

Global stability and traveling waves of an epidemic model with diffusion and latent period

Abstract

In this paper, we investigate the global stability and the existence of traveling waves for a delayed diffusive epidemic model. The disease transmission process is modeled by a specific nonlinear function that covers many common types of incidence rates. In addition, the global stability of the disease-free equilibrium and the endemic equilibrium is established by using the direct Lyapunov method. By constructing a pair of upper and lower solutions and applying the Schauder fixed point theorem, the existence of traveling wave solution which connects the two steady states is obtained and characterized by two parameters that are the basic reproduction number and the minimal wave speed. Furthermore, the models and main results presented in [2, 4, 6, 7] are extended and generalized.

Keywords: Global stability, traveling wave, nonlinear incidence rate, Schauder fixed point theorem.

1 Introduction

In epidemiological models, the study of global stability of equilibria determines the dynamical behavior of disease at long-term. However, the traveling wave solutions describe the transition from a disease-free equilibrium to an endemic equilibrium. In addition, the existence of traveling waves indicates that the disease can spread and invade successfully. Therefore, the existence of traveling waves for fast diseases models have been investigated by many authors (see, for example, [1, 2, 3, 4]). But in reality, various diseases occur in more than one year and there is a renewal of the susceptible individuals [5]. For this reason, Yang et al. [6] studied the existence of traveling waves of a delayed diffusive SIR epidemic model with saturated incidence rate in the case when

the diffusion coefficients are equals. In 2015, Li et al. [7] improved the work of [6] by determining the existence of traveling waves for all diffusion coefficients. To extend and generalize the above models, we propose the following system

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = d_S \Delta S + \Lambda - \mu S(x,t) - f(S(x,t), I(x,t))I(x,t), \\ \frac{\partial I(x,t)}{\partial t} = d_I \Delta I + f(S(x,t-\tau), I(x,t-\tau))I(x,t-\tau) - (\mu + d + r)I(x,t), \\ \frac{\partial R(x,t)}{\partial t} = d_R \Delta R + rI(x,t) - \mu R(x,t), \end{cases} \quad (1)$$

where $S(x,t)$, $I(x,t)$, and $R(x,t)$ represent the densities of susceptible, infected and removed individuals at position x and time t , respectively. The positive constants d_S , d_I and d_R denote the corresponding diffusion coefficients. Λ is the recruitment rate of susceptible population, μ is the natural death rate of the population, d is the death rate due to disease, τ is the latent period of the disease, and r is the recovery rate of the infective population. The disease transmission process is modeled by Hattaf's incidence rate [8] of the form $f(S, I) = \frac{\beta S}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI}$, where $\alpha_1, \alpha_2, \alpha_3 \geq 0$ are constants and $\beta > 0$ is the infection coefficient. We recall that this incidence rate includes the common types such as the bilinear incidence (or mass action incidence) when $\alpha_1 = \alpha_2 = \alpha_3 = 0$, the saturated incidence rate when $\alpha_1 = \alpha_3 = 0$ or $\alpha_2 = \alpha_3 = 0$, the Beddington-DeAngelis functional response introduced in [9, 10] when $\alpha_3 = 0$, and Crowley-Martin functional response presented in [11, 12, 13] when $\alpha_3 = \alpha_1 \alpha_2$.

On the other hand, the delayed diffusive SIR epidemic model presented by Li et al. [7] is a special case of system (1), it suffices to take $\alpha_1 = \alpha_3 = 0$. In addition, we obtain the model proposed by Yang et al. in [6] when $d_S = d_I = d_R$ and $\alpha_1 = \alpha_3 = 0$.

Observing that the first two equations in system (1) do not depend on the function R , and so the third equation can be omitted without loss of generality. System (1) can be rewritten as

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = d_S \Delta S + \Lambda - \mu S(x,t) - f(S(x,t), I(x,t))I(x,t), \\ \frac{\partial I(x,t)}{\partial t} = d_I \Delta I + f(S(x,t-\tau), I(x,t-\tau))I(x,t-\tau) - (\mu + d + r)I(x,t), \end{cases} \quad (2)$$

with homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, \text{ on } \partial\Omega \times (0, +\infty), \quad (3)$$

and initial conditions

$$S(x, \theta) = \phi_1(x, \theta), \quad I(x, \theta) = \phi_2(x, \theta), \quad x \in \bar{\Omega}, \quad \theta \in [-\tau, 0], \quad (4)$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on $\partial\Omega$. From biological point of view, the

Neumann boundary conditions mean that the individuals do not move across the boundary $\partial\Omega$.

This paper is structured as follows. In the next section, we establish the global stability of equilibria for problem (2)-(4). In section 3, we investigate the existence of traveling waves by using the technique of lower and upper solutions and applying the Schauder fixed point theorem. In section 4, the nonexistence of traveling waves is considered. The paper ends with a discussion and conclusion in section 5.

2 Global stability of steady states

Firstly, we discuss the existence of equilibria, which are the steady states of the following reaction system

$$\begin{cases} \frac{\partial S(t)}{\partial t} = \Lambda - \mu S(t) - f(S(t), I(t))I(t), \\ \frac{\partial I(t)}{\partial t} = f(S(t-\tau), I(t-\tau))I(t-\tau) - (\mu + d + r)I(t), \end{cases} \quad (5)$$

which is a special case of the model proposed by Hattaf et al. [14]. Based on the results given in [14], we deduce that the basic reproduction number of (5) as follows

$$R_0 = \frac{\beta\Lambda}{(\mu + \alpha_1\Lambda)(\mu + d + r)}.$$

In addition, if $R_0 \leq 1$, the system (5) has a unique disease-free equilibrium $E_f(\frac{\Lambda}{\mu}, 0)$ corresponding to the extinction of disease. When $R_0 > 1$, the disease-free equilibrium is still present and the system (5) has a unique endemic equilibrium of the form $E^*(S^*, I^*)$ with $S^* = \frac{2(a + \alpha_1\Lambda)}{\beta - \alpha_1a + \alpha_2\mu - \alpha_3\Lambda + \sqrt{\delta}}$,

$$I^* = \frac{\Lambda - \mu S^*}{a}, \quad a = \mu + d + r \text{ and } \delta = (\beta - \alpha_1a + \alpha_2\mu - \alpha_3\Lambda)^2 + 4\alpha_3\mu(a + \alpha_1\Lambda).$$

Secondly, we establish the global stability of steady states by constructing suitable Lyapunov functionals for (2)-(4). These Lyapunov functionals are obtained from the corresponding system (5) by applying the method of Hattaf and Yousfi presented in [15]. Therefore, we construct the Lyapunov functional for system (2)-(4) at E_f as follows

$$W_1 = \int_{\Omega} V_1(S(x, t), I(x, t))dx, \quad (6)$$

where

$$V_1(t) = S(t) - S_0 - \int_{S_0}^{S(t)} \frac{f(S_0, 0)}{f(X, 0)}dX + I(t) + \int_{t-\tau}^t f(S(\theta), I(\theta))I(\theta)d\theta, \quad (7)$$

when $S_0 = \frac{\Lambda}{\mu}$. Calculating the time derivative of W_1 along the solution of system (2)-(4), we have

$$\begin{aligned} \frac{dW_1}{dt} &= \int_{\Omega} \left\{ \mu S_0 \left(1 - \frac{S}{S_0}\right) \left(1 - \frac{f(S_0, 0)}{f(S, 0)}\right) + aI \left(\frac{f(S, I)}{f(S, 0)} R_0 - 1 \right) \right\} dx \\ &\quad - \frac{d_S S_0}{1 + \alpha_1 S_0} \int_{\Omega} \frac{|\nabla S|^2}{S^2} dx \\ &\leq \int_{\Omega} \left\{ \mu S_0 \left(1 - \frac{S}{S_0}\right) \left(1 - \frac{f(S_0, 0)}{f(S, 0)}\right) + aI (R_0 - 1) \right\} dx - \frac{d_S S_0}{1 + \alpha_1 S_0} \int_{\Omega} \frac{|\nabla S|^2}{S^2} dx \\ &= \int_{\Omega} \left\{ -\mu \frac{(S - S_0)^2}{S(1 + \alpha_1 S_0)} + aI (R_0 - 1) \right\} dx - \frac{d_S S_0}{1 + \alpha_1 S_0} \int_{\Omega} \frac{|\nabla S|^2}{S^2} dx. \end{aligned}$$

Since $R_0 \leq 1$, we have $\frac{dW_1}{dt} \leq 0$. Thus, the disease-free equilibrium E_f is stable, and $\frac{dW_1}{dt} = 0$ if and only if $S = S_0$ and $I(R_0 - 1) = 0$. We discuss two cases:

- If $R_0 < 1$, then $I = 0$.
- If $R_0 = 1$. From $S = S_0$ and the first equation of (2), we have

$$\frac{\partial S}{\partial t} = \frac{\partial S_0}{\partial t} = \Lambda - \mu S_0 - f(S_0, I)I = 0.$$

Then, $f(S_0, I)I = 0$. Since $\beta > 0$ and $S_0 > 0$, then $I = 0$.

From the above discussion, we deduce that the largest compact invariant set in $\{(S, I) | \frac{dW_1}{dt} = 0\}$ is just the singleton E_f . From LaSalle invariance principle [16], we conclude that E_f is globally asymptotically stable.

Similarly, we construct a Lyapunov functional W_2 for system (2)-(4) at E^* from the Lyapunov functional V_2 defined by Hattaf et al. in [14] in the case when $R_0 > 1$. Consequently, we obtain the following result.

Theorem 2.1.

- (i) If $R_0 \leq 1$, the disease-free equilibrium E_f of system (2)-(4) is globally asymptotically stable for all diffusion coefficients.
- (ii) If $R_0 > 1$, the endemic equilibrium E^* of system (2)-(4) is globally asymptotically stable for all diffusion coefficients.

Now, we consider system (1) with homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = \frac{\partial R}{\partial \nu} = 0, \text{ on } \partial\Omega \times (0, +\infty), \quad (8)$$

and initial conditions

$$S(x, \theta) = \phi_1(x, \theta), \quad I(x, \theta) = \phi_2(x, \theta), \quad R(x, \theta) = \phi_3(x, \theta), \quad x \in \bar{\Omega}, \quad \theta \in [-\tau, 0]. \quad (9)$$

From the above theorem, we get the following corollary.

Corollary 2.1.

- (i) If $R_0 \leq 1$, the disease-free equilibrium $Q_f(\Lambda/\mu, 0, 0)$ of system (1) with (8)-(9) is globally asymptotically stable for all diffusion coefficients.
- (ii) If $R_0 > 1$, the endemic equilibrium $Q^*(S^*, I^*, R^*)$ with $R^* = \frac{rI^*}{\mu}$ of system (1) with (8)-(9) is globally asymptotically stable for all diffusion coefficients.

3 Existence of traveling waves

In this section, we investigate the existence of traveling wave solution connecting the disease-free equilibrium and the endemic equilibrium. For this, we consider the following model

$$\begin{cases} \frac{\partial S(x, t)}{\partial t} = d_S \frac{\partial^2 S(x, t)}{\partial x^2} + \Lambda - \mu S(x, t) - f(S(x, t), I(x, t))I(x, t), \\ \frac{\partial I(x, t)}{\partial t} = d_I \frac{\partial^2 I(x, t)}{\partial x^2} + f(S(x, t - \tau), I(x, t - \tau))I(x, t - \tau) - (\mu + d + r)I(x, t), \end{cases} \quad (10)$$

Let $\tilde{S}(x, t) = \frac{\mu}{\Lambda} S(x\sqrt{d_I}, t)$, $\tilde{I}(x, t) = \frac{\mu}{\Lambda} I(x\sqrt{d_I}, t + \tau)$, $D = \frac{d_S}{d_I}$, $\tilde{\beta} = \frac{\beta\Lambda}{\mu}$, $\tilde{\alpha}_1 = \frac{\alpha_1\Lambda}{\mu}$, $\tilde{\alpha}_2 = \frac{\alpha_2\Lambda}{\mu}$ and $\tilde{\alpha}_3 = \frac{\alpha_3\Lambda^2}{\mu^2}$. Hence, we rewrite the system (10) in the following form

$$\begin{cases} \frac{\partial \tilde{S}(x, t)}{\partial t} = D \frac{\partial^2 \tilde{S}(x, t)}{\partial x^2} + \mu - \mu \tilde{S}(x, t) - \tilde{f}(\tilde{S}(x, t), \tilde{I}(x, t - \tau))\tilde{I}(x, t - \tau), \\ \frac{\partial \tilde{I}(x, t)}{\partial t} = \frac{\partial^2 \tilde{I}(x, t)}{\partial x^2} + \tilde{f}(\tilde{S}(x, t), \tilde{I}(x, t - \tau))\tilde{I}(x, t - \tau) - (\mu + d + r)\tilde{I}(x, t), \end{cases} \quad (11)$$

where $\tilde{f}(\tilde{S}, \tilde{I}) = \frac{\tilde{\beta}\tilde{S}}{1 + \tilde{\alpha}_1\tilde{S} + \tilde{\alpha}_2\tilde{I} + \tilde{\alpha}_3\tilde{S}\tilde{I}}$. The system (11) has always a disease-free equilibrium $\tilde{E}_f(1, 0)$. Besides, if $R_0 > 1$, the system (11) has also a unique endemic equilibrium $\tilde{E}^*(\frac{\mu}{\Lambda}S^*, \frac{\mu}{\Lambda}I^*)$.

The traveling wave solution of (11) connecting $\tilde{E}_f$ and $\tilde{E}^*$ is a special solution of the form

$$(S(x, t), I(x, t)) = (\phi(x + ct), \psi(x + ct)), \quad \xi = x + ct,$$

with the boundary conditions $(\phi, \psi)(-\infty) = (1, 0)$ and $(\phi, \psi)(+\infty) = (\frac{\mu}{\Lambda}S^*, \frac{\mu}{\Lambda}I^*)$. Here $\phi, \psi \in C^2(\mathbb{R}, \mathbb{R})$ and $c > 0$ is a constant corresponding to the wave speed. On substituting $\phi(x + ct)$ and $\psi(x + ct)$ into (11) and denoting $x + ct$ by ξ , we obtain the following wave equation

$$\begin{aligned} c\phi'(\xi) &= D\phi''(\xi) + \mu - \mu\phi(\xi) - \tilde{f}(\phi(\xi), \psi(\xi - c\tau))\psi(\xi - c\tau), \\ c\psi'(\xi) &= \psi''(\xi) + \tilde{f}(\phi(\xi), \psi(\xi - c\tau))\psi(\xi - c\tau) - (\mu + d + r)\psi(\xi), \end{aligned} \quad (12)$$

for all $\xi \in \mathbb{R}$, together with the boundary conditions

$$(\phi, \psi)(-\infty) = (1, 0) \text{ and } (\phi, \psi)(+\infty) = \left(\frac{\mu}{\Lambda} S^*, \frac{\mu}{\Lambda} I^*\right). \quad (13)$$

Linearizing the second equation of (12) at $\tilde{E}_f$, we get the following characteristic equation

$$\Delta(\lambda, c) = \lambda^2 - c\lambda + \frac{\tilde{\beta}}{1+\tilde{\alpha}_1} e^{-\lambda c\tau} - (\mu + d + r).$$

It is not hard to prove the following result.

Lemma 3.1. *Assume that $R_0 > 1$. Then there exist $c^* > 0$ and $\lambda^* > 0$ such that $\Delta(\lambda^*, c^*) = 0$ and $\frac{\partial \Delta(\lambda^*, c^*)}{\partial \lambda} = 0$. Moreover, we have*

- (i) *If $0 < c < c^*$, then $\Delta(\lambda, c) > 0$, for all $\lambda \geq 0$;*
- (ii) *If $c > c^*$, then $\Delta(\lambda, c) = 0$ has two positive real roots $\lambda_1(c)$ and $\lambda_2(c)$ such that $0 < \lambda_1(c) < \lambda^* < \lambda_2(c)$ and*

$$\Delta(\lambda, c) \begin{cases} > 0, & \lambda \in [0, \lambda_1(c)) \cup (\lambda_2(c), \infty), \\ < 0, & \lambda \in (\lambda_1(c), \lambda_2(c)). \end{cases}$$

Throughout this section, we always assume that $R_0 > 1$ and $c > c^*$. In addition, we always denote $\lambda_i(c)$ by λ_i , $i = 1, 2$.

Now, we define two continuous functions as follows

$$\begin{aligned} \bar{\phi}(\xi) &= 1, \\ \bar{\psi}(\xi) &= \min \left\{ e^{\lambda_1 \xi}, \frac{1 + \tilde{\alpha}_1}{\tilde{\alpha}_2 + \tilde{\alpha}_3} (R_0 - 1) \right\}. \end{aligned}$$

Lemma 3.2. *The function $(\bar{\phi}, \bar{\psi})$ is an upper solution of system (12).*

Proof. Clearly, the function $\bar{\phi}(\xi)$ satisfies the following inequality

$$c\bar{\phi}'(\xi) \geq D\bar{\phi}''(\xi) + \mu - \mu\bar{\phi}(\xi) - \tilde{f}(\bar{\phi}(\xi), \bar{\psi}(\xi - c\tau))\bar{\psi}(\xi - c\tau). \quad (14)$$

It remains to prove that the function $\bar{\psi}(\xi)$ satisfies

$$c\bar{\psi}'(\xi) \geq \bar{\psi}''(\xi) + \tilde{f}(\bar{\phi}(\xi), \bar{\psi}(\xi - c\tau))\bar{\psi}(\xi - c\tau) - (\mu + d + r)\bar{\psi}(\xi), \quad (15)$$

for all $\xi \neq \xi_1 := \frac{1}{\lambda_1} \ln \left(\frac{1 + \tilde{\alpha}_1}{\tilde{\alpha}_2 + \tilde{\alpha}_3} (R_0 - 1) \right)$.

If $\xi < \xi_1$, then $\bar{\psi}(\xi) = e^{\lambda_1 \xi}$ and

$$\begin{aligned} & \bar{\psi}''(\xi) - c\bar{\psi}'(\xi) + \tilde{f}(\bar{\phi}(\xi), \bar{\psi}(\xi - c\tau))\bar{\psi}(\xi - c\tau) - (\mu + d + r)\bar{\psi}(\xi) \\ & \leq \lambda_1^2 e^{\lambda_1 \xi} - c\lambda_1 e^{\lambda_1 \xi} - (\mu + d + r)e^{\lambda_1 \xi} + \frac{\tilde{\beta}}{1+\tilde{\alpha}_1} e^{\lambda_1(\xi - c\tau)} \\ & = e^{\lambda_1 \xi} \Delta(\lambda_1, c). \end{aligned}$$

Since $\Delta(\lambda_1, c) = 0$, we deduce (15).

If $\xi > \xi_1$, then $\bar{\psi}(\xi) = \frac{1 + \tilde{\alpha}_1}{\tilde{\alpha}_2 + \tilde{\alpha}_3}(R_0 - 1)$ and

$$\tilde{f}(\bar{\phi}(\xi), \bar{\psi}(\xi - c\tau))\bar{\psi}(\xi - c\tau) - (\mu + d + r)\bar{\psi}(\xi) = 0.$$

Hence, the inequality (15) holds because $\bar{\psi}(\xi)$ is a constant when $\xi > \xi_1$. This completes the proof. ■

Lemma 3.3. *Let $\sigma \in (0, \lambda_1)$ be sufficiently small. Then the function $\underline{\phi}(\xi) = \max\{1 - \frac{1}{\sigma}e^{\sigma\xi}, 0\}$ satisfies the following inequality*

$$c\underline{\phi}'(\xi) \leq D\underline{\phi}''(\xi) + \mu - \mu\underline{\phi}(\xi) - \tilde{f}(\underline{\phi}(\xi), \bar{\psi}(\xi - c\tau))\bar{\psi}(\xi - c\tau), \quad (16)$$

for all $\xi \neq \xi_2 := \frac{1}{\sigma} \ln \sigma$.

Proof. If $\xi > \xi_2$, then $\underline{\phi}(\xi) = 0$ and the inequality (16) holds.

If $\xi < \xi_2$, then $\underline{\phi}(\xi) = 1 - \frac{1}{\sigma}e^{\sigma\xi}$ and

$$\begin{aligned} & D\underline{\phi}''(\xi) - c\underline{\phi}'(\xi) - \mu\underline{\phi}(\xi) + \mu - \tilde{f}(\underline{\phi}(\xi), \bar{\psi}(\xi - c\tau))\bar{\psi}(\xi - c\tau) \\ & \geq D\underline{\phi}''(\xi) - c\underline{\phi}'(\xi) - \mu\underline{\phi}(\xi) + \mu - \tilde{\beta}\underline{\phi}(\xi)\bar{\psi}(\xi - c\tau) \\ & \geq D\underline{\phi}''(\xi) - c\underline{\phi}'(\xi) - \mu\underline{\phi}(\xi) + \mu - \tilde{\beta}\underline{\phi}(\xi)e^{\lambda_1(\xi - c\tau)} \\ & = -D\sigma e^{\sigma\xi} + ce^{\sigma\xi} + \frac{\mu}{\sigma}e^{\sigma\xi} - \tilde{\beta}e^{\lambda_1(\xi - c\tau)}(1 - \frac{1}{\sigma}e^{\sigma\xi}) \\ & \geq -D\sigma e^{\sigma\xi} + ce^{\sigma\xi} + \frac{\mu}{\sigma}e^{\sigma\xi} - \tilde{\beta}e^{\lambda_1(\xi - c\tau)} \\ & \geq e^{\sigma\xi}(-D\sigma + c - \tilde{\beta}e^{-c\tau\lambda_1}e^{(\lambda_1 - \sigma)\xi}) \\ & \geq e^{\sigma\xi}(-D\sigma + c - \tilde{\beta}e^{-c\tau\lambda_1}\sigma^{\frac{\lambda_1 - \sigma}{\sigma}}). \end{aligned}$$

Since $\lim_{\sigma \rightarrow 0^+} \sigma^{\frac{\lambda_1 - \sigma}{\sigma}} = 0$, we deduce for $\sigma > 0$ sufficiently small that

$$-D\sigma + c - \tilde{\beta}e^{-c\tau\lambda_1}\sigma^{\frac{\lambda_1 - \sigma}{\sigma}} > 0.$$

Hence, the inequality (16) holds ■

Lemma 3.4. *Let $M > 1$ sufficiently large and $0 < \eta < \min\{\sigma, \lambda_1, \lambda_2 - \lambda_1\}$. Then the function $\underline{\psi}(\xi) = \max\{e^{\lambda_1\xi}(1 - Me^{\eta\xi}), 0\}$ satisfies the following inequality*

$$c\underline{\psi}'(\xi) \leq \underline{\psi}''(\xi) + \tilde{f}(\underline{\phi}(\xi), \underline{\psi}(\xi - c\tau))\underline{\psi}(\xi - c\tau) - (\mu + d + r)\underline{\psi}(\xi), \quad (17)$$

for all $\xi \neq \xi_3 := \frac{1}{\eta} \ln \frac{1}{M}$.

Proof. If $\xi > \xi_3$, then $\underline{\psi}(\xi) = 0$ and the inequality (17) holds.

If $\xi < \xi_3$, then $\underline{\psi}(\xi) = e^{\lambda_1 \xi} (1 - M e^{\eta \xi})$. We put $M_1 = e^{(1-\xi_2)\eta}$ and we take

$$M > \max\left\{\frac{(\frac{\tilde{\alpha}_2 + \tilde{\alpha}_3}{1 + \tilde{\alpha}_1} + 1/\sigma)\tilde{\beta}}{-\Delta(\lambda_1 + \eta, c)}, M_1, 1\right\}.$$

Then $\underline{\phi}(\xi) = 1 - \frac{1}{\sigma} e^{\sigma \xi}$ because $\xi_3 < \xi_2$.

On the other hand,

$$\begin{aligned} \tilde{f}(\underline{\phi}(\xi), \underline{\psi}(\xi - c\tau)) \underline{\psi}(\xi - c\tau) &= \frac{\tilde{\beta} \underline{\phi}(\xi) \underline{\phi}(\xi - c\tau)}{1 + \tilde{\alpha}_1 \underline{\phi}(\xi) + \tilde{\alpha}_2 \underline{\psi}(\xi - c\tau) + \tilde{\alpha}_3 \underline{\phi}(\xi) \underline{\psi}(\xi - c\tau)} \\ &= \frac{\tilde{\beta} \underline{\psi}(\xi - c\tau)}{1 + \tilde{\alpha}_1 \underline{\phi}(\xi) + \tilde{\alpha}_2 \underline{\psi}(\xi - c\tau) + \tilde{\alpha}_3 \underline{\phi}(\xi) \underline{\psi}(\xi - c\tau)} - \frac{\tilde{\beta} \frac{1}{\sigma} e^{\sigma \tau} \underline{\psi}(\xi - c\tau)}{1 + \tilde{\alpha}_1 \underline{\phi}(\xi) + \tilde{\alpha}_2 \underline{\psi}(\xi - c\tau) + \tilde{\alpha}_3 \underline{\phi}(\xi) \underline{\psi}(\xi - c\tau)} \\ &= \frac{\tilde{\beta} \underline{\psi}(\xi - c\tau)}{1 + \tilde{\alpha}_1 \underline{\phi}(\xi) + \tilde{\alpha}_2 \underline{\psi}(\xi - c\tau) + \tilde{\alpha}_3 \underline{\phi}(\xi) \underline{\psi}(\xi - c\tau)} - \frac{\tilde{\beta} \underline{\psi}(\xi - c\tau)}{1 + \tilde{\alpha}_1} \\ &\quad + \frac{\tilde{\beta} \underline{\psi}(\xi - c\tau)}{1 + \tilde{\alpha}_1} - \frac{\tilde{\beta} \frac{1}{\sigma} e^{\sigma \tau} \underline{\psi}(\xi - c\tau)}{1 + \tilde{\alpha}_1 \underline{\phi}(\xi) + \tilde{\alpha}_2 \underline{\psi}(\xi - c\tau) + \tilde{\alpha}_3 \underline{\phi}(\xi) \underline{\psi}(\xi - c\tau)} \\ &\geq \frac{\tilde{\beta} \underline{\psi}(\xi - c\tau)}{1 + \tilde{\alpha}_1} + \frac{1}{1 + \tilde{\alpha}_1 \underline{\phi}(\xi) + \tilde{\alpha}_2 \underline{\psi}(\xi - c\tau) + \tilde{\alpha}_3 \underline{\phi}(\xi) \underline{\psi}(\xi - c\tau)} \left[-\frac{(\tilde{\alpha}_2 + \tilde{\alpha}_3) \tilde{\beta} \underline{\psi}(\xi - c\tau)}{1 + \tilde{\alpha}_1} - \tilde{\beta} \frac{1}{\sigma} e^{\sigma \tau} \underline{\psi}(\xi - c\tau) \right] \\ &\geq \frac{\tilde{\beta} \underline{\psi}(\xi - c\tau)}{1 + \tilde{\alpha}_1} - \frac{\tilde{\beta}(\tilde{\alpha}_2 + \tilde{\alpha}_3)}{1 + \tilde{\alpha}_1} \underline{\psi}^2(\xi - c\tau) - \tilde{\beta} \frac{1}{\sigma} e^{(\sigma + \lambda_1)\xi} \\ &\geq \frac{\tilde{\beta} \underline{\psi}(\xi - c\tau)}{1 + \tilde{\alpha}_1} - \frac{\tilde{\beta}(\tilde{\alpha}_2 + \tilde{\alpha}_3)}{1 + \tilde{\alpha}_1} e^{2\lambda_1 \xi} - \tilde{\beta} \frac{1}{\sigma} e^{(\sigma + \lambda_1)\xi}. \end{aligned}$$

Therefore

$$\begin{aligned} &\underline{\psi}''(\xi) - c \underline{\psi}'(\xi) + \tilde{f}(\underline{\phi}(\xi), \underline{\psi}(\xi - c\tau)) \underline{\psi}(\xi - c\tau) - (\mu + d + r) \underline{\psi}(\xi) \\ &\geq \underline{\psi}''(\xi) - c \underline{\psi}'(\xi) + \frac{\tilde{\beta} \underline{\psi}(\xi - c\tau)}{1 + \tilde{\alpha}_1} - \frac{\tilde{\beta}(\tilde{\alpha}_2 + \tilde{\alpha}_3)}{1 + \tilde{\alpha}_1} e^{2\lambda_1 \xi} - \tilde{\beta} \frac{1}{\sigma} e^{(\sigma + \lambda_1)\xi} - (\mu + d + r) \underline{\psi}(\xi) \\ &= -\Delta(\lambda_1 + \eta, c) M e^{(\lambda_1 + \eta)\xi} - \frac{\tilde{\beta}(\tilde{\alpha}_2 + \tilde{\alpha}_3)}{1 + \tilde{\alpha}_1} e^{2\lambda_1 \xi} - \tilde{\beta} \frac{1}{\sigma} e^{(\sigma + \lambda_1)\xi} \\ &\geq \left(-\Delta(\lambda_1 + \eta, c) M - \frac{\tilde{\beta}(\tilde{\alpha}_2 + \tilde{\alpha}_3)}{1 + \tilde{\alpha}_1} - \tilde{\beta} \frac{1}{\sigma} \right) e^{(\lambda_1 + \eta)\xi}. \end{aligned}$$

Since $M > \frac{(\frac{\tilde{\alpha}_2 + \tilde{\alpha}_3}{1 + \tilde{\alpha}_1} + 1/\sigma)\tilde{\beta}}{-\Delta(\lambda_1 + \eta, c)}$, we have $-\Delta(\lambda_1 + \eta, c) M - \frac{\tilde{\beta}(\tilde{\alpha}_2 + \tilde{\alpha}_3)}{1 + \tilde{\alpha}_1} - \tilde{\beta} \frac{1}{\sigma} > 0$.

Thus, the inequality (17) holds. ■

Now, we will apply the Schauder fixed point theorem to show the existence of solutions of (12).

We choose $\rho > \max\{\mu + d + r, \mu + \tilde{\beta}/2\}$ and we denote

$$\begin{aligned} H_1(\phi, \psi)(\xi) &= \rho\phi(\xi) + \mu - \mu\phi(\xi) - \tilde{f}(\phi(\xi), \psi(\xi - c\tau))\psi(\xi - c\tau), \\ H_2(\phi, \psi)(\xi) &= \rho\psi(\xi) + \tilde{f}(\phi(\xi), \psi(\xi - c\tau))\psi(\xi - c\tau) - (\mu + d + r)\psi(\xi). \end{aligned}$$

By the choice of ρ , it is not hard to show that H_1 is monotone increasing in ϕ and monotone decreasing in ψ , and H_2 is monotone increasing in ϕ and ψ .

We define the set as follows

$$\Gamma = \left\{ (\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2) / (\underline{\phi}, \underline{\psi}) \leq (\phi, \psi) \leq (\overline{\phi}, \overline{\psi}) \right\}.$$

Then Γ is nonempty, closed and convex in $C(\mathbb{R}, \mathbb{R}^2)$.

Next, we construct an operator $F : \Gamma \rightarrow C(\mathbb{R}, \mathbb{R}^2)$ defined by

$$F(\phi, \psi)(\xi) = (F_1(\phi, \psi), F_2(\phi, \psi))(\xi),$$

where

$$\begin{aligned} F_1(\phi, \psi)(\xi) &= \frac{1}{D(\lambda_{12} - \lambda_{11})} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} \right\} H_1(\phi, \psi)(x) dx, \\ F_2(\phi, \psi)(\xi) &= \frac{1}{(\lambda_{22} - \lambda_{21})} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{21}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{22}(\xi-x)} \right\} H_2(\phi, \psi)(x) dx, \end{aligned}$$

with

$$\begin{aligned} \lambda_{11} &= \frac{c - \sqrt{c^2 + 4D\rho}}{2D}, \quad \lambda_{12} = \frac{c + \sqrt{c^2 + 4D\rho}}{2D}, \\ \lambda_{21} &= \frac{c - \sqrt{c^2 + 4\rho}}{2}, \quad \lambda_{22} = \frac{c + \sqrt{c^2 + 4\rho}}{2}. \end{aligned}$$

Lemma 3.5. *The operator F maps Γ into Γ .*

Proof. From the monotonicity of H_1 with respect to ϕ and ψ , we get for all $\xi \in \mathbb{R}$,

$$F_1(\underline{\phi}, \overline{\psi})(\xi) \leq F_1(\phi, \psi)(\xi) \leq F_1(\overline{\phi}, \underline{\psi})(\xi). \quad (18)$$

From (16), we obtain

$$\begin{aligned} F_1(\underline{\phi}, \overline{\psi})(\xi) &\geq \frac{1}{D(\lambda_{12} - \lambda_{11})} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} \right\} \\ &\quad \times [\rho\underline{\phi}(x) + c\underline{\phi}'(x) - D\underline{\phi}''(x)] dx, \end{aligned}$$

for all $\xi \neq \xi_2$.

If $\xi > \xi_2$, we have

$$\begin{aligned}
 F_1(\underline{\phi}, \overline{\psi})(\xi) &\geq \frac{1}{D(\lambda_{12} - \lambda_{11})} \int_{-\infty}^{\xi_2} e^{\lambda_{11}(\xi-x)} [\rho \underline{\phi}(x) + c \underline{\phi}'(x) - D \underline{\phi}''(x)] dx \\
 &+ \frac{1}{D(\lambda_{12} - \lambda_{11})} \int_{\xi_2}^{\xi} e^{\lambda_{11}(\xi-x)} [\rho \underline{\phi}(x) + c \underline{\phi}'(x) - D \underline{\phi}''(x)] dx \\
 &+ \frac{1}{D(\lambda_{12} - \lambda_{11})} \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} [\rho \underline{\phi}(x) + c \underline{\phi}'(x) - D \underline{\phi}''(x)] dx \\
 &= \underline{\phi}(\xi) - \frac{e^{\lambda_{11}(\xi-\xi_2)} \underline{\phi}'(\xi_2-)}{\lambda_{12} - \lambda_{11}} \\
 &\geq \underline{\phi}(\xi).
 \end{aligned}$$

Similarly, we show that $F_1(\underline{\phi}, \overline{\psi})(\xi) \geq \underline{\phi}(\xi)$ when $\xi < \xi_2$. Hence, from the continuity of $F_1(\underline{\phi}, \overline{\psi})(\xi)$ and $\underline{\phi}(\xi)$, we get

$$F_1(\underline{\phi}, \overline{\psi})(\xi) \geq \underline{\phi}(\xi), \quad \forall \xi \in \mathbb{R}. \quad (19)$$

Since $H_1(\overline{\phi}, \underline{\psi}) \leq \rho \overline{\phi}$, we have

$$F_1(\overline{\phi}, \underline{\psi})(\xi) \leq 1, \quad \forall \xi \in \mathbb{R}. \quad (20)$$

From (18)-(20), we get

$$\underline{\phi}(\xi) \leq F_1(\underline{\phi}, \underline{\psi})(\xi) \leq \overline{\phi}(\xi), \quad \forall \xi \in \mathbb{R}. \quad (21)$$

From the monotonicity of H_2 with respect to ϕ and ψ , we get for all $\xi \in \mathbb{R}$,

$$F_2(\underline{\phi}, \underline{\psi})(\xi) \leq F_2(\phi, \psi)(\xi) \leq F_2(\overline{\phi}, \overline{\psi})(\xi). \quad (22)$$

By (15), we obtain

$$\begin{aligned}
 F_2(\overline{\phi}, \overline{\psi})(\xi) &\leq \frac{1}{(\lambda_{22} - \lambda_{21})} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{21}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{22}(\xi-x)} \right\} \\
 &\times [\rho \overline{\psi}(x) + c \overline{\psi}'(x) - \overline{\psi}''(x)] dx,
 \end{aligned}$$

for all $\xi \neq \xi_1$.

If $\xi < \xi_1$, we have

$$\begin{aligned}
 F_2(\overline{\phi}, \overline{\psi})(\xi) &\leq \frac{1}{(\lambda_{22} - \lambda_{21})} \int_{-\infty}^{\xi} e^{\lambda_{21}(\xi-x)} [\rho \overline{\psi}(x) + c \overline{\psi}'(x) - \overline{\psi}''(x)] dx \\
 &+ \frac{1}{(\lambda_{22} - \lambda_{21})} \int_{\xi}^{\xi_1} e^{\lambda_{22}(\xi-x)} [\rho \overline{\psi}(x) + c \overline{\psi}'(x) - \overline{\psi}''(x)] dx \\
 &+ \frac{1}{(\lambda_{22} - \lambda_{21})} \int_{\xi_1}^{+\infty} e^{\lambda_{22}(\xi-x)} [\rho \overline{\psi}(x) + c \overline{\psi}'(x) - \overline{\psi}''(x)] dx, \\
 &\leq \overline{\psi}(\xi).
 \end{aligned}$$

If $\xi > \xi_1$, we have

$$\begin{aligned} F_2(\bar{\phi}, \bar{\psi})(\xi) &\leq \frac{1}{(\lambda_{22} - \lambda_{21})} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{21}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{22}(\xi-x)} \right\} (\rho \bar{\psi}(x)) dx \\ &= \bar{\psi}(\xi). \end{aligned}$$

Hence, from the continuity of $F_2(\bar{\phi}, \bar{\psi})(\xi)$ and $\bar{\phi}(\xi)$, we get

$$F_2(\bar{\phi}, \bar{\psi})(\xi) \leq \bar{\psi}(\xi), \quad \forall \xi \in \mathbb{R}. \quad (23)$$

From (17), we obtain

$$\begin{aligned} F_2(\underline{\phi}, \underline{\psi})(\xi) &\geq \frac{1}{(\lambda_{22} - \lambda_{21})} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{21}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{22}(\xi-x)} \right\} \\ &\quad \times [\rho \underline{\psi}(x) + c \underline{\psi}'(x) - \underline{\psi}''(x)] dx, \end{aligned}$$

for all $\xi \neq \xi_3$.

Similarly, we prove that

$$F_2(\underline{\phi}, \underline{\psi})(\xi) \geq \underline{\psi}(\xi), \quad \forall \xi \in \mathbb{R}. \quad (24)$$

From (22)-(24), we obtain

$$\underline{\psi}(\xi) \leq F_2(\phi, \psi)(\xi) \leq \bar{\psi}(\xi), \quad \forall \xi \in \mathbb{R}. \quad (25)$$

By (21) and (25), we conclude that F maps Γ to Γ . ■

Let $u > 0$ is a constant satisfying $u < \min\{-\lambda_{11}, -\lambda_{21}\}$ and define

$$B_u(\mathbb{R}, \mathbb{R}^2) = \left\{ (\phi, \psi) \in C : \sup_{\xi \in \mathbb{R}} \{|\phi(\xi)| e^{-u|\xi|}\} < \infty, \sup_{\xi \in \mathbb{R}} \{|\psi(\xi)| e^{-u|\xi|}\} < \infty \right\}$$

with norm

$$|(\phi, \psi)|_u = \max \left\{ \sup_{\xi \in \mathbb{R}} |\phi(\xi)| e^{-u|\xi|}, \sup_{\xi \in \mathbb{R}} |\psi(\xi)| e^{-u|\xi|} \right\}.$$

Obviously, $B_u(\mathbb{R}, \mathbb{R}^2)$ is a Banach space with the norm $|\cdot|_u$.

Lemma 3.6. *The operator F is continuous with respect to the norm $|\cdot|_u$ in $B_u(\mathbb{R}, \mathbb{R}^2)$.*

Proof. Let $\Theta_1 = (\phi_1, \psi_1) \in \Gamma$ and $\Theta_2 = (\phi_2, \psi_2) \in \Gamma$. We have

$$\begin{aligned}
 & \left| \tilde{f}(\phi_1(\xi), \psi_1(\xi - c\tau))\psi_1(\xi - c\tau) - \tilde{f}(\phi_2(\xi), \psi_2(\xi - c\tau))\psi_2(\xi - c\tau) \right| \\
 & \leq \left| \frac{\tilde{\beta}\phi_1(\xi)(\psi_1(\xi - c\tau) - \psi_2(\xi - c\tau))}{(1 + \tilde{\alpha}_1\phi_1(\xi) + \tilde{\alpha}_2\psi_1(\xi - c\tau) + \tilde{\alpha}_3\phi_1(\xi)\psi_1(\xi - c\tau))(1 + \tilde{\alpha}_1\phi_2(\xi) + \tilde{\alpha}_2\psi_2(\xi - c\tau) + \tilde{\alpha}_3\phi_2(\xi)\psi_2(\xi - c\tau))} \right. \\
 & \quad + \frac{\tilde{\beta}\psi_2(\xi - c\tau)(\phi_1(\xi) - \phi_2(\xi))}{(1 + \tilde{\alpha}_1\phi_1(\xi) + \tilde{\alpha}_2\psi_1(\xi - c\tau) + \tilde{\alpha}_3\phi_1(\xi)\psi_1(\xi - c\tau))(1 + \tilde{\alpha}_1\phi_2(\xi) + \tilde{\alpha}_2\psi_2(\xi - c\tau) + \tilde{\alpha}_3\phi_2(\xi)\psi_2(\xi - c\tau))} \\
 & \quad + \frac{\tilde{\beta}\tilde{\alpha}_2\psi_1(\xi - c\tau)\psi_2(\xi - c\tau)(\phi_1(\xi) - \phi_2(\xi))}{(1 + \tilde{\alpha}_1\phi_1(\xi) + \tilde{\alpha}_2\psi_1(\xi - c\tau) + \tilde{\alpha}_3\phi_1(\xi)\psi_1(\xi - c\tau))(1 + \tilde{\alpha}_1\phi_2(\xi) + \tilde{\alpha}_2\psi_2(\xi - c\tau) + \tilde{\alpha}_3\phi_2(\xi)\psi_2(\xi - c\tau))} \\
 & \quad \left. + \frac{\tilde{\beta}\tilde{\alpha}_1\phi_1(\xi)\phi_2(\xi)(\psi_1(\xi - c\tau) - \psi_2(\xi - c\tau))}{(1 + \tilde{\alpha}_1\phi_1(\xi) + \tilde{\alpha}_2\psi_1(\xi - c\tau) + \tilde{\alpha}_3\phi_1(\xi)\psi_1(\xi - c\tau))(1 + \tilde{\alpha}_1\phi_2(\xi) + \tilde{\alpha}_2\psi_2(\xi - c\tau) + \tilde{\alpha}_3\phi_2(\xi)\psi_2(\xi - c\tau))} \right| \\
 & \leq \frac{\tilde{\beta}}{\tilde{\alpha}_2} |\phi_1(\xi) - \phi_2(\xi)| + \tilde{\beta}(1 + \tilde{\alpha}_1) |\psi_1(\xi - c\tau) - \psi_2(\xi - c\tau)|.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 & |F_1(\phi_1, \psi_1)(\xi) - F_1(\phi_2, \psi_2)(\xi)| \\
 & \leq \frac{\rho - \mu}{D(\lambda_{12} - \lambda_{11})} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} \right\} |\phi_1(x) - \phi_2(x)| dx \\
 & \quad + \frac{\tilde{\beta}}{D\tilde{\alpha}_2(\lambda_{12} - \lambda_{11})} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} \right\} |\phi_1(x) - \phi_2(x)| dx \\
 & \quad + \frac{\tilde{\beta}(1 + \tilde{\alpha}_1)}{D(\lambda_{12} - \lambda_{11})} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} \right\} |\psi_1(x - c\tau) - \psi_2(x - c\tau)| dx.
 \end{aligned}$$

We have

$$\begin{aligned}
 \left\{ \int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} \right\} e^{-u|\xi|} e^{u|x|} dx & \leq \left\{ \int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} \right\} e^{u|\xi-x|} dx \\
 & \leq \frac{1}{\lambda_{12} - u} - \frac{1}{\lambda_{11} + u}.
 \end{aligned}$$

Then

$$\begin{aligned}
 & |F_1(\phi_1, \psi_1)(\xi) - F_1(\phi_2, \psi_2)(\xi)| e^{-u|\xi|} \\
 & \leq \frac{\rho - \mu}{D(\lambda_{12} - \lambda_{11})} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} \right\} e^{-u|\xi|} e^{u|x|} |\phi_1(x) - \phi_2(x)| e^{-u|x|} dx \\
 & \quad + \frac{\tilde{\beta}}{D\tilde{\alpha}_2(\lambda_{12} - \lambda_{11})} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} \right\} e^{-u|\xi|} e^{u|x|} |\phi_1(x) - \phi_2(x)| e^{-u|x|} dx \\
 & \quad + \frac{\tilde{\beta}(1 + \tilde{\alpha}_1)}{D(\lambda_{12} - \lambda_{11})} \left\{ \int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} \right\} e^{-u|\xi|} e^{u|x-c\tau|} |\psi_1(x - c\tau) - \psi_2(x - c\tau)| e^{-u|x-c\tau|} dx.
 \end{aligned}$$

Thus,

$$|F_1 - F_2|_u \leq \frac{(\lambda_{11} - \lambda_{12} + 2u)(\rho - \mu + \frac{\tilde{\beta}}{\tilde{\alpha}_2} + \tilde{\beta}(1 + \tilde{\alpha}_1)e^{uc\tau})}{D(\lambda_{12} - \lambda_{11})(\lambda_{12} - u)(\lambda_{11} + u)} |\Theta_1 - \Theta_2|_u.$$

Therefore, $F_1 : \Gamma \rightarrow \Gamma$ is continuous with respect to the norm $|\cdot|_u$. Using the same technique, we show that $F_2 : \Gamma \rightarrow \Gamma$ is continuous with respect to the norm $|\cdot|_u$. Consequently, F is continuous with respect to the norm $|\cdot|_u$ in $B_u(\mathbb{R}, \mathbb{R}^2)$. ■

Lemma 3.7. *The operator F is compact with respect to the norm $|\cdot|_u$ in $B_u(\mathbb{R}, \mathbb{R}^2)$.*

Proof. Let $(\phi, \psi) \in \Gamma$. We have

$$\begin{aligned} \left| \frac{d}{d\xi} F_1(\phi, \psi)(\xi) \right| &\leq \frac{(\rho + \mu)|\lambda_{11}|}{D(\lambda_{12} - \lambda_{11})} \int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} dx + \frac{(\rho + \mu)\lambda_{12}}{D(\lambda_{12} - \lambda_{11})} \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} dx \\ &= \frac{2(\rho + \mu)}{D(\lambda_{12} - \lambda_{11})}. \end{aligned}$$

Since $H_2(\phi, \psi)(\xi) \leq (\rho + \tilde{\beta}) \frac{1 + \tilde{\alpha}_1}{\tilde{\alpha}_2 + \tilde{\alpha}_3} (R_0 - 1)$, we have

$$\begin{aligned} \left| \frac{d}{d\xi} F_2(\phi, \psi)(\xi) \right| &\leq \frac{(\rho + \tilde{\beta})(1 + \tilde{\alpha}_1)(R_0 - 1)|\lambda_{21}|}{(\lambda_{22} - \lambda_{21})(\tilde{\alpha}_2 + \tilde{\alpha}_3)} \int_{-\infty}^{\xi} e^{\lambda_{21}(\xi-x)} dx \\ &\quad + \frac{(\rho + \tilde{\beta})(1 + \tilde{\alpha}_1)(R_0 - 1)\lambda_{22}}{(\lambda_{22} - \lambda_{21})(\tilde{\alpha}_2 + \tilde{\alpha}_3)} \int_{\xi}^{+\infty} e^{\lambda_{22}(\xi-x)} dx \\ &= \frac{2(\rho + \tilde{\beta})(1 + \tilde{\alpha}_1)(R_0 - 1)}{(\lambda_{22} - \lambda_{21})(\tilde{\alpha}_2 + \tilde{\alpha}_3)}. \end{aligned}$$

For each $n \in N$, we define an operator F^n by

$$F^n(\phi, \psi)(\xi) = \begin{cases} F(\phi, \psi)(\xi), & \xi \in [-n, n], \\ F(\phi, \psi)(-n), & \xi \in (-\infty, -n), \\ F(\phi, \psi)(n), & \xi \in (n, +\infty). \end{cases}$$

Clearly, F^n is uniformly bounded and equicontinuous for $(\phi, \psi) \in \Gamma$. From Arzela-Ascoli theorem, we deduce that $F^n : \Gamma \rightarrow \Gamma$ is compact with respect to the super norm in $C(\mathbb{R}, \mathbb{R}^2)$. Thus, $F^n : \Gamma \rightarrow \Gamma$ is compact with respect to

the norm $|\cdot|_u$ in $B_u(\mathbb{R}, \mathbb{R}^2)$.

On the other hand, $\{F^n\}_0^{+\infty}$ is a compact series and

$$\begin{aligned} |F^n(\phi, \psi)(\xi) - F(\phi, \psi)(\xi)|_u &= \sup_{\xi \in \mathbb{R}} |F^n(\phi, \psi)(\xi) - F(\phi, \psi)(\xi)| e^{-u|\xi|} \\ &= \sup_{\xi \in (-\infty, -n] \cup [n, +\infty)} |F^n(\phi, \psi)(\xi) - F(\phi, \psi)(\xi)| e^{-u|\xi|} \\ &\leq \sup_{\xi \in (-\infty, -n] \cup [n, +\infty)} \left[1 + \frac{1}{\tilde{\alpha}_2 + \tilde{\alpha}_3} (R_0 - 1)\right] e^{-u|\xi|} \\ &\leq \left[1 + \frac{1 + \tilde{\alpha}_1}{\tilde{\alpha}_2 + \tilde{\alpha}_3} (R_0 - 1)\right] e^{-un} \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

Hence, $\{F^n\}_0^{+\infty}$ converge to F in Γ with respect to the norm $|\cdot|_u$. By Proposition 2.1 in [17], we conclude that $F : \Gamma \rightarrow \Gamma$ is also compact with respect to the norm $|\cdot|_u$. This completes the proof. ■

Theorem 3.1. *Assume that $R_0 > 1$. For every $c > c^*$, system (11) admits a traveling wave solution $(\phi(x + ct), \psi(x + ct))$ with speed c connecting the disease-free equilibrium $\tilde{E}_f(1, 0)$ and the endemic equilibrium $\tilde{E}^*(\frac{\mu}{\Lambda} S^*, \frac{\mu}{\Lambda} I^*)$. Moreover, we have $\lim_{\xi \rightarrow -\infty} \psi(\xi) e^{-\lambda_1 \xi} = 1$.*

Proof. From Lemmas 3.6 and 3.7, we deduce that the operator F satisfies the conditions of Schauder's fixed point theorem. Hence F admits a fixed point $(\phi, \psi) \in \Gamma$. It is easy to verify that this fixed point is a solution of (12). It remains to show that the fixed point (ϕ, ψ) satisfies the boundary conditions (13).

We have $(\phi, \psi) \in \Gamma$, then $1 - \frac{1}{\sigma} e^{\sigma \xi} \leq \phi(\xi) \leq 1$ and $0 \leq \psi \leq e^{\lambda_1 \xi}$. So, $\phi(-\infty) = 1$, $\psi(-\infty) = 0$.

Now, we will show that $(\phi, \psi)(+\infty) = (\frac{\mu}{\Lambda} S^*, \frac{\mu}{\Lambda} I^*)$.

Clearly, $(S(x, t), I(x, t)) = (\phi(x + ct), \psi(x + ct))$ is solution of (11).

From Theorem 2.1 (ii), we deduce that the system (11) is globally asymptotically stable when $R_0 > 1$. Hence $\phi(+\infty) = \frac{\mu}{\Lambda} S^*$ and $\psi(+\infty) = \frac{\mu}{\Lambda} I^*$.

We have $\psi \in \Gamma$, then $1 - M e^{\eta \xi} \leq \psi(\xi) e^{-\lambda_1 \xi} \leq 1$. Hence, $\lim_{\xi \rightarrow -\infty} \psi(\xi) e^{-\lambda_1 \xi} = 1$.

This completes the proof. ■

From the above theorem, we get the following result.

Corollary 3.8. *Assume that $R_0 > 1$. For every $c > c^*$, system (1) admits a traveling wave solution connecting the disease-free equilibrium $Q_f(\Lambda/\mu, 0, 0)$ and the endemic equilibrium $Q^*(S^*, I^*, R^*)$.*

Proof. It remains to show that $R(-\infty) = 0$ and $R(+\infty) = R^*$.
The wave equation of $R(\xi)$ is given by

$$cR'(\xi) = d_R R''(\xi) + rI(\xi) - \mu R(\xi).$$

Let $\eta_1 < 0 < \eta_2$ such that $d_R \eta^2 - c\eta - \mu = 0$. Then

$$R(\xi) = \frac{r}{d_R(\eta_2 - \eta_1)} \left(\int_{-\infty}^{\xi} e^{\eta_1(\xi-x)} I(x) dx + \int_{\xi}^{+\infty} e^{\eta_2(\xi-x)} I(x) dx \right).$$

From L'Hopital rule, we obtain

$$\begin{aligned} R(+\infty) &= \frac{r}{d_R(\eta_2 - \eta_1)} \lim_{\xi \rightarrow +\infty} \left(\frac{\int_{-\infty}^{\xi} e^{-\eta_1 x} I(x) dx}{e^{-\eta_1 \xi}} + \frac{\int_{\xi}^{+\infty} e^{-\eta_2 x} I(x) dx}{e^{-\eta_2 \xi}} \right) \\ &= \frac{r}{d_R(\eta_2 - \eta_1)} \left(\frac{-I^*}{\eta_1} + \frac{I^*}{\eta_2} \right) = \frac{rI^*}{\mu} = R^*. \end{aligned}$$

Using the same technique, we get $R(-\infty) = 0$. ■

4 Nonexistence of traveling waves

In this section, we focus on the nonexistence of nontrivial traveling wave solutions of our model.

Theorem 4.1. *If $R_0 > 1$ and $c \in (0, c^*)$, or $R_0 < 1$, then system (12) has no nontrivial positive solution satisfying the boundary conditions (13).*

Proof. By contradiction, we assume that there exists a positive solution (ϕ, ψ) of problem (12)-(13).

First, we consider the case of $R_0 > 1$ and $c \in (0, c^*)$.

From Lemma 3.1, there exists a $\varepsilon > 0$ sufficiently small such that the equation

$$\lambda^2 - \bar{c}\lambda + \frac{\tilde{\beta}}{1 + \tilde{\alpha}_1} (1 - 2\varepsilon) e^{-\lambda \bar{c}\tau} - (\mu + d + r) = 0,$$

has no real solution for $\bar{c} \in (0, \frac{c+2c^*}{3})$.

We have $\phi(-\infty) = 1$, then there exists $M_\varepsilon > 0$ such that $1 - \varepsilon \leq \phi(\xi) \leq 1$, for any $\xi < -M_\varepsilon$.

Hence, for $\xi < -M_\varepsilon$, we have

$$\begin{aligned} c\psi'(\xi) &\geq \psi''(\xi) + \frac{\tilde{\beta}(1 - \varepsilon)\psi(\xi - c\tau)}{1 + \tilde{\alpha}_1 + (\tilde{\alpha}_2 + \tilde{\alpha}_3)\psi(\xi - c\tau)} - (\mu + d + r)\psi(\xi) \\ &\geq \psi''(\xi) + \frac{\tilde{\beta}(1 - \varepsilon)\psi(\xi - c\tau)}{[1 + \tilde{\alpha}_1 + (\tilde{\alpha}_2 + \tilde{\alpha}_3)\psi(\xi - c\tau)]^{h+1}} - (\mu + d + r)\psi(\xi), \quad \forall h > 0. \end{aligned}$$

Then,

$$c\psi'(\xi) \geq \psi''(\xi) + g(\psi(\xi - c\tau)) - (\mu + d + r)\psi(\xi),$$

where

$$g(u) = \inf_{v \in \left(u, \frac{1+\tilde{\alpha}_1}{\tilde{\alpha}_2+\tilde{\alpha}_3}(R_0-1)\right)} \frac{\tilde{\beta}(1-\epsilon)v}{[1 + \tilde{\alpha}_1 + (\tilde{\alpha}_2 + \tilde{\alpha}_3)v]^{h+1}}.$$

So, the positive function $u(x, t) = \psi(\xi + ct)$ satisfies

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} \geq \frac{\partial^2 u(x, t)}{\partial x^2} + g(u(x, t - \tau)) - (\mu + d + r)u(x, t), & x \in \mathbb{R}, t > 0, \\ u(x, s) = \psi(x + cs) > 0, & x \in \mathbb{R}, s \in [-\tau, 0]. \end{cases}$$

Hence, $u(x, t)$ is an upper solution of the following initial value problem

$$\begin{cases} \frac{\partial w(x, t)}{\partial t} = \frac{\partial^2 w(x, t)}{\partial x^2} + g(w(x, t - \tau)) - (\mu + d + r)w(x, t), & x \in \mathbb{R}, t > 0, \\ w(x, s) = \psi(x + cs) > 0, & x \in \mathbb{R}, s \in [-\tau, 0]. \end{cases}$$

From the theory of asymptotic spreading [18], we have

$$\liminf_{t \rightarrow +\infty} w(x, t) > 0, \quad |x| \leq \frac{c + c^*}{2}t.$$

It follows, from the comparison principal [19], that

$$\liminf_{t \rightarrow +\infty} u(x, t) \geq \liminf_{t \rightarrow +\infty} w(x, t) > 0, \quad |x| \leq \frac{c + c^*}{2}t. \quad (26)$$

For $-x = \frac{c+c^*}{2}t$, we have $x + ct \rightarrow -\infty$ as $t \rightarrow +\infty$. Then

$$\lim_{t \rightarrow +\infty} u(x, t) = 0, \quad -x = \frac{c + c^*}{2}t,$$

which contradicts (26). Therefore, system (12) has no nontrivial positive solution satisfying (13) when $R_0 > 1$ and $c \in (0, c^*)$.

Secondly, for any speed $c > 0$, we consider the case of $R_0 < 1$. The wave equation of $\psi(\xi)$ is given by

$$c\psi'(\xi) = \psi''(\xi) + \tilde{f}(\phi(\xi), \psi(\xi - c\tau))\psi(\xi - c\tau) - (\mu + d + r)\psi(\xi).$$

Let $\lambda'_{21} < 0 < \lambda'_{22}$ such that $\lambda^2 - c\lambda - (\mu + d + r) = 0$. Then, we have

$$\begin{aligned} \psi(\xi) &= \frac{1}{(\lambda'_{22} - \lambda'_{21})} \left(\int_{-\infty}^{\xi} e^{\lambda'_{21}(\xi-x)} \tilde{f}(\phi(x), \psi(x - c\tau))\psi(x - c\tau) dx \right. \\ &\quad \left. + \int_{\xi}^{+\infty} e^{\lambda'_{22}(\xi-x)} \tilde{f}(\phi(x), \psi(x - c\tau))\psi(x - c\tau) dx \right) \\ &= \frac{1}{(\lambda'_{22} - \lambda'_{21})} \left(\int_0^{+\infty} e^{\lambda'_{21}x} \tilde{f}(\phi(\xi - x), \psi(\xi - x - c\tau))\psi(\xi - x - c\tau) dx \right. \\ &\quad \left. + \int_{-\infty}^0 e^{\lambda'_{22}x} \tilde{f}(\phi(\xi - x), \psi(\xi - x - c\tau))\psi(\xi - x - c\tau) dx \right). \end{aligned}$$

Integrating the two sides, we have

$$\begin{aligned}
 \int_{-\infty}^{+\infty} \psi(\xi) d\xi &= \frac{1}{\mu + d + r} \int_{-\infty}^{+\infty} \tilde{f}(\phi(x), \psi(x - c\tau)) \psi(x - c\tau) d\xi \\
 &\leq \frac{\tilde{\beta}}{(1 + \tilde{\alpha}_1)(\mu + d + r)} \int_{-\infty}^{+\infty} \psi(\xi - c\tau) d\xi \\
 &= R_0 \int_{-\infty}^{+\infty} \psi(\xi) d\xi \\
 &< \int_{-\infty}^{+\infty} \psi(\xi) d\xi.
 \end{aligned}$$

This is a contradiction. Consequently, there is no nontrivial traveling wave when $R_0 > 1$ and $c \in (0, c^*)$, or $R_0 < 1$. ■

From Theorem 4.1, we get easily the following result.

Corollary 4.1. *If $R_0 > 1$ and $c \in (0, c^*)$, or $R_0 < 1$, system (1) has no non-trivial positive traveling wave solution connecting the disease-free equilibrium $Q_f(\Lambda/\mu, 0, 0)$ and the endemic equilibrium $Q^*(S^*, I^*, R^*)$.*

5 Discussion and conclusion

In this work, we have proposed and analyzed a delayed diffusive epidemic model with Hattaf's incidence rate that includes the traditional bilinear incidence rate, the saturated incidence rate, the Beddington-DeAngelis functional response and the Crowley-Martin functional response. Our model has two steady states that are the disease-free equilibrium Q_f and the endemic equilibrium Q^* . We have proved that if the basic reproduction number $R_0 \leq 1$, Q_f is globally asymptotically stable and the model admits no traveling wave solutions. Biologically speaking, the disease extinct and can not spread when $R_0 \leq 1$. However, if $R_0 > 1$, Q^* is globally asymptotically stable and the existence of traveling waves depend to another parameter, c^* , namely the minimal wave speed of propagation of the disease. More precisely, we have shown that if $R_0 > 1$ and for every speed $c > c^*$, the model has a traveling wave solution connecting the two steady states, and if $R_0 > 1$ and $c < c^*$, the model has no traveling wave solutions. Biologically, this result means that the disease persists in the population if $R_0 > 1$ and can spread for a speed greater than c^* but cannot be spread for any speed smaller than c^* .

On the other hand, we remark that

$$\begin{aligned}
 \frac{dc^*(\tau)}{d\tau} &= -\frac{\beta\Lambda c^*}{(\mu + \alpha_1\Lambda)e^{\lambda^*c^*\tau} + \beta\Lambda\tau} < 0, \\
 \frac{dc^*(\tau)}{d\beta} &= \frac{\mu}{\lambda^*(\mu + \alpha_1\Lambda)e^{\lambda^*c^*\tau} + \lambda^*\beta\Lambda\tau} > 0.
 \end{aligned}$$

Then the minimal wave speed c^* is a decreasing function on τ and is an increasing function on the β . Hence, we conclude that the latent period slows the spreading disease, while the infection rate increase the spreading disease.

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