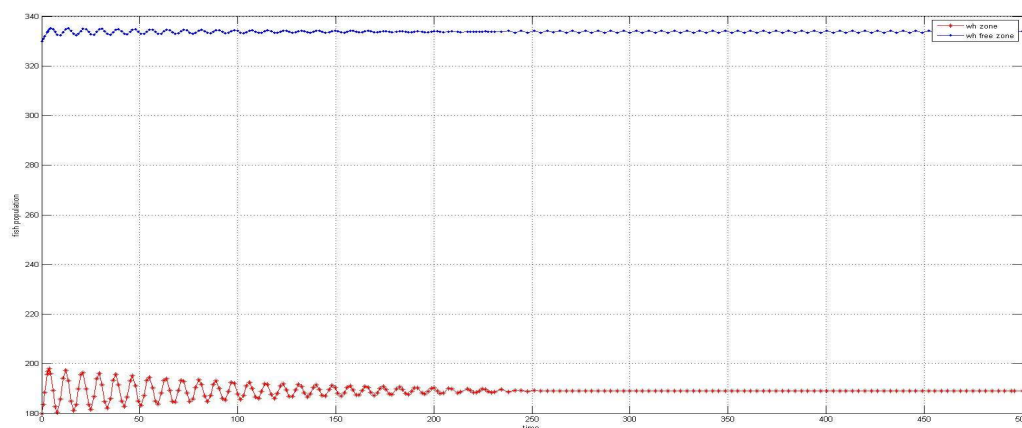


Fig.2

Fig.2 shows the time series evolution of fish population in the two zones showing stable oscillation of the population

towards $E_4(\gamma^*, \delta^*)$ delay = 1.53 (< 1.549)

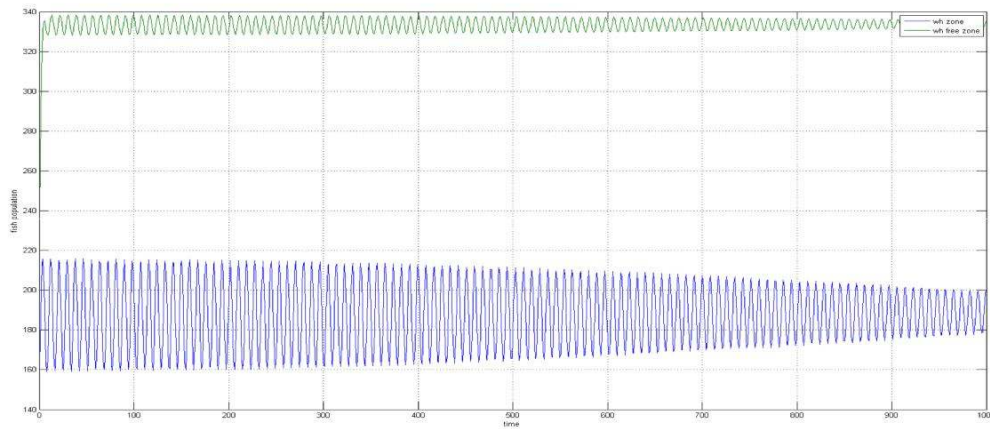


Fig.3

Fig. 3 shows the time series evolution of fish population in the two zones showing unstable oscillation of the population towards $E_4(\gamma^*, \delta^*)$ delay = 1.549(τ_0)

Example 3: Consider the following parameters for the model (2.2)

$$m = 600; n = 300; e = 10; c = 0.004; \alpha = 1.5; \beta = 0.6; \rho = 0.33; \psi = 0.20; \\ r = 0.46; s = 0.37; W(0) = 180; F(0) = 330.$$

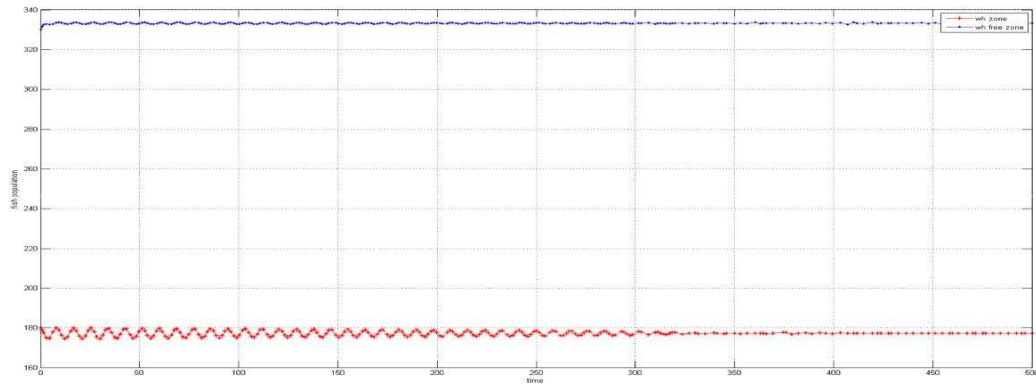


Fig.4

Fig. 4 shows the time series evolution of fish population in the two zones showing stable oscillation of the population towards $E_4(\gamma^*, \delta^*)$ delay = 1.48(< 1.495)

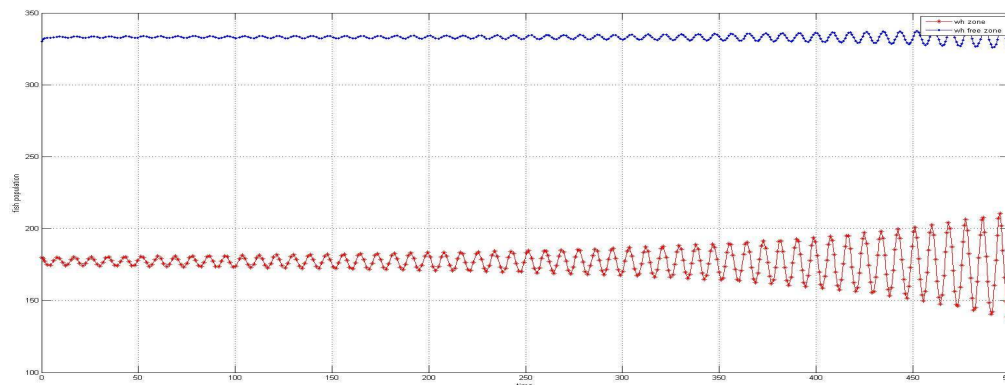


Fig.5

Fig. 5 shows the time series evolution of fish population in the two zones showing unstable oscillation of the population towards $E_4(\gamma^*, \delta^*)$ delay = $1.495(\tau_0)$

6 Conclusions:

In this paper we investigated the effect of water hyacinth on the equilibria of fish biomass densities. The equilibria of the model were analysed.

1. The study has proved that water hyacinth have severe impact on fish stock.
2. Slight increments on the model parameters associated with water hyacinth completely changed the equilibria of the model.
3. Fish population in the water hyacinth zone was decreasing in the equilibria.
4. It is observed that the delay of certain dimensions can induce instability oscillations via hopf bifurcation with switching of stability. Also observed that switching of stability occurs.
5. Water hyacinth may lead to fish extinction, therefore efforts should be done to eradicate it with whatever means which are environmentally friendly.

References

- [1] T. P. Albright, T. G. Moorhouse, T. J. McNabb, The Rise and Fall of water hyacinth in Lake Victoria and the Kagera River Basin, 1989-2001, Journal of Aquatic plant Management 42 (2004) 72-84.
- [2] B. Dubey, P. Chandra, P. Sinha, A model for fishery resource with reserve area, Nonlinear Analysis: Real World Applications 4 (2003) 625-637.
- [3] B. Gopal, Aquatic Plant Studies, Elsevier Publisher, Amsterdam-Oxford, 1987.
- [4] E. Kateregga, T. Sterner, Indicators for an invasive species: Water hyacinth in Lake Victoria, Ecological Indicators 7 (2007) 362-370.
- [5] E. Kateregga, T. sterne, Lake Victoria Fish Stock and Effects of Water hyacinths, The Journal of Environment and Development 18 (2009) 62-78.
- [6] LVEMP (Lake Victoria Environmental Management Programme), Uganda. Annual Report, Entebbe, Uganda: Ministry of Water, Lands and Environment, 1997.

- 214 [7] W. F. Masifwa, T. Twongo, P. Denny, The impact of water hyacinth, *Eichhornia crassipes* (Mart) Solms on
215 the abundance and diversity of aquatic macro invertebrates along the shores of northern Lake Victoria,
216 Uganda, *Hydrobiologia* 451 (2001) 79-88.
- 217 [8] J. P. Mpele, Y. Nkansah-Gyekye, O. D. Makinde, A dynamic model for a three species open-access fishery
218 with taxation as a control instrument of harvesting efforts, The case of Lake Victoria, *Communications in*
219 *Mathematical Biology and Neuroscience* 1 (2014).
- 220 [9] R. J. Muli, Environmental problems in Lake Victoria (East Africa): What the international community can do,
221 *Lakes and Reservoirs: Research Management* 2 (1996) 47-53.
- 222 [10] J. D. Murray, *Mathematical Biology*, 3rd edition, 1993.
- 223 [11] P. B. O. Ochumba, Observation on blue-green algae blooms in the open waters of Lake Victoria, Kenya,
224 *African Journal of Ecology* 27 (1989) 23-34.
- 225 [12] T. Twongo, J. Balirwa, The water hyacinth Problem and the Biological Control Option in the Highland Region
226 of the Upper Nile Basin: Uganda's experience, Nile conference, Arusha-Tanzania, 1995.
- 227 [13] N.W.O. Wawire, The impact of Water Hyacinth (*Eichhornia crassipes*) on Productivity, Profitability and
228 Species Composition of the Lake Victoria fishery, Kenya, PhD thesis, Moi University, Eldoret-Kenya, 2003.