

An ROBDD Algorithm for The Reliability of Double-Threshold Systems

Abstract

A double-threshold system (DTS) is a system that is successful if and only if the weighted arithmetic sum of its successes/failures equals or exceeds a certain threshold T_1 and is smaller than or equal to a certain threshold T_2 . Generally a DTS is neither symmetric nor coherent. The DTS reduces for positive weights to a weighted k -to- l -out-of- n : G system, whose symmetric special case is the k -to- l -out-of- n : G system. Another important special case of the DTS is the threshold system (TS), commonly known for positive weights as the weighted k -out-of- n system. The paper presents the fundamental properties of the DTS. Recursive relations covering a DTS are given together with various possible sets of boundary conditions. Based on these, a novel recursive algorithm for computing the reliability of a DTS is described, and then demonstrated via an illustrative example using the signal flow graph technique together with probability map interpretation. The DTS recursive algorithm developed herein is an extension of earlier algorithms for (single-) threshold systems and for k -out-of- n systems. The current algorithm as well as these former algorithms are shown to be equivalent to implementation of the Reduced Ordered Binary Decision Diagram (ROBDD).

Keywords: Double-Threshold, k -to- l -out-of- n , weighted k -out-of- n , recursive relations, boundary conditions, signal flow graph, reliability

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Assumptions

1. Both the components and the system are 2-state, i.e. either successful or failed.

2. Component states are statistically independent.
3. Component reliabilities are not necessarily equal.
4. The success/failure function of the system is a double-threshold switching function in the successes/failures of its components.

Acronyms

TS	Threshold System, also called Single-Threshold System or weighted k -out-of- n system.
DTS	Double-Threshold System, a system that reduces for positive weights to a weighted k -to- l -out-of- n : G system.

Notation

n	number of system components.
$X_i, \bar{X}_i$	indicator variables for successful and unsuccessful operation of component i . These are switching random variables that take only one of the two discrete values 0 and 1; $X_i = 1$ and $\bar{X}_i = 0$ iff i is good, and $X_i = 0$ and $\bar{X}_i = 1$ if i is failed.
$S, \bar{S}$	indicator variables for successful and unsuccessful operation of the system, called system success and system failure, respectively.
W_i	weight of the variable X_i (weight of component i); a real-valued constant.
T_1, T_2	threshold discriminations (or merely thresholds) of a double-threshold function or system; real-valued constants.
$D(n; \mathbf{X}; \mathbf{W}; T_1, T_2)$	$D(n; X_1, X_2, \dots, X_n; W_1, W_2, \dots, W_n; T_1, T_2)$ a double-threshold switching function of n variables $\mathbf{X}$ (or a double-threshold system of n components with successes $\mathbf{X}$), weights $\mathbf{W}$ and thresholds T_1 and T_2 .
$P\{.\}$	probability of the event $\{.\}$.
p_i, q_i	reliability and unreliability of component i : $p_i = P\{X_i = 1\}$; $q_i = P\{X_i = 0\} = 1.0 - p_i$. Both p_i and q_i take real values in the closed continuous interval $[0.0, 1.0]$.
R, U	reliability and unreliability of the system: $R = P\{S = 1\}$; $U = P\{S = 0\} = 1.0 - R$.
$\lfloor x \rfloor$	floor of a real number x .
$\lceil x \rceil$	ceiling of a real number x .
$\mathbf{X}, \mathbf{p}, \mathbf{W}$	n -dimensional vectors of component successes, reliabilities, and weights; $\mathbf{X} = [X_1 \ X_2 \ \dots \ X_n]^T$, etc.

$\mathbf{X} \mid j_i$	the value of $\mathbf{X}$ when X_i is set equal to j , where $j = 0$ or 1 .
$\mathbf{p} \mid j_i$	the value of $\mathbf{p}$ when p_i is set equal to j , where $j = 0$ or 1 .
$I\{A\}$	indicator variable for proposition A ; $I\{A\}$ equals 1 if A is true and is 0 otherwise.
$\mathbf{W}/W_i$	an $(n - 1)$ -dimensional vector obtained by excluding W_i from $\mathbf{W}$; $\mathbf{W}/W_i = [W_1 \ W_2 \ \dots \ W_{i-1} \ W_{i+1} \ \dots \ W_n]^T$

Nomenclature

Duality: strictly speaking, the dual of a switching function is obtained by complementing the function and all its switching arguments (inverting both inputs and output) [1, 2]. In the reliability literature, "duality" is sometimes freely used to indicate "similarity" or "analogy".

Monotone: a monotone system is one whose reliability function is a non-decreasing function in each component reliability, i.e.,

$$R(\mathbf{p}|1_m) - R(\mathbf{p}|0_m) = \partial R(\mathbf{p})/\partial p_m \geq 0.0, \ 1 \leq m \leq n. \quad (0.1)$$

Relevant: component m is relevant to the system if there exists a valid value for p such that $\partial R(p)/\partial p_m \neq 0.0$.

Coherent: a coherent system is a monotone system whose components are all relevant [2, 3]. If the reliability function $R(p)$ of a coherent system with identical components is plotted versus p within the square $0.0 \leq p \leq 1.0, 0.0 \leq R(p) \leq 1.0$, then it satisfies $R(0.0) = 0.0$, and $R(1.0) = 1.0$, and exhibits an S-shape, i. e., the curve $R(p)$ versus p is monotonically non-decreasing and if it crosses the diagonal (p versus p), it does so only once and from below, [4].

k -out-of- n :G system: a system that is good if and only if (iff) at least k out of its n components are good.

k -out-of- n :F system: a system that is failed iff at least k out of its n components are failed.

k -out-of- n (partially-redundant) system: a collective name for k -out-of- n :G and k -out-of- n :F systems; a k -out-of- n :F system is equivalent to an $(n - k + 1)$ -out-of- n :G system. A k -out-of- n system is a coherent system in the practical case of $1 \leq k \leq n$, while it is only monotone for the hypothetical or fictitious limiting cases of $k = 0$ and $k = (n + 1)$.

s-p complex: a coherent system is series-parallel (s-p) complex iff it has no components in series or in parallel [5]. A k -out-of- n system is s-p complex for $1 < k < n$, and hence cannot be treated (even partially) by series-parallel reductions.

Pivoting: by pivoting on component m , system reliability $R(\mathbf{p})$ can be written as

$$R(\mathbf{p}) = q_m * R(\mathbf{p}|0_m) + p_m * R(\mathbf{p}|1_m), \quad (0.2)$$

where $R(p|0_m)$ and $R(p|1_m)$ are the reliabilities of the minors or subsystems of the original system with respect to component m . Pivoting is also called factoring and is equivalent to the total probability theorem [6] in the algebraic domain or to the Boole-Shannon expansion [7] in the Boolean domain.

1 Introduction

The single threshold system, or simply the threshold system, which can be neither symmetric nor coherent, is an important generalization of the k -out-of- n :G(F) system. A threshold system is defined as a system composed of n statistically independent 2-state components such that the success or failure of the system is a threshold (linearly separable) switching function [8–10] in the successes or failures of the system components. This system is successful if and only if the weighted arithmetic

sum of its component successes equals or exceeds a certain threshold. Therefore, a threshold system is characterized by $(n + 1)$ coefficients, namely, its threshold and the set of n component weights. An important special case of the threshold system is the weighted k -out-of- n :G system, which is a coherent non-symmetric system of strictly positive weights and a threshold equal to k [10–13]. If further, all the weights are equal to 1, the weighted k -out-of- n :G system reduces to the k -out-of- n :G system. Therefore, the k -out-of- n :G system can be defined as a threshold system with a common positive weight for its components and a threshold equal to k multiplied by this common weight [8].

This paper deals with a generalization of the single-threshold system, which we call a double-threshold system. A double-threshold system is a system whose success or failure (but not both) is a double-threshold switching function in the successes/failures of its components, i.e., it is defined to be successful or failed if and only if the weighted arithmetic sum of its successes/failures equals or exceeds a certain threshold T_1 and is smaller than or equal to a certain threshold T_2 . Generally, a double threshold system is neither symmetric nor coherent. Its symmetric special case is the k -to- l -out-of- n system (for which the component weights are equal, and its coherent special case is the single-threshold system of positive weights (for which the upper threshold T_2 is ignored or considered infinite), which is commonly referred to in the literature as a weighted k -out-of- n system.

A switching function is characterized by 2^n real coefficients while a threshold function is defined by $(n + 1)$ coefficients, and a double-threshold function is characterized by $(n + 2)$ real coefficients. While the class of double-threshold functions is a somewhat restricted subset of all switching functions, it is still large enough to represent many systems of practical significance.

A prominent example of a double-threshold system is the k -to- l -out-of- n :G system [14–22], which is good iff the number of good components among its n components is between k and l (inclusive), i.e.

$$\{S(\mathbf{X}) = 1\} \Leftrightarrow \left\{ k \leq \sum_{i=1}^n X_i \leq l \right\}. \quad (1.1)$$

Another prominent example is the mirror-image or dual of the above system, namely, the k -to- l -out-of- n :F system which fails if at least k and at most l of its n components fail, i.e.,

$$\{\bar{S}(\mathbf{X}) = 1\} \Leftrightarrow \left\{ k \leq \sum_{i=1}^n \bar{X}_i \leq l \right\}. \quad (1.2)$$

The success function for each of the k -to- l -out-of- n systems is a *symmetric* switching function beside being a double-threshold one. These systems are useful non-coherent models of some multiprocessor computer as well as communication and transportation systems. The k -to- l -out-of- n :G system requires a minimum number k of good components to function successfully. However, when the number of good components exceeds a maximum l , competition among good components for limited resources available causes system failure.

This paper is intended to be a review or tutorial exposition, that we hope to make of significant practical utility. We endeavor for clarity and simplicity to allow all readers to easily follow the discussion. Thus, we deliberately include certain details and explanation of terminology that experts might consider obvious or even trivial. In some papers, the absence and obscurity of such details has occasionally led to misunderstanding and pitfalls.

The paper lists and proves (or at least justifies) the fundamental properties of double-threshold switching functions and systems. In particular, recursive relations governing double-threshold systems are given together with various possible sets of boundary conditions. Based on these, a novel recursive algorithm for computing the reliability of a double-threshold system is described, and then demonstrated via an illustrative example using the signal flow graph technique. We start our reliability

analysis in the switching (Boolean) domain rather than the probability (algebraic) domain, then transform the result to the algebraic domain. We try to use the plethora of techniques available for switching (Boolean) algebra [23–29]. We will use the symbol ($\vee$) instead of ($+$) for the *OR* switching operation and $R(\mathbf{p})$ or $R(p)$ for system reliability for non-identical component reliabilities $\mathbf{p}$ or a common component reliability p , i.e., we will make the time dependence of R implicit through the time dependence of $\mathbf{p}$ or p .

The organization of the rest of this paper is as follows. Section 2 represents the formal definition and certain fundamental properties of the DTS. The recursive relations governing the reliability of a DTS are given in Section 3 together with different appropriate sets of Boundary Conditions. Based on these, a novel recursive algorithm for computing the reliability of DTS is given. This algorithm is illustrated for a 5-component system in Section 4, and further illustrated by a (Mason) signal flow graph, and then verified and interpreted on a probability map. Section 6 concludes the paper.

2 Formal Definition and Fundamental Properties

By definition, a switching function:

$$S(X) = S(X_1, X_2, \dots, X_n) \quad (2.1)$$

is a threshold function iff there exists a set of real numbers $W_1, W_2, \dots, W_n$, called weights, and T , called a threshold, such that:

$$S(\mathbf{X}) = \begin{cases} 1 & \text{iff } \sum_{i=1}^n W_i X_i \geq T \\ 0 & \text{iff } \sum_{i=1}^n W_i X_i < T. \end{cases} \quad (2.2)$$

A double-threshold system (DTS) is a system whose success is a double-threshold function and can be defined by:

$$S(\mathbf{X}) = 1 \quad \text{iff} \quad T_1 \leq \sum_{i=1}^n W_i X_i \leq T_2 \quad (2.3)$$

and denoted by $D(n; \mathbf{X}; \mathbf{W}; T_1; T_2)$.

We can use the above designation to express the success of certain special cases of the DTS as follows:

Success of the k -to- l -out-of- n : G system

$$= D(n; \mathbf{X}; 1, 1, \dots, 1; k; l), \quad (2.4)$$

Success of the k -to- n -out-of- n : G system

$$\begin{aligned} &= \text{success of the } k\text{-out-of-}n\text{:}G \text{ system} \\ &= D(n; \mathbf{X}; 1, 1, \dots, 1; k; n), \end{aligned} \quad (2.5)$$

Success of the k -to- n -out-of- n : F system

$$\begin{aligned} &= \text{success of the } k\text{-out-of-}n\text{:}F \text{ system} \\ &= D(n; \mathbf{X}, 1, 1, \dots, 1; n - k + 1; n) \\ &= \text{success of the } (n - k + 1)\text{-out-of-}n\text{:}G \text{ system.} \end{aligned} \quad (2.6)$$

Alternatively, a DTS might be defined as a system whose failure is a double-threshold function i.e.,

$$\bar{S}(\mathbf{X}) = 1 \text{ iff } T_1 \leq \sum_{i=1}^n W_i \bar{X}_i \leq T_2. \quad (2.7)$$

Note that the definitions in (2.3) and (2.7) are incompatible. Therefore, we will restrict our forthcoming discussion to the DTS defined by (2.3).

We now list a few fundamental properties of a Double-Threshold system.

Property 1

The DTS has its success function as

$$S(\mathbf{X}) = D(n; \mathbf{X}; \mathbf{W}, T_1, T_2), \quad (2.8)$$

satisfying a conjunction of two propositions (which are consistent since $T_1 \leq T_2$), namely:

$$\left\{ T_1 \leq \sum_{i=1}^n W_i X_i \right\} \text{ AND } \left\{ \sum_{i=1}^n W_i X_i \leq T_2 \right\} \quad (2.9)$$

The failure function is given by (2.7) and hence, this failure function satisfies a disjunction of two disjoint (mutual-exclusive) propositions, namely:

$$\left\{ T_1 > \sum_{i=1}^n W_i X_i \right\} \text{ OR } \left\{ \sum_{i=1}^n W_i X_i > T_2 \right\} \quad (2.10)$$

Therefore, a DTS is defined to have either its success or its failure (but not both) as a double-threshold function. This situation is different from that of a threshold system, where success and failure are both threshold functions. Therefore, while a k -out-of- $n:F$ system can be identified as an $(n - k + 1)$ -out-of- $n:G$ system, there is no similar result for the k -to- l -out-of- $n:F$ system which fails when the number of its failed components N_F is such that $\{k \leq N_F \leq l\}$, and hence succeeds when the number of its successful components $N_S = n - N_F$ is such that $\{N_S > (n - k)\}$ or $\{N_S < (n - l)\}$. These two sets of values for N_S do not join to form a set of consecutive values $[m_1, m_2]$ that describes an m_1 -to- m_2 -out-of- $n:G$ system.

Property 2

It is possible to express the success of a DTS (2.8) with negated weights and negated interchanged thresholds, i.e., as:

$$S(\mathbf{X}) = D(n; \mathbf{X}; -\mathbf{W}, -T_2, -T_1). \quad (2.11)$$

For example, the following three expressions denote the same DTS.

$$6 \leq X_1 + 3X_2 + 5X_3 + 5X_4 + 7X_5 \leq 12 \quad (2.12a)$$

$$S(\mathbf{X}) = D(5; \mathbf{X}; 1, 3, 5, 5, 7; 6; 12) \quad (2.12b)$$

$$S(\mathbf{X}) = D(5; \mathbf{X}; -1, -3, -5, -5, -7; -12; -6). \quad (2.12c)$$

Property 3

The system success is a double-threshold function in either the component successes, the component failures, or some mixture thereof. In fact, the indicator variable of one component can be replaced by one minus its complement in the double-threshold relation, thereby changing the system success into:

$$S(\mathbf{X}) = D(n; X_1, X_2, \dots, \bar{X}_i, \dots, X_n; W_1, W_2, \dots, -W_i, \dots, W_n; T_1 - W_i, T_2 - W_i). \quad (2.13)$$

This means that we can modify the system description by replacing the success of component i by its failure and correspondingly negating its weight and deducing that weight from both thresholds. Similarly, all component successes can be complemented provided S is rewritten as

$$S(\mathbf{X}) = D\left(n; \bar{\mathbf{X}}; -\mathbf{W}; T_1 - \sum_{i=1}^n W_i; T_2 - \sum_{i=1}^n W_i\right). \quad (2.14)$$

This allows an alternative system description in which component failures are used instead of component successes as arguments for system success provided component weights are negated and the sum of the original weights is deduced from both thresholds.

Property 4

If the system success $S(\mathbf{X})$ is a symmetric switching function, its weights can be assumed to be all equal to unity without loss of generality. If the system is coherent, the real thresholds T_1 and T_2 can be replaced by integer values equal to the floor $k = \lfloor T_2 \rfloor$ and the ceiling $l = \lceil T_1 \rceil$, respectively. Equations (2.4)-(2.6) express some symmetric cases of the DTS.

Property 5

If the component m of the DTS is known to be failed or good, the system reduces to a subsystem, with a double-threshold success, via

$$S(\mathbf{X}|0_m) = D(n-1; \mathbf{X}/X_m; \mathbf{W}/W_m; T_1; T_2) \quad (2.15)$$

$$S(\mathbf{X}|1_m) = D(n-1; \mathbf{X}/X_m; \mathbf{W}/W_m; T_1 - W_m; T_2 - W_m). \quad (2.16)$$

The original double-threshold function $S(\mathbf{X})$ is related to its sub-functions/restrictions in (2.15) and (2.16) via the Boole-Shannon expansion.

$$S(\mathbf{X}) = (\bar{X}_m \wedge S(\mathbf{X}|0_m)) \vee (X_m \wedge S(\mathbf{X}|1_m)). \quad (2.17)$$

The RHS of equation (2.17) is in probability-ready form [2, 28–30], since any ORed quantities are disjoint and any ANDed quantities are statistically independent. It can be transformed immediately to a probability expression by replacing switching variables by their expectations and logical multiplication and addition (ANDing and ORing) by their arithmetic counterparts.

3 Recursive Relations and A Recursive Algorithm

The reliability of a DTS system can be computed via the recursive relation obtained by transforming equation (2.17) to the probability domain as follows:

$$\begin{aligned} R(n; \mathbf{p}; \mathbf{W}; T_1; T_2) = & q_n R(n-1; \mathbf{p}/p_n; \mathbf{W}/W_n; T_1; T_2) \\ & + p_n R(n-1; \mathbf{p}/p_n; \mathbf{W}/W_n; T_1 \\ & - W_n; T_2 - W_n), \end{aligned} \quad (3.1)$$

together with an appropriate set of boundary conditions obtained for subsystems of an acceptably small number of components. For subsystems of 0 components, the boundary conditions are shown in Figure 1a, namely

$$R(0; ; ; T_1; T_2) = I(T_1 \leq 0)I(0 \leq T_2). \quad (3.2)$$

For subsystems of a single complement n , the boundary conditions are shown in Figure 1b, namely

$$\begin{aligned} R(1; p_n; W_n; T_1; T_2) = & I(T_1 \leq 0)I(0 \leq T_2)q_n \\ & + I(T_1 \leq W_n)I(W_n \leq T_2)p_n. \end{aligned} \quad (3.3)$$

For subsystems of two components i and j , the boundary conditions are shown in Figure 1c, namely

$$\begin{aligned} R(2; p_i, p_j; W_i, W_j; T_1; T_2) = & I(T_1 \leq 0)I(0 \leq T_2)q_i q_j \\ & + I(T_1 \leq W_i)I(W_i \leq T_2)p_i q_j \\ & + I(T_1 \leq W_j)I(W_j \leq T_2)q_i p_j \\ & + I(T_1 \leq W_i + W_j)I(W_i + W_j \leq T_2)p_i p_j. \end{aligned} \quad (3.4)$$

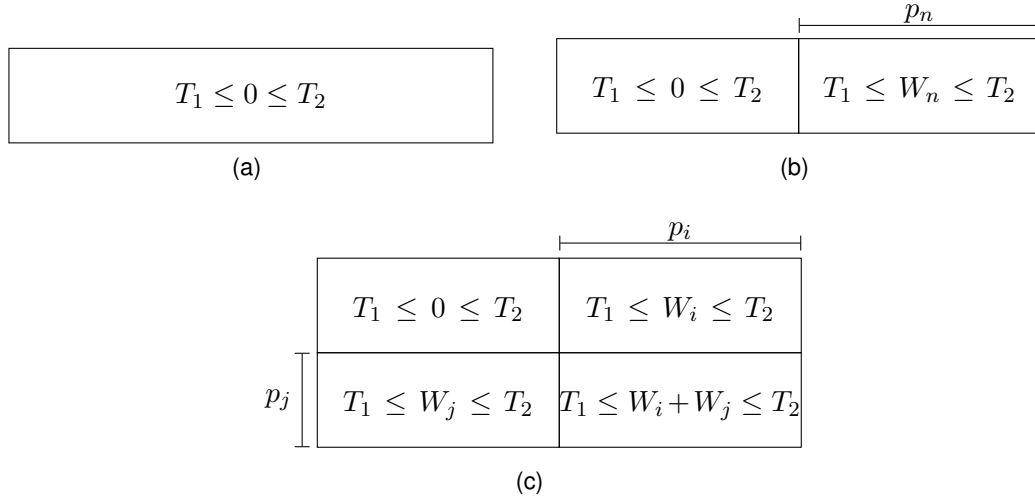


Figure 1: Variable-entered Karnaugh maps representing possible boundary conditions for a double-threshold system of (a) 0 components, (b) one component n , and (c) two components i and j .

Similarly, boundary conditions could be written for subsystems of three or more components in the same manner.

The repeated implementation of the recursive relation (3.1) doubles the number and decreases the sizes of the DTSs involved. The recursion might be terminated at the level $n_f = 0$ (Boundary Conditions (3.2)), at the level $n_f = 1$ (Boundary Conditions (3.3)), at the level of $n_f = 2$ (Boundary Conditions (3.4)), or elsewhere. The decomposition or expansion tree is a complete binary tree of $[2^{(n-n_f)} - 1]$ nodes. Therefore, the temporal complexity of the present algorithm is exponential.

In the special case of the coherent DTS, we can improve the efficiency of the current algorithm by pruning the decomposition tree, i.e., by terminating the recursion as soon as a value of 0 or 1 is achieved. This results from the fact that for a coherent system, the component weights are strictly positive, and hence

$$0 \leq \sum_{i=1}^n W_i X_i \leq \sum_{i=1}^n W_i \quad (3.5)$$

which when added to the definition (2.3), result in the following boundary conditions

$$R(n; \mathbf{p}; \mathbf{W}; T_1, T_2) = 1.0 \quad \text{if } \{T_1 \leq 0\} \text{ AND } \left\{ \sum_{i=1}^n W_i \leq T_2 \right\} \quad (3.6)$$

$$R(n; \mathbf{p}; \mathbf{W}; T_1, T_2) = 0.0 \quad \text{if } \left\{ \sum_{i=1}^n W_i < T_1 \right\} \text{ OR } \{T_2 < 0\} \quad (3.7)$$

With boundary conditions (3.6) and (3.7) used, the expansion tree ceases to be a complete binary tree. It also ceases to be a strict binary tree and becomes an acyclic graph if some of the resulting intermediate nodes are identified to be the same and profitably combined. The algorithm described herein is a generalization of efficient or optimal algorithms for computing the reliabilities of k -out-of- n systems [1, 2, 31–35], k -to- l -out-of- n system [16, 17], and threshold systems [8, 10].

4 Illustrative Example

Consider the double-threshold system defined by (2.12). Figure (2) is a Karnaugh-map representation of the pseudo-Boolean function $F(X) = \sum_{i=1}^n W_i X_i$ [36], and double-threshold success $S(X)$. Figure 3 is a Mason Signal Flow graph representing the expansion tree implementing the recursion (3.1) with boundary conditions (3.6) and (3.7). Since some intermediate nodes are combined, the expansion tree becomes an acyclic tree or an ROBDD graph [2]. In Figure 3, each shaded or intermediary node at level m has two inputs: one with transmission q_m from a node of the same two thresholds, and another with transmission p_m from a node with its two thresholds each decreased by the weight W_m . The black nodes are source nodes of value 1.0, while the white nodes are of value 0.0 (and might be deleted). Note that the non-coherent nature of the system is manifested by the appearance of the black (success) nodes as one group sandwiched between two groups of white (failure) nodes. The sandwiching phenomenon appears in the SFGs for other non-coherent systems such as the k -to- l -out-of- n : G system [16, 17], but it is absent in those of coherent systems such as certain single-threshold systems [8] or k -out-of- n systems [1, 2, 10]. For those coherent systems, the black and white nodes cluster separately in distinct groups. Finally, we obtain the following compact symbolic expression for system reliability

$$\begin{aligned} R(5; \mathbf{p}; 1, 3, 5, 5, 7, 6; 12) = \\ p_5((p_4 q_3 + q_4 p_3) q_2 q_1 + q_4 q_3) \\ + q_5[(p_4 q_3 + q_4 p_3)(p_2 + q_2 p_1) + p_4 p_3 q_2] \end{aligned} \quad (4.1)$$

Correctness of expression (4.1) is verified on a special version of the Karnaugh map [24, 37], called the probability map [24, 37] as shown in Figure 4. It is clear from this figure that the current algorithm achieves an almost minimal disjoint coverage of the real transform of system success [38].

Compactness of the symbolic expression for R in (4.1) is a good asset. It results in a reduced computational time and reduced round-off error when evaluating R numerically. Its availability allows the computation of other pertinent quantities [39]. The dependence of R in (4.1) on time (t) is implicit through the component reliabilities $p_m(t)$. If the $p_m(t)$'s are given (e.g., $p_m(t) = \exp(-\lambda_m t)$ for a constant failure rate), then the system failure rate is computed via differentiation

$$f = -\frac{dR(t)}{dt}, \quad (4.2)$$

and the mean-time-to-failure (MTTF) of the system is computed via integration

$$MTTF = \int_0^\infty R(t) dt \quad (4.3)$$

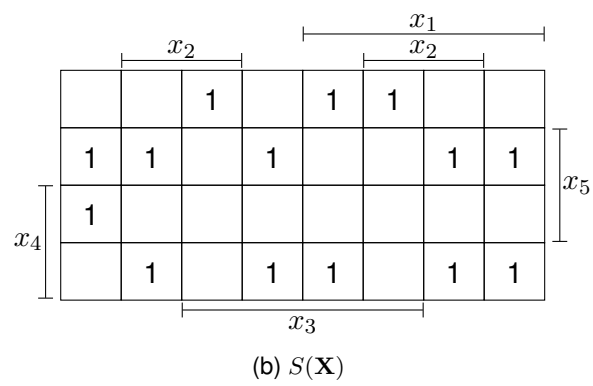
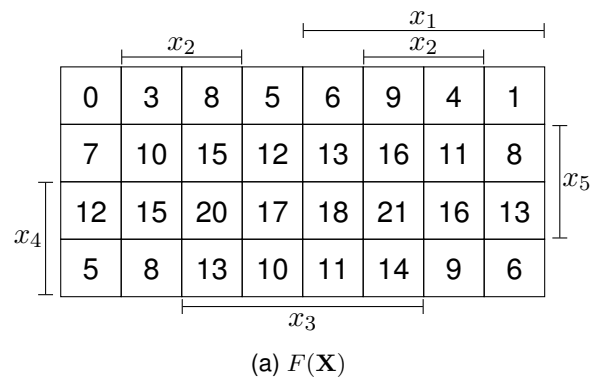


Figure 2: Karnaugh maps for $F(\mathbf{X})$ and $S(\mathbf{X})$

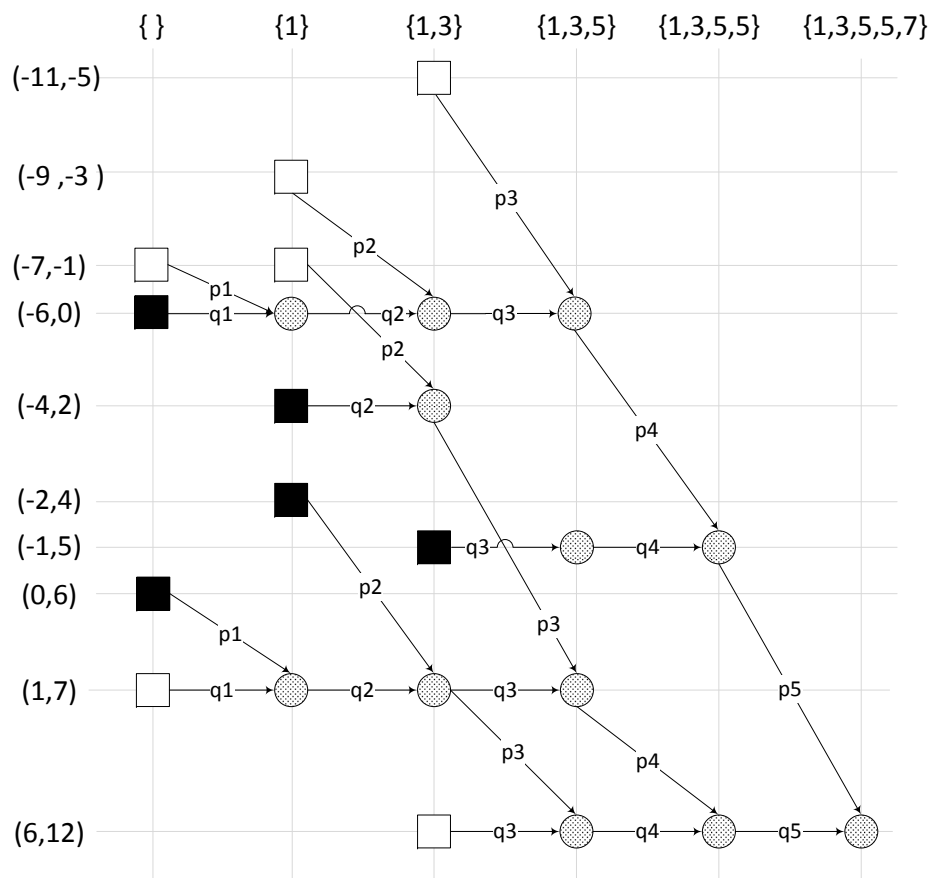


Figure 3: Signal flow graph with nodes $R(n; \mathbf{p}; \mathbf{W}; T_1, T_2)$ and sink node $R(5; \mathbf{p}; 1, 3, 5, 5, 7; 6, 12)$. Nodes in the same vertical level n share the same $\mathbf{W}$ as shown, while (T_1, T_2) for every node is written on the same horizontal level and $\mathbf{p}$ is implied. The indicated grid of $\mathbf{W}$ and (T_1, T_2) values is very helpful in identifying leaf nodes of values 0 (shown white) or of values 1 (shown black).

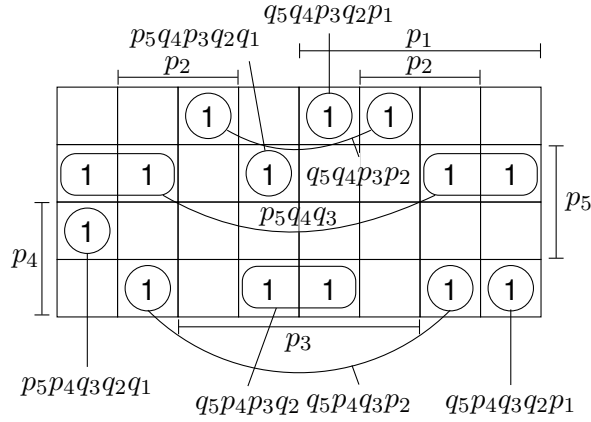


Figure 4: Interpretation of the result of Figure 3 on a probability map

5 Interpretation as an ROBDD

The current algorithm includes as a special case an earlier algorithm for computing the reliability for a (single-)threshold or weighted-out-of-system, which in turn, includes as a special case the AR algorithm for computing the reliability of a k -out-of- n system. We show now that all these algorithms are implementations of the Reduced-Ordered-Binary-Decision-Diagram (ROBDD) strategy. The k -out-of- n algorithm restricts the Boolean function to be monotone and symmetric, but the single- and double-threshold algorithms do not impose such a restriction, though they demand that the Boolean function be of a particular type.

The ROBDD strategy was proposed in [40] as an extension of the BDD methodology of [41]. The ROBDD deals with general switching (two-valued Boolean) functions, and is now considered the state-of-the-art data structure for handling such functions, with extensive applications in reliability [42–47]. Our three algorithms, including the current one, have exactly the same features as the ROBDD algorithm, namely:

1. Both these algorithms and the ROBDD algorithm are based on the Boole-Shannon expansion in the Boolean domain.

$$f(\mathbf{X}) = \overline{X}_m (f(\mathbf{X})/\overline{X}_m) \vee X_m (f(\mathbf{X})/X_m), \quad (5.1)$$

where

$$f(\mathbf{X})/\overline{X}_m = f(\mathbf{X})|_{X_m=0} \quad (5.2)$$

$$f(\mathbf{X})/X_m = f(\mathbf{X})|_{X_m=1} \quad (5.3)$$

are called quotients, ratio, cofactors, sub-functions, or restrictions of $f(\mathbf{X})$. This expansion translates in the probability domain to the following expression for system reliability.

$$R(\mathbf{p}) = q_m R(\mathbf{p}|p_m = 0) + p_m R(\mathbf{p}|p_m = 1) \quad (5.4)$$

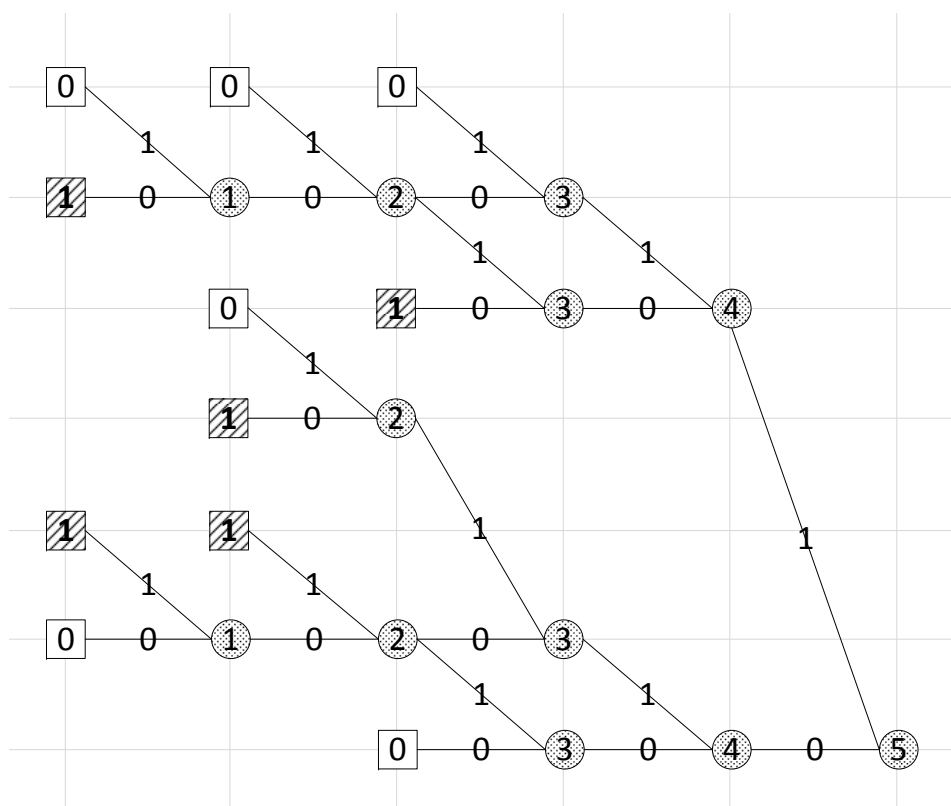


Figure 5: The ROBDD structure isomorphic to the SFG of 3. For simplicity separate leaf nodes of 0 and 1 are used. Actually, there is a single leaf node of 0 and another of 1. Note that the grid of $\mathbf{W}$ and (T_1, T_2) of Fig. (3) is missing here.

where $q_m = 1.0 - p_m$. Equation (5.4) is simply a restatement of (0.2). Our earlier equation (3.1) is the particular instance of (5.4) when applied to the double-threshold success. Equation (5.4) is simply an expression of the Total Probability Theorem [2, 6] or Factoring Theorem [39, 48, 49]

2. Both algorithms visit the variables in a certain order, typically monotonically ascending or monotonically descending.
3. Both algorithms reduce the resulting expansion tree (which is exponential in size) to a root acycli graph that is both canonical and hopefully compact or sub-exponential. The reduction rule [47] requires merging isomorphic sub-stress, and deletion of useless nodes whose outgoing edges points to the same child node.

The equivalence of the current algorithm to an ROBDD algorithms is manifested by the isomorphism of Fig. 3 to Fig. 5 which depicts the same computation via an ROBDD. For simplicity, Fig. ?? shows the leaf nodes of 0 and 1 as split nodes.

6 Conclusion

This paper introduces a novel class of useful reliability models, called the double-threshold system, which includes many well-known models as special cases. The paper presents the system definition, properties, recursive relations, and boundary conditions. Subsequently, a novel recursive algorithm for computing the system reliability is presented and illustrated with a 5-component example via the pictorial tools of the signal flow graph and probability map.

References

- [1] A.M.A Rushdi. *Reliability of k-out-of-n Systems, Chapter 5 in Misra, K. B. (Editor), New Trends in System Reliability Evaluation*, volume 16. Elsevier Science Publishers, Amsterdam, The Netherlands, pp. 185-227, 1993.
- [2] A.M.A Rushdi. Partially-redundant systems: Examples, reliability, and life expectancy. *International Magazine on Advances in Computer Science and Telecommunications*, 1(1):1–13, 2010.
- [3] Bo Bergman, Elja Arjas, Marvin Rausand, Bent Natvig, Kjell A Doksum, and Tore Schweder. On reliability theory and its applications [with discussion and reply]. *Scandinavian Journal of Statistics*, pages 1–41, 1985.
- [4] Richard E Barlow and Frank Proschan. *Mathematical Theory of Reliability*, volume 44. SIAM, New York, NY, USA, 1996.
- [5] Barlow R.E. and Iyer S. Computational complexity of coherent systems and the reliability polynomial. *Probability in the Engineering and Informational Sciences*, 2(04):461–469, 1988.
- [6] K.S. Trivedi. *Probability and Statistics with Reliability, Queuing, and Computer Science Applications*. 2nd ed. Wiley New York, 2002.
- [7] Frank Markham Brown. *Boolean Reasoning: the Logic of Boolean Equations*. Courier Dover Publications, 2003.
- [8] A.M.A Rushdi. Threshold systems and their reliability. *Microelectronics and Reliability*, 30(2):299–312, 1990.
- [9] Saburo Muroga. *Threshold logic and its applications*, volume 13. Wiley-Interscience New York, 1971.
- [10] Ali Muhammad Ali Rushdi and Alaa Mohammad Alturki. Reliability of coherent threshold systems. *Journal of Applied Sciences*, 15(3):431–443, 2015.
- [11] W. Kuo and M.J. Zuo. *The k-out-of-n System Model, Chapter 7 in Optimal Reliability Modeling: Principles and Applications*, volume 16. Wiley, New York, NY, USA, pp. 231-280, 2003.

- [12] Jer-Shyan Wu and Rong-Jaye Chen. An algorithm for computing the reliability of weighted-k-out-of-n systems. *Reliability, IEEE Transactions on*, 43(2):327–328, 1994.
- [13] Serkan Eryilmaz. Capacity loss and residual capacity in weighted k-out-of-n: G systems. *Reliability Engineering & System Safety*, 136:140–144, 2015.
- [14] KD Heidtmann. A class of noncoherent systems and their reliability analysis, 11th int. In *Symp. Fault Tolerant Computing (4th edn), FTCS-11*, 1981.
- [15] SP Jain and Krishna Gopal. Reliability of k-to-l-out-of-n systems. *Reliability engineering*, 12(3):175–179, 1985.
- [16] A.M.A Rushdi. Efficient computation of k-to-l-out-of-n system reliability. *Reliability Engineering*, 17(3):157–163, 1987.
- [17] A.M.A Rushdi and F.M.A. Dehlawi. Optimal computation of k-to-l-out-of-n system reliability. *Microelectronics and Reliability*, 27(5):875–896, 1987.
- [18] HOANG Pham. Optimal design for a class of noncoherent systems. *Reliability, IEEE Transactions on*, 40(3):361–363, 1991.
- [19] Yow-mow Chen. Transient analysis of reliability and availability in k-to-l-out-of-n: G system. *Reliability Engineering & System Safety*, 35(3):179–182, 1992.
- [20] Hoang Pham. Optimal system-profit design of k-to-l-out-of-n systems. *Microelectronics and Reliability*, 32(5):711–718, 1992.
- [21] A.M.A Rushdi and A.O. Al-Thubaity. Efficient computation of the sensitivity of k-out-of-n system reliability. *Microelectronics and reliability*, 33(13):1963–1979, 1993.
- [22] Shambhu J. Upadhyaya and Hoang Pham. Analysis of noncoherent systems and an architecture for the computation of the system reliability. *Computers, IEEE Transactions on*, 42(4):484–493, 1993.
- [23] Kaufmann A., Grouchko D., and Cruon R. *Mathematical Methods for the Study of the Reliability of Systems*. Academic Press, New York, NY, USA, 1977.
- [24] A.M.A Rushdi. Symbolic reliability analysis with the aid of variable-entered karnaugh maps. *Reliability, IEEE Transactions on*, 32(2):134–139, 1983.
- [25] A.M.A Rushdi. How to hand-check a symbolic reliability expression. *Reliability, IEEE Transactions on*, 32(5):402–408, 1983.
- [26] A.M.A Rushdi. A switching-algebraic analysis of consecutive-k-out-of-n: F systems. *Microelectronics and Reliability*, 27(1):171–174, 1987.
- [27] A.M.A Rushdi. A switching-algebraic analysis of circular consecutive-k-out-of-n: F systems. *Reliability Engineering & System Safety*, 21(2):119–127, 1988.
- [28] A.M.A Rushdi and Omar M Ba-Rukab. A doubly-stochastic fault-tree assessment of the probabilities of security breaches in computer systems. In *Proceedings of the Second Saudi Science Conference, Part Four: Computer, Mathematics, and Statistics*, pages 1–17, 2004.
- [29] A.M.A Rushdi and Omar M Ba-Rukab. Fault-tree modelling of computer system security. *International Journal of Computer Mathematics*, 82(7):805–819, 2005.

- [30] A.M.A Rushdi and Azza S Goda. Symbolic reliability analysis via shannon's expansion and statistical independence. *Microelectronics and Reliability*, 25(6):1041–1053, 1985.
- [31] A.M.A Rushdi. Comments on: An efficient non-recursive algorithm for computing the reliability of k-out-of-n systems. *Reliability, IEEE Transactions on*, 40(1):60–61, 1991.
- [32] A.M.A Rushdi and Abdulghani M Al-Qasimi. Efficient computation of the pmf and the cdf of the generalized binomial distribution. *Microelectronics and Reliability*, 34(9):1489–1499, 1994.
- [33] A.M.A Rushdi and A. E. Alsulami. Cost elasticities of reliability and MTTF for k-out-of-n systems. *Journal of Mathematics and Statistics*, 3(3):122–128, 2007.
- [34] A.M. Al-Qasimi and A.M.A Rushdi. A tutorial on how to efficiently calculate and format tables of the binomial distribution. *Journal of King Abdul-Aziz University: Engineering Sciences*, 19(1):3–17, 2008.
- [35] A.M.A Rushdi. Utilization of symmetric switching functions in the computation of k-out-of-n system reliability. *Microelectronics and Reliability*, 26(5):973–987, 1986.
- [36] A.M.A Rushdi. Indexes of a telecommunication network. *Reliability, IEEE Transactions on*, 37(1):57–64, 1988.
- [37] Richard B Hurley. Probability maps. *Reliability, IEEE Transactions on*, 12(3):39–44, 1963.
- [38] A.M.A Rushdi. On computing the spectral coefficients of a switching function. *Microelectronics and Reliability*, 27(6):965–979, 1987.
- [39] Ali Muhammad Ali Rushdi and Ahmad Kamal Hassan. Reliability of migration between habitat patches with heterogeneous ecological corridors. *Ecological Modelling*, 304:1–10, 2015.
- [40] Randal E Bryant. Graph-based algorithms for boolean function manipulation. *IEEE Transactions on computers*, 100(8):677–691, 1986.
- [41] Sheldon B. Akers. Binary decision diagrams. *IEEE Transactions on computers*, 100(6):509–516, 1978.
- [42] Krishna B Misra. *Handbook of performability engineering*. Springer Science & Business Media, 2008.
- [43] Antoine Rauzy. Binary decision diagrams for reliability studies. In *Handbook of Performability Engineering*, pages 381–396. Springer, 2008.
- [44] Kim Bjorkman. Solving dynamic flowgraph methodology models using binary decision diagrams. *Reliability Engineering & System Safety*, 111:206–216, 2013.
- [45] Yuchang Mo. A multiple-valued decision-diagram-based approach to solve dynamic fault trees. *IEEE Transactions on Reliability*, 63(1):81–93, 2014.
- [46] Yuchang Mo, Liudong Xing, and Suprasad V Amari. A multiple-valued decision diagram based method for efficient reliability analysis of non-repairable phased-mission systems. *IEEE Transactions on Reliability*, 63(1):320–330, 2014.
- [47] Yuchang Mo, Liudong Xing, Suprasad V Amari, and Joanne Bechta Dugan. Efficient analysis of multi-state k-out-of-n systems. *Reliability Engineering & System Safety*, 133:95–105, 2015.

- [48] A Satyanarayana and Mark K Chang. Network reliability and the factoring theorem. *Networks*, 13(1):107–120, 1983.
- [49] Ali Muhammad Ali Rushdi and Ahmad Kamal Hassan. An exposition of system reliability analysis with an ecological perspective. *Ecological Indicators*, 63:282–295, 2016.