

ON SIMILAR PARTNER CURVES IN DUAL SPACE

ABSTRACT. In this paper , we give the new family of curves and call it a family of similar curves in three-dimensional dual space ID^3 with variable transformation . Also we give some characterizations of this family and some theorems. We show that the family of curves with wanishing curvatures forms a family of similar curves in three-dimensional dual space ID^3 .

1. INTRODUCTION

In the differential geometry, special curves which satisfy some relationships between their curvatures and torsions have an important role. The most fastinating of these curves are general helix, slant helix, B_2 -slant helix, Salkowski and anti- Salkowski curves [3, 4, 5, 6]. Another way to classification and chracterization of curves is the relation between the Frenet vectors of the curves. For example (in 1845) Saint Venant [7] proposed the question upon the surface generated by the principal normal of a curve, a second curve can exist which has for its principal normal of a curve. This question was answered by Bertrand in 1850 in a paper [8] in which he showed that a linear relationship with constant coefficients shall exist between the first and second curvatures of the given original curve. In other words, if we denote first and second curvatures of a given curve by k_1 and k_2 respectively, then for $\lambda, \mu \in \mathbb{R}$ we have $\lambda k_1 + \mu k_2 = 1$. Since the time of Bertrand's paper, pairs of curves of this kind have been called Conjugate Bertrand Curves , or more commonly Bertrand Curves [9, 10, 11].

In 1888, C. Bioche give a new theorem in order to obtaining Bertrand curves by using the intersection of two curves C_1 and C_2 in Euclidean 3-space. Later, in 1960, J. F. Burke in [12] give a theorem related with Bioche's thorem on Bertrand curves.

Recently, similar curves have been introduced a new type of special curves in E^3 for which the arc-length parameters have a relationship and between the space curves α and α^* such that, at the corresponding points of the curves, the tangent lines of α coincides with the tangent of α^* , then α is called a similar curve, and α^* similar partner curve of α [14].

Dual numbers had been introduced by W.K. Clifford (1849-1870) as a tool for his geometrical investigations. After him E.Study used dual numbers and dual vectors in his research on line geometry and kinematics. He devoted special attention to the representation of oriented lines by dual unit vectors and defined the famous mapping: The set of oriented lines in an Euclidean three-dimension space E^3 is one to one correspondence with the points of a dual space ID^3 of triples of dual numbers [13].

In this work, we give some characterizations of these family and some theorems. We show that the family of curves with wanishing curvatures forms a family of similar curves in three-dimensional dual space ID^3 .

2. PRELIMINARIES

A dual number is a number of the form $\hat{x} = x + \varepsilon x^*$, where x, x^* are real numbers

called the components of $\hat{x}$, and $\varepsilon = (0, 1)$ arbitrary dual unit satisfy, the relation $\varepsilon^2 = 0$. Let $ID = \{\hat{x} \mid \hat{x} = x + \varepsilon x^*, x, x^* \in IR\}$. Addition of two elements in ID and scalar multiplication of an element in ID by a real number are defined in the usual way. ID is a vector space with respect to addition and scalar multiplication. The product $\hat{x} \otimes \hat{y}$ is the element in ID obtained by multiplying $x + \varepsilon x^*$ and $y + \varepsilon y^*$ as if they were polynomials and then using the relation $\varepsilon^2 = 0$ to simplify the result.

$$\hat{x} \otimes \hat{y} = \hat{x} \hat{y} = (x + \varepsilon x^*)(y + \varepsilon y^*) = xy + \varepsilon(xy^* + x^*y).$$

Then the dual number $x + \varepsilon x^*$ divided by the dual number $y + \varepsilon y^*$ provided $y \neq 0$ can be defined as

$$\frac{\hat{x}}{\hat{y}} = \frac{x + \varepsilon x^*}{y + \varepsilon y^*} = \frac{x}{y} + \varepsilon \frac{x^*y - xy^*}{y^2}.$$

The set of

$$\begin{aligned} ID^3 &= ID \times ID \times ID \\ &= \left\{ \vec{\hat{x}} \mid \vec{\hat{x}} = (x_1 + \varepsilon x_1^*) + (x_2 + \varepsilon x_2^*) + (x_3 + \varepsilon x_3^*) \right\} \\ &= \left\{ \vec{\hat{x}} \mid \vec{\hat{x}} = (x_1, x_2, x_3) + \varepsilon (x_1^*, x_2^*, x_3^*) \right\} \\ &= \left\{ \vec{\hat{x}} \mid \vec{\hat{x}} = \vec{x} + \varepsilon \vec{x}^* \right\} \end{aligned}$$

is a module on the ring ID . For any $\vec{\hat{x}} = \vec{x} + \varepsilon \vec{x}^*, \vec{\hat{y}} = \vec{y} + \varepsilon \vec{y}^* \in ID^3$, the scalar or inner product and the vector product of $\vec{\hat{x}}$ and $\vec{\hat{y}}$ are defined by, respectively,

$$\langle \vec{\hat{x}}, \vec{\hat{y}} \rangle = \langle \vec{x}, \vec{y} \rangle + \varepsilon \left(\langle \vec{x}, \vec{y}^* \rangle + \langle \vec{x}^*, \vec{y} \rangle \right),$$

$$\vec{\hat{x}} \wedge \vec{\hat{y}} = (\hat{x}_2 \hat{y}_3 - \hat{x}_3 \hat{y}_2, \hat{x}_3 \hat{y}_1 - \hat{x}_1 \hat{y}_3, \hat{x}_1 \hat{y}_2 - \hat{x}_2 \hat{y}_1),$$

where $\hat{x}_i = x_i + \varepsilon x_i^*, \hat{y}_i = y_i + \varepsilon y_i^* \in ID, 1 \leq i \leq 3$. If $x \neq 0$, the norm $\|\vec{\hat{x}}\|$ of $\vec{\hat{x}} = \vec{x} + \varepsilon \vec{x}^*$ is defined by

$$\|\vec{\hat{x}}\| = \sqrt{\langle \vec{\hat{x}}, \vec{\hat{x}} \rangle} = \|\vec{x}\| + \varepsilon \frac{\langle \vec{x}, \vec{x}^* \rangle}{\|\vec{x}\|}.$$

A dual vector $\vec{\hat{x}}$ with norm 1 is called a dual unit vector. Let $\vec{\hat{x}} = \vec{x} + \varepsilon \vec{x}^* \in ID$. If every $x_i(t)$ and $x_i^*(t), 1 \leq i \leq 3$ real valued functions, are differentiable the dual space curve

$$\begin{aligned} \hat{x} &: I \subset IR \rightarrow ID^3 \\ t &\rightarrow \vec{\hat{x}}(t) = x_1(t) + \varepsilon x_1^*(t) + x_2(t) + \varepsilon x_2^*(t) + x_3(t) + \varepsilon x_3^*(t) \\ &= \vec{x}(t) + \varepsilon \vec{x}^*(t) \end{aligned}$$

in ID^3 is differentiable. We call the real part $\vec{x}(t)$ the indicatrix of $\vec{\hat{x}}(t)$. The dual arc-length of the curve $\vec{\hat{x}}(t)$ from t_1 to t is defined as

$$(1) \quad \hat{s} = \int_{t_1}^t \|\vec{\hat{x}}(t)\| dt = \int_{t_1}^t \|\vec{x}(t)\| dt + \varepsilon \int_{t_1}^t \langle \vec{t}, \vec{x}^* \rangle s + \varepsilon s^*,$$

where $\vec{t}$ is a unit tangent vector of $\vec{x}(t)$. From now on we will take the arc-length s of $\vec{x}(t)$ as the parameter instead of t . Now we will obtain equations relatively to the derivatives of dual Frenet vectors throughout in ID^3 . Let

$$\begin{aligned} \hat{x} &: I \subset IR \rightarrow ID^3 \\ s &\rightarrow \vec{\hat{x}}(s) = \vec{x}(s) + \varepsilon \vec{x}^*(s) \end{aligned}$$

be a dual curve with the arc-length parameter s of the indicatrix. Then ,

$$\frac{d\vec{\hat{x}}}{d\hat{s}} = \frac{d\vec{x}}{ds} \frac{ds}{d\hat{s}} = \vec{\hat{t}}$$

is called the dual unit tangent vector of $\vec{\hat{x}}(s)$. With the aid of (1), we have

$$\frac{d\vec{\hat{t}}}{d\hat{s}} = \frac{d\vec{t}}{ds} \frac{ds}{d\hat{s}} = \hat{\kappa} \vec{\hat{n}}.$$

The norm of the vector $\frac{d\vec{\hat{t}}}{d\hat{s}}$ is called curvature function of $\vec{\hat{x}}(s)$. We impose the restriction that the function $\hat{\kappa}: I \rightarrow ID$ is never pure dual. Then , the dual unit vector $\vec{\hat{n}} = \frac{1}{\hat{\kappa}} \frac{d\vec{\hat{t}}}{d\hat{s}}$ is called the principal normal of $\vec{\hat{x}}(s)$. The dual vector $\vec{\hat{b}}$ is called the binormal of $\vec{\hat{x}}(s)$. The dual vectors $\vec{\hat{t}}, \vec{\hat{n}}, \vec{\hat{b}}$ are called the dual Frenet frame of $\vec{\hat{x}}(s)$ at the point $\hat{x}(s)$. The equalities relative to derivative of dual Frenet vectors $\vec{\hat{t}}, \vec{\hat{n}}, \vec{\hat{b}}$ throughout the dual space curve are written in the matrix form

$$(2) \quad \frac{d}{d\hat{s}} \begin{bmatrix} \vec{\hat{t}} \\ \vec{\hat{n}} \\ \vec{\hat{b}} \end{bmatrix} = \begin{bmatrix} 0 & \hat{\kappa} & 0 \\ -\hat{\kappa} & 0 & \hat{\tau} \\ 0 & -\hat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \vec{\hat{t}} \\ \vec{\hat{n}} \\ \vec{\hat{b}} \end{bmatrix}$$

where $\hat{\kappa} = \kappa + \varepsilon \kappa^*$ is pure dual curvature and $\hat{\tau} = \tau + \varepsilon \tau^*$ is pure dual torsion. (2) are called the Frenet formulae of dual curve in ID^3 .

3. THE SIMILAR CURVES IN ID^3 WITH VARIABLE TRANSFORMATIONS

Definition 1. Let ID^3 be the dual space with the standard inner product $\langle \cdot, \cdot \rangle$. If there exist a corresponding relationship between the dual space curves $\hat{\alpha}$ and $\hat{\alpha}_*$ such that, at the corresponding points of the dual curves, the tangent lines of $\hat{\alpha}(\hat{s})$ coincides with the tangent lines of $\hat{\alpha}_*(\hat{s}_*)$, then $\hat{\alpha}$ is called a dual similar curve, and $\hat{\alpha}_*$ a dual similar partner curve of $\hat{\alpha}$. The pair $\{\hat{\alpha}, \hat{\alpha}_*\}$ is said to be a dual similar curve pair. Let there exists a variable transformation

$$\hat{s} = \int \lambda d\hat{s}_*$$

of the arc-length $\hat{s}$ and $\hat{s}_*$. Denote by $\left\{ \vec{\hat{t}}(\hat{s}), \vec{\hat{n}}(\hat{s}), \vec{\hat{b}}(\hat{s}) \right\}$ and $\left\{ \vec{\hat{t}}_*(\hat{s}_*), \vec{\hat{n}}_*(\hat{s}_*), \vec{\hat{b}}_*(\hat{s}_*) \right\}$ the Frenet frame along $\hat{\alpha}$ and $\hat{\alpha}_*$, the following theorems, we give a necessary and sufficient condition for a dual space curve to be a dual similar curves. If we integrate the equality

$$(3) \quad \vec{\hat{t}} = \vec{\hat{t}}_*$$

we obtain the equation (3) and the following important theorem:

Theorem 1. The position vectors of the family dual similar curves in ID^3 with variable transformation can be written in the following form .

$$\hat{\alpha}(\hat{s}) = \int \vec{t}(\hat{s}(\hat{s}_*)) d(\hat{s}_*) = \int \vec{t}(\hat{s}) \lambda_{\alpha}^{\beta} d(\hat{s}_*).$$

Theorem 2. Let $\hat{\alpha}(\hat{s})$ be a dual curve parameterized by arc-length $\hat{s}$. Provided that $\hat{\alpha}(\hat{\varphi})$ be another parametrization of the curve with parameter $\hat{\varphi} = \int \hat{\kappa}(\hat{s}) d(\hat{s})$.

Then the unit tangent vector $\vec{t}$ of $\hat{\alpha}(\hat{s})$ satisfies a dual vector differential equation of third order as follows:

$$(4) \quad \left[\frac{1}{\gamma} \left[\vec{t} \right]'' \right]' + \left[\frac{1+\gamma^2}{\gamma} \right] (\vec{t})' + \frac{\gamma'}{\gamma} \vec{t} = 0.$$

where

$$\gamma(\hat{\varphi}) = \frac{\hat{\tau}(\hat{\varphi})}{\hat{\kappa}(\hat{\varphi})}, \quad (\vec{t})' = (d\vec{t})/d\hat{\varphi}, \quad (\vec{t})'' = (d^2\vec{t})/d\hat{\varphi}^2$$



Let be derivatives given in (2) according to φ ,

$$(5) \quad \begin{aligned} \frac{d\vec{t}}{d\varphi} &= \hat{\kappa} \frac{1}{\hat{\kappa}} \vec{n} = \vec{n} \\ \frac{d\vec{n}}{d\varphi} &= -\hat{\kappa} \frac{1}{\hat{\kappa}} \vec{t} + \hat{\tau} \frac{1}{\hat{\kappa}} \vec{b} = -\vec{t} + \gamma \vec{b} \\ \frac{d\vec{b}}{d\varphi} &= -\hat{\tau} \frac{1}{\hat{\kappa}} \vec{b} = -\gamma \vec{n} \end{aligned}$$

where $\gamma(\varphi) = \frac{\hat{\tau}(\hat{\varphi})}{\hat{\kappa}(\hat{\varphi})}$. Then, corresponding matrix form of (5) can be obtained

$$(6) \quad \begin{bmatrix} \vec{t}' \\ \vec{n}' \\ \vec{b}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \gamma \\ 0 & -\gamma & 0 \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix}.$$

From the first and second equation of new Frenet derivatives (6) we obtain

$$(7) \quad \vec{b} = \frac{1}{\gamma(\hat{\varphi})} \left(\vec{t}'' + \vec{t} \right).$$

Substituting the above equation in the last equation (6), we obtain a dual vector differential equation of third order (4) as desired. Where the representation of the position vector of an arbitrary space can be determined as follows

$$\hat{\alpha}(\hat{s}) = \int \frac{1}{\hat{\kappa}(\hat{\varphi})} \vec{t}(\hat{\varphi}) d\hat{\varphi} + C \quad \text{and} \quad \varphi = \int \hat{\kappa}(\hat{\varphi})$$

is a constant vector.

Theorem 3. Let $\hat{\alpha}(\hat{s})$ and $\hat{\alpha}_*(\hat{s}_*)$ be dual curves in ID^3 . Then $\hat{\alpha}(\hat{s})$ and $\hat{\alpha}_*(\hat{s}_*)$ are dual similar curves in ID^3 with variable transformation if and only if the principal normal vectors are the same for all curves

$$(8) \quad \vec{\hat{n}}(\hat{s}) = \vec{\hat{n}}_*(\hat{s}_*)$$

under the particular variable transformation

$$(9) \quad \lambda(\hat{s}) = \hat{\kappa}_*(\hat{s}_*) \text{ and } \lambda(\hat{s}) = \hat{\tau}_*(\hat{s}_*)$$

of arc-lengths.

Proof. ($\Rightarrow$) Let $\hat{\alpha}(\hat{s})$ and $\hat{\alpha}_*(\hat{s}_*)$ be dual similar curves in ID^3 with variable transformation. Then differentiating the equation $\vec{\hat{t}} = \vec{\hat{t}}_*$ with respect to $(\hat{s}_*)$ we obtain

$$\frac{d\hat{s}}{d\hat{s}_*} \hat{\kappa}(\hat{s}) \vec{\hat{n}}(\hat{s}) = \hat{\kappa}_*(\hat{s}_*) \vec{\hat{n}}_*(\hat{s}_*).$$

□

The above equation leads to the two equation (8) and (9).

($\Leftarrow$) Conversely, Let $\hat{\alpha}(\hat{s})$ and $\hat{\alpha}_*(\hat{s}_*)$ be dual similar curves in ID^3 with variable transformation satisfying (8) and (9). By multiplying (8) with $\hat{\kappa}_*$ and integrate the result equality with respect to $\hat{s}_*$, we obtain

$$(10) \quad \int \hat{\kappa}_*(\hat{s}_*) \vec{\hat{n}}_*(\hat{s}_*) d\hat{s}_* = \int \hat{\kappa}_*(\hat{s}_*) \vec{\hat{n}}_*(\hat{s}_*) \frac{d\hat{s}_*}{d\hat{s}} d\hat{s}.$$

From the equations (8), (9) and (10) take the form

$$\vec{\hat{t}}_*(\hat{s}_*) = \int \hat{\kappa}_*(\hat{s}_*) \vec{\hat{n}}_*(\hat{s}_*) d\hat{s}_* = \int \hat{\kappa}(\hat{s}) \vec{\hat{n}}(\hat{s}) d\hat{s} = \vec{\hat{t}}(\hat{s})$$

which means that $\hat{\alpha}(\hat{s})$ and $\hat{\alpha}_*(\hat{s}_*)$ are dual similar curves in ID^3 with variable transformation and proof is completed.

Theorem 4. Let $\hat{\alpha}(\hat{s})$ and $\hat{\alpha}_*(\hat{s}_*)$ be dual curves in ID^3 . Then $\hat{\alpha}(\hat{s})$ and $\hat{\alpha}_*(\hat{s}_*)$ are dual similar curves in ID^3 with variable transformation if and only if the binormal vectors are the same for all curves

$$b_{\beta}(\hat{s}) = b_{\alpha}(s_{\alpha}),$$

under arbitrary variable transformation $\hat{s}_* = \hat{s}_*(\hat{s})$ of the arc-lengths.

Proof. ($\Rightarrow$) $\hat{\alpha}(\hat{s})$ and $\hat{\alpha}_*(\hat{s}_*)$ are two dual similar curves in ID^3 with variable transformation. Then there are a variable transformation of arc-length such that the dual tangent vectors and the dual principal normal vectors are the same. From equation $\left\{ \vec{\hat{t}}_*(\hat{s}_*) = \vec{\hat{t}}(\hat{s}) \right\}$ and $\left\{ \vec{\hat{n}}(\hat{s}) = \vec{\hat{n}}_*(\hat{s}_*) \right\}$ we obtain

$$\vec{\hat{b}}_*(\hat{s}_*) = \vec{\hat{t}}_*(\hat{s}_*) \times \vec{\hat{n}}_*(\hat{s}_*) = \vec{\hat{t}}(\hat{s}) \times \vec{\hat{n}}(\hat{s}) = \vec{\hat{b}}(\hat{s}).$$

□

($\Leftarrow$:) Let $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha}_*(\widehat{s}_*)$ are the same dual binormal vector under the arbitrary variable transformation $\widehat{s}_* = \widehat{s}_*(\widehat{s})$ of the arc-lengths. Differentiating the equation $\left\{ \widehat{b}_*(\widehat{s}_*) = \widehat{b}(\widehat{s}) \right\}$ with respect to $\widehat{s}_*$ we obtain

$$\begin{aligned} -\widehat{\tau}_* \widehat{n}_*(\widehat{s}_*) &= -\widehat{\tau} \widehat{n}(\widehat{s}) \frac{d\widehat{s}}{d\widehat{s}_*} \\ \widehat{n}_*(\widehat{s}_*) &= \widehat{n}(\widehat{s}) \end{aligned} \quad (11)$$

from equation (11) we obtain

$$\widehat{t}_*(\widehat{s}_*) = \widehat{n}_*(\widehat{s}_*) \times \widehat{b}_*(\widehat{s}_*) = \widehat{n}(\widehat{s}) \times \widehat{b}(\widehat{s}) = \widehat{t}(\widehat{s}).$$

Theorem 5. Let $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha}_*(\widehat{s}_*)$ are dual curves in ID^3 . Then $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha}_*(\widehat{s}_*)$ are dual similar curves with variable transformation if and only if ratios of torsion and curvature are the same for all curves

$$(12) \quad \frac{\widehat{\kappa}_*}{\widehat{\kappa}} = \frac{\widehat{\tau}_*}{\widehat{\tau}},$$

under the particular variable transformations $\left(\lambda_{\beta}^{\widehat{\alpha}} = \frac{d\widehat{s}}{d\widehat{s}_*} = \frac{\widehat{\kappa}(\widehat{s})}{\widehat{\kappa}_*(\widehat{s}_*)} \right)$ keeping equal total curvatures, i.e.,

$$(13) \quad \widehat{\varphi}_*(\widehat{s}_*) = \int \widehat{\kappa}_* d\widehat{s}_* = \int \widehat{\kappa} d\widehat{s} = \widehat{\varphi}(\widehat{s}) \text{ of the arc-lengths.}$$

Proof. ($\Rightarrow$) Let $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha}_*(\widehat{s}_*)$ are two dual curves in ID^3 . Then there exists a variable transformation of the arc-lengths such that the tangent and the binormal vectors are the same. Differentiating the equations $\left\{ \widehat{t}_*(\widehat{s}_*) = \widehat{t}(\widehat{s}) \right\}$ and $\left\{ \widehat{b}_*(\widehat{s}_*) = \widehat{b}(\widehat{s}) \right\}$ we have

$$\begin{aligned} \widehat{\kappa}_*(\widehat{s}_*) \widehat{n}_*(\widehat{s}_*) &= \widehat{\kappa}(\widehat{s}) \widehat{n}(\widehat{s}) \frac{d\widehat{s}}{d\widehat{s}_*} \\ -\widehat{\tau}_*(\widehat{s}_*) \widehat{n}_*(\widehat{s}_*) &= -\widehat{\tau}(\widehat{s}) \widehat{n}(\widehat{s}) \frac{d\widehat{s}}{d\widehat{s}_*} \end{aligned}$$



which leads to the following two equations

$$(14) \quad \widehat{\kappa}_*(\widehat{s}_*) = \widehat{\kappa}(\widehat{s}) \frac{d\widehat{s}}{d\widehat{s}_*}$$

$$(15) \quad \widehat{\tau}_*(\widehat{s}_*) = \widehat{\tau}(\widehat{s}) \frac{d\widehat{s}}{d\widehat{s}_*}$$

The variable transformation (12) is the equation (14) after integration. Dividing the above two equations (14) and (15), we obtain the equation (12) under the variable transformations (13).

($\Leftarrow$:) Let $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha}_*(\widehat{s}_*)$ are two dual curves in ID^3 such that the equation (12) is satisfied under the variable transformation (13) of the arclengths. Let now

consider Theorem (2), the dual tangent vectors $\vec{t}_*(\hat{s}_*)$ and $\vec{t}(\hat{s})$ of the two curves satisfy the following vector differential equations of third order as follows:

$$(16) \quad \left[\frac{1}{\gamma(\hat{\varphi}(\hat{s}))} \left[\vec{t}(\hat{\varphi}(\hat{s})) \right]'' \right]' + \left[\frac{1 + (\gamma(\hat{\varphi}(\hat{s})))^2}{\gamma(\hat{\varphi}(\hat{s}))} \right] (\vec{t})' + \frac{(\gamma(\hat{\varphi}(\hat{s})))'}{\gamma(\hat{\varphi}(\hat{s}))} \vec{t}(\hat{\varphi}(\hat{s})) = 0.$$

$$(17) \quad \left[\frac{1}{\gamma_*(\hat{\varphi}(\hat{s}_*))} \left[\vec{t}(\hat{\varphi}(\hat{s}_*)) \right]'' \right]' + \left[\frac{1 + (\gamma_*(\hat{\varphi}(\hat{s}_*)))^2}{\gamma_*(\hat{\varphi}(\hat{s}_*))} \right] (\vec{t})' + \frac{(\gamma_*(\hat{\varphi}(\hat{s}_*)))'}{\gamma_*(\hat{\varphi}(\hat{s}_*))} \vec{t}(\hat{\varphi}(\hat{s}_*)) = 0.$$

where

$$\gamma(\hat{\varphi}(\hat{s})) = \frac{\hat{\tau}(\hat{\varphi}(\hat{s}))}{\hat{\kappa}(\hat{\varphi}(\hat{s}))}, \quad \gamma_*(\hat{\varphi}(\hat{s})) = \frac{\hat{\tau}_*(\hat{\varphi}(\hat{s}_*))}{\hat{\kappa}_*(\hat{\varphi}(\hat{s}_*))}, \quad \hat{\varphi} = \int \hat{\kappa}(\hat{s}) \text{ and } \hat{\varphi}_* = \int \hat{\kappa}_*(\hat{s}_*) .$$

The equation (16) causes

$$\gamma_*(\hat{\varphi}(\hat{s})) = \gamma(\hat{\varphi}(\hat{s}))$$

under the variable transformations $\varphi_\alpha = \varphi_\beta$. So that the two equations (16) and (17) under the equation (12) and the transformation (13) are the same. Hence the solution is the same i.e., the tangent vectors are the same which completes the proof of the theorem. 

Lemma 1. *The straight line alone forms a family of dual similar curves in ID^3 with variable transformations.*

Example 1. *Let the dual curve be straight line then $\tau = 0$. Under the variable transformation λ the curvature does not change.*

Example 2. *Euler Spirals, were discovered independently  by three researchers. In 1694, Bernoulli wrote the equations for the Euler spiral for the first time, but did not draw the spirals or compute them numerically. In 1774, Euler re-described their properties, and derived a series expansion to the curve's integrals. Later, In 1871, he also computed the spiral's end points. The curves were re-discovered 1891 for the third time by Talbot, who used them to design railway tracks [16]. An Euler spiral is curve whose curvature changes linearly with its curve length (the curvature of a circular curve is equal to the reciprocal of the radius). Euler spirals are also commonly referred to as spiro, clothoids or Cornu spirals. Moreover, Euler spiral in railroad/highway engineering for connecting and transiting the geometry between a tangent and a circular curve. Let us consider the Euler Spiral $\vec{\gamma}(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ of E^3*

$$\begin{cases} \gamma_1(\hat{s}) = \frac{3}{5} \int \sin(\hat{s}^2 + 1) d\hat{s} \\ \gamma_2(\hat{s}) = \frac{3}{5} \int \cos(\hat{s}^2 + 1) d\hat{s} \\ \gamma_3(\hat{s}) = \frac{4}{5} \hat{s} \end{cases}$$

Then the dual unit tangent vector $\vec{t}$ of $\vec{\gamma}(s)$ satisfies a vector differential equation of third order given by

$$\left[\frac{1}{\gamma(\hat{\theta})} \left[\vec{t}(\hat{\theta}) \right]'' \right]' + \left[\frac{1 + \gamma^2(\hat{\theta})}{\gamma(\hat{\theta})} \right] (\vec{t})' + \frac{\gamma'(\hat{\theta})}{\gamma(\hat{\theta})} \vec{t}(\hat{\theta}) = 0.$$

 We are calculated this curve's curvature function with help of Mathematica Programme $\hat{\kappa} = \frac{6\hat{s}}{5}$ and $\hat{\tau} = -\frac{8\hat{s}}{5}$. The Frenet -Serret frame of the curve $\vec{\gamma} = \vec{\gamma}(s)$ may be written by the aid Mathematica Programme as follows: 

$$\left\{ \begin{array}{l} \vec{t}(\hat{s}) = (\frac{3}{5} \sin(\hat{s}^2 + 1), \frac{3}{5} \cos(\hat{s}^2 + 1), \frac{4}{5}), \\ \vec{n}(\hat{s}) = (\cos(\hat{s}^2 + 1), -\sin(\hat{s}^2 + 1), 0), \\ \vec{b}(\hat{s}) = (\frac{4}{5} \sin(\hat{s}^2 + 1), \cos(\hat{s}^2 + 1), -\frac{3}{5}). \end{array} \right\}$$

Lemma 2. The family of Euler Spiral  is a family of dual similar curves with variable transformations. We can deduce the position vector of Euler Spiral curve using the definition of dual similar curves with variable transformation as follows:

$$\begin{aligned} \vec{\gamma}(\hat{v}) &= (\gamma_1(\hat{v}), \gamma_2(\hat{v}), \gamma_3(\hat{v})) \\ &= (\frac{3}{5} \int \sin(\hat{v}^2 + 1) d\hat{v}, \frac{3}{5} \int \cos(\hat{v}^2 + 1) d\hat{v}, \frac{4}{5} \hat{v}) \end{aligned}$$

where $\hat{s} = \hat{v}$ is the arc-length of Euler Spiral and the curvature is $\hat{\kappa}_x(\hat{v}) = \frac{6\hat{v}}{5}$. The dual unit tangent vectors of Euler Spiral takes the form:

$$\vec{t}(\hat{v}) = (\frac{3}{5} \sin(\hat{v}^2 + 1), \frac{3}{5} \cos(\hat{v}^2 + 1), \frac{4}{5}),$$

from theorem(1), we can write as the following,

$$\vec{\beta}(\hat{s}_*) = \int \left[\frac{3}{5} \sin[(\hat{v}(\hat{s}_*))^2 + 1], \frac{3}{5} \cos[(\hat{v}(\hat{s}_*))^2 + 1], \frac{4}{5} \right] d\hat{s}_*$$

where $\hat{s}_* = \hat{s}$. From the equation (9), we obtain

$$d\hat{s} = \frac{\hat{\kappa}_*}{\hat{\kappa}} d\hat{s}_*$$

or

$$\hat{s}(\hat{s}_*) = \int \frac{\hat{\kappa}_*}{\hat{\kappa}} d\hat{s}_*.$$

If we put the curvature $\hat{\kappa}_* = \hat{\kappa}(\hat{s})$, $\hat{s}_* = \hat{s}$, we have $\hat{v}(\hat{s}_*) = \int \hat{\kappa}(\hat{s}) d\hat{s}$. Then the position vector of Euler Spiral with arbitrary curvature $\hat{\kappa}(\hat{s})$ takes the following form:

$$\vec{x}(\hat{v}) = \int (\frac{3}{5} \sin(\int \hat{\kappa}(\hat{s}) d\hat{s} + 1), \frac{3}{5} \cos(\int \hat{\kappa}(\hat{s}) d\hat{s} + 1), \frac{4}{5}) d\hat{s}.$$

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REFERENCES

- [1] Hamilton, W.R., Element of Quaternions, I, II and III,  New YORK, 1899.
- [2] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press Inc., London, 1983.
- [3] Izumiya, S., Takeuchi, N., New special curves and developable surfaces, Turk. J.Math. Vol. 28 (2004), 153-163.
- [4] Önder, M., Kazaz, M., Kocayigit, H., Kılıç  -slant helix in Euclidean 4-space E^4 , Int. J.Cont. Math. Sci. vol. 3, no.29, 1433-1440, 2008.
- [5] Salkowski, E., Zur Transformation von Raumkurven, Mathematice Annalen 66(4), 517-557, 1909.

- [6] Monterde, J., Salkowski curves revisited: A family of curve with constant curvature and non-constant torsion, *Comput. Aided Geo. Design*, 26-271-278, 2009.
- [7] B. Saint Venant, Memoire sur les lignes courbes non planes, *Journal de l'Ecole Polytechnique*. 18,1-76, 1845.
- [8] J. M. Bertrand, Memoire sur la theorie des courbes a double courbure, *comptes Rendus*, vol.36, 1850.
- [9] Do Carmo, M.P., *Differential Geometry of Curves and Surfaces*, Pearson Education, 1976.
- [10] Hacisalihpğlu, H. H., On the pitch of a closed ruled surface, *Mech. Mach. Theory*, 7, no. 3,291-305, 1972.
- [11] Kuhlne, W., *Differential geometry: curves-surfaces-manifolds*. Braum- scheig, Wiesbaden, 1999.
- [12] John F. Burke, Bertrand Curves Associated with a Pair of Curves, *Mathematics Magazine*, Vol. 34, No. 1, pp. 60-62, 1960.
- [13] Ch. Bioche, Sur les courbes de M. Bertrand, *Bull. Soc. Math. France* 17, 109-112, 1889.
- [14] A.T. Ali, Position vectors of general helices in Euclidean 3-space, *Bull. Math. Anal. Appl.*3(2), (2010), 198–205.
- [15] Guggenheimer, W., *Differential Geometry*, McGraw-Hill, New York,1963.
- [16] G. Harary, A. TAL, 3D Euler Spirals for 3D Curve Completion, *Symposium on Computational Geometry*: 107-108, 2010.