

Eta-Einstien S -Manifolds and Spectral Geometry^{*†}

Najma Abdul Rehman

Abstract

We examine the spectral geometry of η -Einstien S -manifolds and compute the the spectral coefficients for S -space form and obtain the results analogue to Patodi's results for Riemannian manifolds and J. Park results for η -Einstien Sasakian manifolds. We show that how an η -Einstien S -manifold and S -space forms are spectrally determined.

1 Introduction

The spectral geometry of the Laplace-Beltrami operator has been studied by many authors [4], [5], [6], [10] and has developed rapidly during the last decade. Mainly, topic of Spectral Geometry is developed by Patodi's work [10] on heat kernel asymptotics for operators of Laplace type on manifolds with boundary. Gilkey [6] computed some spectral invariants concerning the asymptotic expansion of the trace of the heat kernel for an elliptic differential operator acting on the space of sections of a vector bundle. Later on Donnelly [4] and Sacks [5] extended the Patodi's results to complex manifolds. After J. Park extended these result to η -Einstien contact manifolds [9].

Generalizing the notion of η -Einstien contact manifolds, M. Kobayashi and S. Tsuchiya defined the notion of η -Einstien S -manifolds [11], [8]. For more detail on eta-Einstien S -manifolds see [2].

Let Δ_p be the Laplace Beltrami operator acting on the space of smooth p -forms, $p=0, 1, 2$ over a compact m -dimensional Riemannian manifold M . Patodi [10] set up the following result for the space forms.

Theorem 1.1. [10] *Let (M_i, g_i) be the compact Riemannian manifolds without boundary. Assume that $\text{Spec}(\Delta_p, M_1) = \text{Spec}(\Delta_p, M_2)$, for $p = 0, 1, 2$. Then*

- (1) *The manifold M_1 has constant scalar curvature c if and only if the manifold M_2 has constant scalar curvature c .*
- (2) *The manifold M_1 is Einstein if and only if the manifold M_2 is Einstein.*
- (3) *The manifold M_1 has constant sectional curvature c if and only if the manifold M_2 has constant sectional curvature c .*

We extend these results to η -Einstien S -manifolds which are the generalization of both complex and contact structure. For $s = 0$, our results are for manifolds with complex structure, for $s = 1$, we get results for η -Einstien Sasakian manifolds [9].

^{*}2010 AMS Mathematics Subject Classification: 58J50, 53D10, 53D35

[†]Keywords and phrases: S -Manifolds, η -Einstien manifolds, Spectral Geometry, Laplace operator.

In this paper, after introduction, the second section is related to S-manifolds. Third section is related to main results related to curvature tensor and heat trace asymptotics on η -Einstein S-manifolds.

2 S-manifolds

As a generalization of both almost complex (in even dimension) and almost contact (in odd dimension) structures, Yano introduced in [1] the notion of f -structure on a smooth manifold of dimension $2n + s$, i.e. a tensor field of type (1,1) and rank $2n$ satisfying $f^3 + f = 0$. The existence of such a structure is equivalent to a reduction of the structural group of the tangent bundle to $U(n) \times O(s)$. Let N be a $(2n + s)$ -dimensional manifold with an f -structure of rank $2n$. If there exist s global vector fields $\xi_1, \xi_2, \dots, \xi_s$ on N such that:

$$f\xi_\alpha = 0, \quad \eta_\alpha \circ f = 0, \quad f^2 = -I + \sum \xi_\alpha \otimes \eta_\alpha, \quad (1)$$

where η_α are the dual 1-forms of ξ_α , we say that the f -structure has complemented frames. For such a manifold there exists a Riemannian metric g such that

$$g(X, Y) = g(fX, fY) + \sum \eta_\alpha(X)\eta_\alpha(Y)$$

for any vector fields X and Y on N .

An f -structure f is normal, if it has complemented frames and

$$[f, f] + 2 \sum \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f, f]$ is Nijenhuis torsion of f .

Let F be the fundamental 2-form defined by $F(X, Y) = g(X, fY)$, $X, Y \in T(N)$. A normal f -structure for which the fundamental form F is closed, $\eta_1 \wedge \dots \wedge \eta_s \wedge (d\eta_\alpha)^n \neq 0$ for any α , and $d\eta_1 = \dots = d\eta_s = F$ is called to be an S -structure. A smooth manifold endowed with an S -structure will be called an S -manifold. These manifolds were introduced by Blair in [1].

We have to remark that that if we take $s = 1$, S -manifolds are natural generalization of Sasakian manifolds. In the case $s \geq 2$ some interesting examples are given in [1, 7].

If N is an S -manifold, then the following formulas are true (see [1]):

$$\bar{\nabla}_X \xi_\alpha = -fX, \quad X \in T(N), \quad \alpha = 1, \dots, s, \quad (2)$$

$$(\bar{\nabla}_X f)Y = \sum \{g(fX, fY)\xi_\alpha + \eta_\alpha(Y)f^2X\}, \quad X, Y \in T(N), \quad (3)$$

where $\bar{\nabla}$ is the Riemannian connection of g . Let L be the distribution determined by the projection tensor $-f^2$ and let M be the complementary distribution which is determined by $f^2 + I$ and spanned by $\xi_1, \dots, \xi_s$. It is clear that if $X \in L$ then $\eta_\alpha(X) = 0$ for any α , and if $X \in M$, then $fX = 0$. A plane section π on N is called an invariant f -section if it is determined by a vector $X \in L(x)$, $x \in N$, such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature of π is called the f -sectional curvature. If N is an S -manifold of constant f -sectional curvature k , then its curvature tensor has the

form

$$\begin{aligned}
R(X, Y, Z, W) = & \sum_{\alpha, \beta} \{g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W) + \\
& + g(fY, fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY, fW)\eta_\alpha(X)\eta_\beta(Z)\} + \\
& + \frac{1}{4}(k + 3s)\{g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)\} + \\
& + \frac{1}{4}(k - s)\{F(X, W)F(Y, Z) - F(X, Z)F(Y, W) - 2F(X, Y)F(Z, W)\}, \quad (4)
\end{aligned}$$

$X, Y, Z, W \in T(N)$. Such a manifold $N(k)$ will be called an S -space form. The Euclidean space E^{2n+s} and the hyperbolic space H^{2n+s} are examples of S -space forms.

Definition 2.1.  $(M_m, \eta_\alpha, g, f, \xi_\alpha)$ is said to be η -Einstein if the Ricci tensor ρ of M is of the form

$$\rho = ag + b \sum_{\alpha=1}^s \eta_\alpha \otimes \eta_\alpha, \quad (5)$$

where a, b are constants on M .

3 Main results

Let M be a $(2m + s)$ -dimensional S -space form then from (4), the Ricci tensor ρ is given by

$$\rho = \frac{4s + (k + 3s)(2m - 1) + 3(k - 1)}{4}g + \frac{(2m + s - 2)(4 - k - 3s) - 3(k - s)}{4}\eta_\alpha \otimes \eta_\alpha. \quad (6)$$

Hence M is an eta-Einstein manifold.

Define the tensor fields $S_{\alpha, \beta}$ and T_c of M respectively by

$$\begin{aligned}
S_{a, b}(X, Y) &= \rho(X, Y) - (ag(X, Y) + b \sum_{\alpha=1}^s \eta_\alpha(X) \otimes \eta_\alpha(X)), \\
T_c(X, Y, Z, W) &= R(X, Y, Z, W) - \left\{ \sum_{\alpha, \beta} \{g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W) \right. \\
&\quad + g(fY, fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY, fW)\eta_\alpha(X)\eta_\beta(Z)\} \\
&\quad + \frac{1}{4}(k + 3s)\{g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)\} \\
&\quad + \frac{1}{4}(k - s)\{F(X, W)F(Y, Z) - F(X, Z)F(Y, W) - \\
&\quad \left. - 2F(X, Y)F(Z, W)\} \right\}, \quad (7)
\end{aligned}$$

for vector fields X, Y, Z, W on M , where a, b are smooth functions on M and k is a constant f -sectional curvature.

Let $\{e_i\}$ be an orthonormal basis of $T_p M$ at any point $p \in M$. Next in this paper, we shall use the following notations:

$$R_{ijkl} = g(R(e_i, e_j)e_k, e_l), \quad \rho_{i, j} = \rho(e_i, e_j), \quad f_{ij} = g(fe_i, e_j), \quad (8)$$

$$\nabla_i f_{jk} = g((\nabla_{e_i} f)e_j, e_k), \quad \nabla_i \eta_{\alpha j} = g((\nabla_{e_i} \xi_\alpha, e_j) \quad (9)$$

and so on, where indices run over the range $1, 2, \dots, 2m + s$.
The equations (2), (3) can be written as:

$$\nabla_i f_{jk} = \sum_{\alpha} g_{ij} \eta_{\alpha k} - \eta_{\alpha j} g_{ik} \quad (10)$$

$$\nabla_i \eta_{\alpha j} = -f_{ij} \quad (11)$$

By the definition of the tensor field $S_{\alpha, \beta}$, (10), (11) and some computations, we have

$$|S_{a,b}|^2 = |\rho|^2 + (2m + s)a^2 + sb^2 - 2a\tau + 2abs - 4bms \quad (12)$$

Here we see that M is η -Einstein if and only if $S_{a,b} = 0$ and

$$a = \frac{\tau}{2m} - s, \quad b = s(2m + s) - \frac{s\tau}{2m}. \quad (13)$$

Now first we shall prove the following lemma for further computations of $\|T_k\|^2$ of the vector field T_k on S-manifold M. In case of Sasakian manifolds, it was proved by J. Park

Lemma 3.1. *On S-manifold, we have*

$$R_{ijkl}\{f_{ki}f_{jl} - f_{kj}f_{il} + 2f_{ji}f_{kl}\} = 6\tau - 12ms + 12m(2m + s - 2)$$

Proof. It is clear from the properties of curvature tensor that

$$R_{ijkl}f_{ki}f_{jl} = \frac{1}{2}(R_{ijkl} - R_{kjil})f_{ki}f_{jl} = \frac{1}{2}(R_{ijkl} + R_{jkil})f_{ki}f_{jl} = -\frac{1}{2}R_{kijl}f_{ki}f_{jl} \quad (14)$$

Similarly

$$-R_{ijkl}f_{kj}f_{il} = -\frac{1}{2}R_{jkil}f_{jk}f_{il} \quad (15)$$

$$R_{ijkl}f_{ji}f_{kl} = -R_{ijkl}f_{ij}f_{kl} \quad (16)$$

from (14), (15) and (16) we have

$$R_{ijkl}\{f_{ki}f_{jl} - f_{kj}f_{il} + 2f_{ji}f_{kl}\} = -3R_{ijkl}f_{ij}f_{kl} \quad (17)$$

Also

$$\nabla_l \nabla_i f_{jk} = \sum_{\alpha} g_{ij} \nabla_l \eta_{\alpha k} - g_{ik} \nabla_l \eta_{\alpha j} = \sum_{\alpha} -g_{ij} f_{lk} + g_{ik} f_{lj} \quad (18)$$

$$\nabla_l \nabla_i f_{jk} - \nabla_i \nabla_l f_{jk} = \sum_{\alpha} -g_{ij} f_{lk} + g_{ik} f_{lj} + g_{lj} f_{ik} + g_{lk} f_{ij}. \quad (19)$$

Apply Ricci identity to (19) and taking sum by setting $i = k$ in the resulting equality, we get

$$-R_{lij\beta}f_{\beta i} - \rho_{l\beta}f_{j\beta} = (2m + s - 2)f_{lj}, \quad (20)$$

and

$$\begin{aligned} -R_{lij\beta}f_{\beta i}f_{lj} - \rho_{l\beta}f_{j\beta}f_{lj} &= (2m + s - 2)f_{lj}f_{lj} \\ &= (2m + s - 2)(2m). \end{aligned}$$

Hence

$$\begin{aligned} -R_{il\beta j}f_{\beta i}f_{jl} &= -\rho_{l\beta}(g_{l\beta} - \eta_l\eta_\beta) + (2m + s - 2)(2m) \\ &= -\tau + 2ms + (2m + s - 2)(2m) \end{aligned} \quad (21)$$

From (14) and (21)

$$\frac{1}{2}R_{jkil}f_{jk}f_{il} = -\tau + 2ms + (2m + s - 2)(2m) \quad (22)$$

From (17) and (22)

$$R_{ijkl}\{f_{ki}f_{jl} - f_{kj}f_{il} + 2f_{ji}f_{kl}\} = 6\tau - 12ms + 12m(2m + s - 2) \quad \square$$

Let M be a compact Riemannian manifold of real dimension m without boundary and let D be a operator of Laplace type on the space of smooth sections to a smooth vector bundle over M . Let e^{-tD} be the fundamental solution of the heat equation. This operator is of trace class and as $t \downarrow 0$ there is a complete asymptotic expansion with locally computable coefficients in the form:

$$Tr_{L^2} e^{-tD} \sim \sum_{n \geq 0} t^{(n-m)/2} a_n(D). \quad (23)$$

Consider the following result [3].

Theorem 3.1. *Let D be an operator of Laplace type on the space of sections $C^\infty(V)$ to a vector bundle V over a compact manifold M . Let I be the identity endomorphism of V . We have*

$$\begin{aligned} a_0(D) &= (4\pi)^{-m/2} \int_M Tr\{I\}, \\ a_2(D) &= (4\pi)^{-m/2} \frac{1}{6} \int_M Tr(6E + \tau I), \\ a_4(D) &= (4\pi)^{-m/2} \frac{1}{360} \int_M Tr\{60E + 60\tau E + 180E^2 + 30\Omega^2 + (12\tau + 5\tau^2 - 2|\rho|^2 + \\ &\quad + 2|R|^2)I\} \end{aligned} \quad (24)$$

Now we will extend Theorem 1.1 for s -manifolds with proof.

Theorem 3.2. *Let $M_i = (M_i, \eta_i, g_i, f_i, \xi_i)$ be m_i -dimensional compact S -manifolds without boundary with $m_i \geq 5$. Assume that $Spec(\Delta_p, M_1) = Spec(\Delta_p, M_2)$ for $p = 0, 1, 2$, then:*

- (1) $m_1 = m_2$ and $Vol(M_1) = Vol(M_2)$.
- (2) M_1 has constant scalar curvature k if and only if the manifold M_2 has constant scalar curvature k .
- (3) M_1 is η -Einstein if and only if M_2 is η -Einstein.
- (4) M_1 is S -space form with constant f -sectional curvature k if and only if M_2 is S -space form with constant f -sectional curvature k .

The values $p=0, 1, 2$ are not particularly special.

Proof. Let $M(\eta, g, f, \xi)$ be a $(2m+s)$ -dimensional compact S-manifold without boundary. From (3.1) Theorem for $D = \Delta_p$ ($p=0, 1, 2$), we have

$$\begin{aligned} Tr_{L^2}(e^{-t}\Delta_0) &= (4\pi t)^{-m/2}\{Vol(M) + O(t)\}, \\ a_2(\Delta_0, M) &= \frac{1}{6}(4\pi)^{-m/2} \int_M \tau, \\ a_2(\Delta_1, M) &= \frac{1}{6}(4\pi)^{-m/2} \int_M (2m+s)\tau. \end{aligned} \quad (25)$$

The work of Patodi shows that there exist universal constants so:

$$a_4(\Delta_p, M) = (4\pi)^{-m/2} \int_M \{c_{m,p}^1 \tau^2 + c_{m,p}^2 \rho^2 + c_{m,p}^1 R^2 + c_{m,p}^1 \tau\}, \quad p = 0, 1, 2. \quad (26)$$

Let $M_i = (M_i, \eta_i, g_i, f_i, \xi_i)$ be $m_i + s$ -dimensional compact S-manifolds without boundary ($i = 1, 2$). Assume that $Spec(\Delta_p, M_1) = Spec(\Delta_p, M_2)$ for $p=0, 1, 2$. Let R_i , ρ_i and τ_i denote the curvature tensor, the Ricci tensor and the scalar curvature of M_i ($i = 1, 2$) respectively. Then by Theorem 3.1(1) we have $m_1 = m_2$ and $Vol(M_1) = Vol(M_2)$. Also

$$\begin{aligned} \tau_1 &= (4\pi)^{m/2} Vol(M_1)^{-1} \{ma_2(\Delta_0, M_1) - a_2(\Delta_1, M_1)\} \\ &= (4\pi)^{m/2} Vol(M_2)^{-1} \{ma_2(\Delta_0, M_2) - a_2(\Delta_1, M_2)\} \\ &= \tau_2. \end{aligned} \quad (27)$$

Further suppose that M_1 is an η -Einstein S-manifold with coefficient functions α_1 and β_1 . Here α_1 and β_1 are constant and hence the scalar curvature τ_1 of M_1 is also constant given as $\tau_1 = (2m+s)\alpha_1 + \beta_1$. Thus from second assertion of theorem the scalar curvature τ_2 of M_2 is also constant and $\tau_1 = \tau_2$. Since $Vol(M_1) = Vol(M_2)$, the integrals of τ^2 are equal. Since $\tau_{ii} = 0$, from a_4 , we have

$$\int_{M_1} (c_{m,p}^2 \rho_1^2 + c_{m,p}^3 R_1^2) = \int_{M_2} (c_{m,p}^2 \rho_2^2 + c_{m,p}^3 R_2^2), \quad (28)$$

for $p = 1, 2$ these two equations are independent [10]. Consequently

$$\int_{M_1} \rho_1^2 = \int_{M_2} \rho_2^2, \quad \int_{M_1} R_1^2 = \int_{M_2} R_2^2. \quad (29)$$

Thus from (12), we have

$$0 = \int_{M_1} |S_{a_1, b_1}^1|^2 = \int_{M_1} |\rho_1|^2 + (2m+s)a_1^2 + sb_1^2 - 2a_1\tau_1 + 2a_1b_1s - 4b_1ms \quad (30)$$

$$= \int_{M_2} |\rho_2|^2 + (2m+s)a_1^2 + sb_1^2 - 2a_1\tau_1 + 2a_1b_1s - 4b_1ms. \quad (31)$$

Here we may note that

$$a_1 = \frac{\tau_1}{2m} - s = \frac{\tau_2}{2m} - s \quad (32)$$

$$b_1 = s(2m+s) - \frac{s\tau_1}{2m} = s(2m+s) - \frac{s\tau_2}{2m}. \quad (33)$$

Here we may take

$$S_{a,b}^2 = \rho_2 - (a_2 g_2 + b_2 \sum_{\alpha=1}^s \eta_{\alpha 2} \otimes \eta_{\alpha 2}), \quad (34)$$

where

$$a_2 = \frac{\tau_2}{2m} - s \quad b_2 = s(2m + s) - \frac{s\tau_2}{2m}, \quad (35)$$

then we have

$$\int_{M_2} |S_{a_2, b_2}^2|^2 = \int_{M_2} |\rho_2|^2 + (2m + s)a_2^2 + sb_2^2 - 2a_2\tau_2 + 2a_2b_2s - 4b_2ms. \quad (36)$$

This implies

$$a_1 = a_2 \quad b_1 = b_2. \quad (37)$$

Therefore all above equation implies

$$0 = \int_{M_2} |S_{a_2, b_2}^2|^2, \quad (38)$$

then M_2 is an η -Einstein manifold with the same coefficients in the defining equation. This completes the proof of Theorem 3.2 (3). Lastly, suppose that M_1 is s-space form with constant f-sectional curvature k . Then from (6), M_1 is η -Einstein with constant coefficients $a_1 = \frac{4s+(k+3s)(2m-1)+3(k-1)}{4}$ and $b_1 = \frac{(2m+s-2)(4-k-3s)-3(k-s)}{4}$  thus the scalar cutvature is

$$\tau_1 = \frac{m}{2} \{(2m + 2)k + 6m + 8s - 6\}. \quad (39)$$

Therefore from assertion (3) and hypothesis that $\text{Spec}(\Delta_p, M_1) = \text{Spec}(\Delta_p, M_2)(p = 0, 1, 2)$, we see that M_2 is η -Einstein manifold with constant coefficients a_2 and b_2 such that $a_2 = a_1$, $b_2 = b_1$ and hence $\tau_2 = \tau_1$. Now we define the tensor field for the S-manifold M_2 as:

$$(T_c^2)_{ijkl} = R_{ijkl} - K_{ijkl}, \quad (40)$$

where

$$\begin{aligned} K_{ijkl} = & \sum_{\alpha, \beta} \{g(fe_i, e_l)\eta_\alpha(e_j)\eta_\beta(e_k) - g(fe_i, e_k)\eta_\alpha(e_j)\eta_\beta(e_l) + \\ & + g(fe_j, fe_k)\eta_\alpha(e_i)\eta_\beta(e_l) - g(fe_j, fe_l)\eta_\alpha(e_i)\eta_\beta(e_k)\} + \\ & + \frac{1}{4}(k + 3s)\{g(fe_i, fe_l)g(fe_j, fe_k) - g(fe_i, fe_k)g(fe_j, fe_l)\} + \\ & + \frac{1}{4}(k - s)\{F(e_i, e_l)F(e_j, e_k) - F(e_i, e_k)F(e_j, e_l) - 2F(e_i, e_j)F(e_k, e_l)\}, \end{aligned} \quad (41)$$

after some computations, we have

$$|K|^2 = m\{(2m + 2)k^2 + 6m + 8s - 6\}. \quad (42)$$

Now using lemma

$$R_{ijkl}K_{ijkl} = 2k\tau - m(k-s)(6m-6+8s), \quad (43)$$

from (40), (42) and (43), we obtain

$$|T_c^2|^2 = |R_2|^2 - 4k\tau + m\{(2m+2)k^2 + 6m + 8s - 6\} + 2m(k-s)(6m-6+8s). \quad (44)$$

Also, since M_1 is m -dimensional S-space form with constant f -sectional curvature k , we have

$$|R_1|^2 = m\{(2m+2)k^2 + 6m + 8s - 6\}, \quad (45)$$

and

$$0 = |T_c^1|^2 = |R_1|^2 - 4k\tau_1 + m\{(2m+2)k^2 + 6m + 8s - 6\} + 2m(k-s)(6m-6+8s). \quad (46)$$

Thus from (29), (44), (46) and $\tau_1 = \tau_2$ we obtain

$$\begin{aligned} 0 &= \int_{M_1} |T_c^1|^2 \\ &= \int_{M_1} |R_1|^2 - 4k\tau_1 + m\{(2m+2)k^2 + 6m + 8s - 6\} + 2m(k-s)(6m-6+8s) \\ &= \int_{M_2} |R_2|^2 - 4k\tau_2 + m\{(2m+2)k^2 + 6m + 8s - 6\} + 2m(k-s)(6m-6+8s) \\ &= \int_{M_2} |T_c^2|^2, \end{aligned} \quad (47)$$

and hence $T_c^2 = 0$ on M_2 . Therefore we see that M_2 is also an $(2m+s)$ -dimensional S-space form with constant f -sectional curvature k . This completes the proof of Theorem. $\square$

References

- [1] D.E. Blair, *Geometry of manifolds with structural group $U(n) \times O(s)$* , J. Differential Geom. Volume 4, (1970). 155-167.
- [2] Letizia Brunetti, *The η -Einstein condition on indefinite S-manifolds*, Math. Slovaca, No. 2, 64, (2014), 469-478.
- [3] J. Branson and P. Gilkey, *The asymptotics of the Laplacian on a manifold with boundary*, Comm. Partial Differential Equations, 15, (1990), 245-272.
- [4] H. Donnelly, *A spectral condition determining the Kaehler property*, Amer. Math. Soc., 47, (1975), 187-194.
- [5] P. Gilkey and J. Sacks, *Spectral geometry and manifolds of constant holomorphic sectional curvature*, Proc. Sympos. Pure Math., 27, (1975), 281-285.
- [6] P. Gilkey, *The Spectral Geometry of operator of Dirac and Laplace type*, Hand book of Global Analysis, 2008, 289-326.
- [7] I. Hasegawa. Y. Oknyama and T. Abe, *On the p -th Sasakian manifolds*, J. Hokkaido Univ. Ed. Sect. II A, 37(1), (1986), 1-16.

-  [8] M. Kobayashi and S. Tsuchiya, *Invariant submanifolds of an f -manifold with complemented frames*, Kodai Math. Sem. Rep., 24, (1972), 430-450.
-  [9] J. Park, *Spectral geometry of eta-Einstein Sasakian manifolds*, Journal of Geometry and Physics, Volume 62, Issue 11, 2012, 2140-2146.
-  [10] V. K. Patodi, *Curvature and fundamental solution of the heat operator*, J. Indian Math. Soc. 34, (1970), 269-285.
-  [11] K. Yano, *On a structure defined by a tensor field of type $(1, 1)$ satisfying $f^3 + f = 0$* Tensor, 14, (1963), 99-109.
-  Gomsats Institute of Information Technology Sahiwal Campus, Sahiwal, Pakistan
-  –