

The series expansion of the Greens function of a differential operator with block-triangular operator potential

Abstract

Aims/ objectives: In this paper we consider the Sturm-Liouville equation with block-triangular, increasing at infinity operator potential. For him, built a fundamental system of solutions, one of which is decreasing at infinity, and the second is growing. We present asymptotics these solutions. Using the decreasing at infinity of the solutions initial equation and the left equation of the corresponding him, for non-self-adjoint operator generated by such differential expression obtained by series expansion of the Green's function.

Keywords: Operator, block-triangular coefficients, spectrum, Green of function

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1 Introduction

Results turning out in self - adjoint and non-self-adjoint cases differentiate substantially. V. A. Marchenko introduced a notion of generalized spectral function R for a Sturm-Liouville operator with arbitrary complex valued potential on the semiaxis [VM1], [VM2]. The spectral distribution R determines formulas of expansion in eigenfunctions and also allows one to solve the inverse problem of spectral analysis in the non-self-adjoint case. The theory of non-self-adjoint singular differential operators, generated by scalar differential expressions, has been well studied. An overview on the theory of non-self-adjoint singular ordinary differential operators is provided in V.E. Lyantse's Appendix I to the monograph of M.A. Naimark [MN3]. In this regard the papers of M.A. Naimark [MN4], V.E. Lyantse [L5], V.A. Marchenko [VM1], [VM2], F.S. Rofe-Beketov [RB6], J.T. Schwartz [S7], T. Kato [K8] should be noted. In the study of the connection between spectral and oscillation properties of non-self-adjoint operators we have to consider the operators (see [KhR9]), which possess the

property that their discrete spectrum is real. These are the operators with triangular and block-triangular coefficients. For an operator with a triangular matrix potential decaying at infinity which first moment is bounded, due to the inverse scattering problem, the spectral structure was established in [BRB10], [RBZ11]. The theory of equations with block - triangular matrix coefficients is relatively new, the first results were published in 2012. For these operators, in addition, must be carried out property, which is that the discrete spectrum non-self-adjoint coincides with the union of discrete spectra of self-adjoint operators corresponding diagonal elements. For the problem on a finite interval, this condition is always satisfied, however, it can not be performed for the singular non-self-adjoint operators. In [KhR12], [KhR13] for the equation with block-triangular, increasing at infinity potential obtained sufficient conditions under which this property holds, and set the structure of the spectrum of such operators. Here the rate of growth elements, not on the main diagonal, is subordinated to the rate of growth of the diagonal elements. In case of infringement of this condition, the operator can have spectral singularities. In recent works, the author of this article generalized some of these results to equations with block-triangular operator coefficients. For such equations decreasing and increasing at infinity operator solutions is obtained, their asymptotics is set, Green's function and resolvent is built. In this paper, for the Sturm-Liouville equation with a block-triangular, increasing at infinity operator potential we obtained a series expansion of the Greens function.

2 Preliminary Notes

Let H_k , $k = 1, 2, \dots, r$ finite-dimensional or infinite-dimensional separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $|\cdot|$, $\dim H_k \leq \infty$. Denote by $\mathbf{H} = H_1 \oplus H_2 \oplus \dots \oplus H_r$. Element $h \in \mathbf{H}$ will be written in the form $h = \text{col}(h_1, h_2, \dots, h_r)$, where $h_k \in H_k$, $k = \overline{1, r}$, I_k , I - identity operators in H_k and $\mathbf{H}$ accordingly. We denote by $L_2(\mathbf{H}, (0, \infty))$ the Hilbert space of vector-valued functions $y(x)$ with values in $\mathbf{H}$ with inner product

$$\langle y, z \rangle = \int_0^\infty (y(x), z(x)) dx$$

and the corresponding norm $\|\cdot\|$.

Consider the equation with block-triangular operator potential

$$l[y] = -y'' + V(x)y = \lambda y, \quad 0 \leq x < \infty, \quad (2.1)$$

where

$$V(x) = v(x) \cdot I + U(x), \quad U(x) = \begin{pmatrix} U_{11}(x) & U_{12}(x) & \dots & U_{1r}(x) \\ 0 & U_{22}(x) & \dots & U_{2r}(x) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & U_{rr}(x) \end{pmatrix}, \quad (2.2)$$

$v(x)$ is a real scalar function, that $0 < v(x) \rightarrow \infty$ monotonically, as $x \rightarrow \infty$, and it has monotone

absolutely continuous derivative. Also, $U(x)$ is a relatively small perturbation, e. g. $|U(x)| \cdot v^{-1}(x) \rightarrow 0$ as $x \rightarrow \infty$ or $|U|v^{-1} \in L^\infty(\mathbb{R}_+)$. The diagonal blocks $U_{kk}(x)$, $k = \overline{1, r}$ are assumed as bounded self-adjoint operators in H_k , $U_{kl} : H_l \rightarrow H_k$.

In case where

$$v(x) \geq Cx^{2\alpha}, \quad C > 0, \alpha > 1. \quad (2.3)$$

we suppose that coefficients of the equation (2.1) satisfy relations:

$$\int_0^{\infty} |U(t)| \cdot v^{-\frac{1}{2}}(t) dt < \infty \quad (2.4)$$

$$\int_0^{\infty} v'^2(t) \cdot v^{-\frac{3}{2}}(t) dt < \infty, \quad \int_0^{\infty} v''(t) \cdot v^{-\frac{3}{2}}(t) dt < \infty. \quad (2.5)$$

Let us consider the functions

$$\gamma_0(x) = \frac{1}{\sqrt[4]{4v(x)}} \cdot \exp\left(-\int_0^x \sqrt{v(u)} du\right), \quad (2.6)$$

$$\gamma_{\infty}(x) = \frac{1}{\sqrt[4]{4v(x)}} \cdot \exp\left(\int_0^x \sqrt{v(u)} du\right). \quad (2.7)$$

It is easy to see that $\gamma_0(x) \rightarrow 0$, $\gamma_{\infty}(x) \rightarrow \infty$ as $x \rightarrow \infty$. These solutions constitute a fundamental system of solutions of the scalar differential equation

$$-z'' + (v(x) + q(x))z = 0, \quad (2.8)$$

where $q(x)$ determined by a formula (cf. with the monograph [T14])

$$q(x) = \frac{5}{16} \left(\frac{v'(x)}{v(x)} \right)^2 - \frac{1}{4} \frac{v''(x)}{v(x)}. \quad (2.9)$$

In such a way that for all $x \in [0, \infty)$ one has

$$W(\gamma_0, \gamma_{\infty}) := \gamma_0(x) \cdot \gamma'_{\infty}(x) - \gamma'_0(x) \cdot \gamma_{\infty}(x) = 1.$$

In case of $v(x) = x^{2\alpha}$, $0 < \alpha \leq 1$, we suppose that the coefficients of the equation (2.1) satisfy the relation

$$\int_a^{\infty} |U(t)| \cdot t^{-\alpha} dt < \infty, \quad a > 0. \quad (2.10)$$

Now functions $\gamma_0(x, \lambda)$ and $\gamma_{\infty}(x, \lambda)$ are defined as follows

$$\gamma_0(x, \lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha} - \lambda)}} \cdot \exp\left(-\int_a^x \sqrt{u^{2\alpha} - \lambda} du\right)$$

$$\gamma_{\infty}(x, \lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha} - \lambda)}} \cdot \exp\left(\int_a^x \sqrt{u^{2\alpha} - \lambda} du\right)$$

These functions also form a fundamental system of solutions of the scalar differential equation, which

is obtained by replacing $v(x)$ with $v(x) - \lambda$ in the formulas (2.8) and (2.9).

In [KhR12] the asymptotics of the functions $\gamma_0(x, \lambda)$ and $\gamma_\infty(x, \lambda)$ was established as $x \rightarrow \infty$. Here, for an equation with matrix coefficients, and in the further for equations with operator coefficients, the following theorem is proved.

Theorem 2.1. *If for equation (2.1) conditions (2.3)-(2.5) are satisfied for $\alpha > 1$ or condition (2.10) for $0 < \alpha \leq 1$, then the equation has a unique decreasing at infinity operator solution $\Phi(x, \lambda)$, satisfying the conditions*

$$\lim_{x \rightarrow \infty} \frac{\Phi(x, \lambda)}{\gamma_0(x, \lambda)} = I \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\Phi'(x, \lambda)}{\gamma_0'(x, \lambda)} = I. \quad (2.11)$$

Also, there exists increasing at infinity operator solution $\Psi(x, \lambda)$, satisfying the conditions

$$\lim_{x \rightarrow \infty} \frac{\Psi(x, \lambda)}{\gamma_\infty(x, \lambda)} = I \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\Psi'(x, \lambda)}{\gamma_\infty'(x, \lambda)} = I. \quad (2.12)$$

Corollary 2.2. *If $\alpha = 1$, i.e. $v(x) = x^2$, then, under condition (2.10), the solutions $\Phi(x, \lambda)$ and $\Psi(x, \lambda)$ have known asymptotics, as in the quality $\gamma_0(x, \lambda)$ and $\gamma_\infty(x, \lambda)$ you can take the following functions*

$$\gamma_0(x, \lambda) = x^{\frac{\lambda-1}{2}} \cdot \exp\left(-\frac{x^2}{2}\right), \quad \gamma_\infty(x, \lambda) = x^{-\frac{\lambda+1}{2}} \cdot \exp\left(\frac{x^2}{2}\right).$$

3 Greens function

Let the following boundary condition be given at $x = 0$

$$\cos A \cdot y'(0) - \sin A \cdot y(0) = 0, \quad (3.1)$$

where A is block-triangular operator of the same structure as the potential $V(x)$ (2.2) of the differential

equation (2.1), and A_{kk} , $k = \overline{1, r}$ are the bounded self-adjoint operators in H_k , which satisfy the conditions

$$-\frac{\pi}{2} I_k << A_{kk} \leq \frac{\pi}{2} I_k. \quad (3.2)$$

Together with the problem (2.1), (3.1) we consider the separated system

$$l_k[y_k] = -y_k'' + (w(x)I_k + U_{kk}(x))y_k = \lambda y_k, \quad k = \overline{1, r}$$

with the boundary conditions

$$\cos A_{kk} \cdot y_k'(0) - \sin A_{kk} \cdot y_k(0) = 0, \quad k = \overline{1, r}. \quad (3.3)$$

Let L' denote the minimal differential operator generated by differential expression $l[y]$ and the boundary condition (3.1), and let L_k' , $k = \overline{1, r}$ denote the minimal differential operator on $L_2(H, (0, \infty))$ generated by differential expression $l_k[y_k]$ and the boundary conditions (3.3). Taking into account the conditions on coefficients, as well as sufficient smallness of perturbations $U_{kk}(x)$, and conditions (3.2), we conclude that, for every symmetric operator L_k' , $k = \overline{1, r}$, there is a case of limit point at infinity. Hence their self-adjoint extensions L_k are the closures of operators L_k' respectively. The operators L_k are semi-bounded below, and their spectra are discrete. Let L denote the operator extension L' , by requiring that $L_2(\mathbf{H}, (0, \infty))$ - the domain of operator L .

We number the eigenvalues of the operator L in ascending order, taking into account their multiplicities:

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$$

Along with the equation (2.1) we consider the equation

$$l_1[y] = -y'' + V^*(x)y = \lambda y \quad (3.4)$$

($V^*(x)$ is adjoint to the operator $V(x)$). If the space $\mathbf{H}$ is finite-dimensional, then the equation (3.4)

can be rewritten as

$$\tilde{l}[\tilde{y}] = -\tilde{y}'' + \tilde{y}V(x) = \lambda \tilde{y}, \quad (3.5)$$

where $\tilde{y} = (\tilde{y}_1 \ \tilde{y}_2 \dots \tilde{y}_r)$ and the equation is called the left. For operator -functions $Y(x, \lambda)$, $Z(x, \lambda) \in$

$B(\mathbf{H})$ let

$$W\{Z^*, Y\} = Z^{*'}(x, \bar{\lambda})Y(x, \lambda) - Z^*(x, \bar{\lambda})Y'(x, \lambda).$$

If $Y(x, \lambda)$ - operator solution of the equation (2.1), and $Z(x, \lambda)$ - operator solution of equation (3.4),

the Wronskian does not depend on x .

Now we denote $Y(x, \lambda)$ and $Y_1(x, \lambda)$ the solutions of the equations (2.1) and (3.4), respectively, satisfying the initial conditions

$$Y(0, \lambda) = \cos A, \ Y'(0, \lambda) = \sin A, \ Y_1(0, \lambda) = (\cos A)^*, \ Y_1'(0, \lambda) = (\sin A)^*, \ \lambda \in \mathbb{C}. \quad (3.6)$$

Because the operator function $Y_1^*(x, \bar{\lambda})$ satisfies equation

$$-Y_1^{*''}(x, \bar{\lambda}) + Y_1^*(x, \bar{\lambda}) \cdot V(x) = \lambda Y_1^*(x, \bar{\lambda}),$$

the operator function $\tilde{Y}(x, \lambda) =: Y_1^*(x, \bar{\lambda})$ is a solution to the left of the equation

$$-\tilde{Y}''(x, \lambda) + \tilde{Y}(x, \lambda) \cdot V(x) = \lambda \tilde{Y}(x, \lambda) \quad (3.7)$$

and satisfies the initial conditions $\tilde{Y}(0, \lambda) = \cos A$, $\tilde{Y}'(0, \lambda) = \sin A$, $\lambda \in \mathbb{C}$. Operator solutions

of equation (3.4) decreasing and increasing at infinity will be denoted by $\Phi_1(x, \lambda)$, $\Psi_1(x, \lambda)$, and the corresponding solutions of the equation (3.7) denote by $\tilde{\Phi}(x, \lambda)$ and $\tilde{\Psi}(x, \lambda)$. For the system operator solutions $Y(x, \lambda)$, $\tilde{\Phi}(x, \lambda) \in B(\mathbf{H})$ of the equations (2.1) and (3.7), respectively, will take the form of Wronskian $W\{\tilde{\Phi}, Y\} = \tilde{\Phi}'(x, \lambda)Y(x, \lambda) - \tilde{\Phi}(x, \lambda)Y'(x, \lambda)$. Let us designate

$$G(x, t, \lambda) = \begin{cases} Y(x, \lambda) \left(W\{\tilde{\Phi}, Y\} \right)^{-1} \tilde{\Phi}(t, \lambda) & 0 \leq x \leq t \\ -\tilde{\Phi}(x, \lambda) \left(W\{\tilde{Y}, \tilde{\Phi}\} \right)^{-1} \tilde{Y}(t, \lambda) & x \geq t \end{cases}. \quad (3.8)$$

It is proved in [KhR13] that the operator function $G(x, t, \lambda)$ possesses all the classical properties of

the Greens function.

Theorem 3.1. *The operator function $G(x, t, \lambda)$ is the Green's function of the differential operator L , i.e.:*

1. *The function $G(x, t, \lambda)$ is continuous for all values $x, t \in [0, \infty)$;*
2. *For any fixed t , the function $G(x, t, \lambda)$ has a continuous derivative with respect to x on each of the intervals $[0, t)$ and (t, ∞) , and at $x = t$ it has the jump*

$$G_x'(x+0, x, \lambda) - G_x'(x-0, x, \lambda) = -I.$$

3. For a fixed t , the function $G(x, t, \lambda)$ of the variable x is an operator solution of equation (2.1) on each of the intervals $[0, t)$, (t, ∞) , and it satisfies the boundary condition (3.1), and at a fixed x function $G(x, t, \lambda)$ of the variable t is an operator solution of the equation (3.7) on each of the intervals $[0, x)$, (x, ∞) , and it satisfies the boundary condition $(\cos A)^* y'(0) - (\sin A)^* y(0) = 0$.

By the definition (3.8), function $G(x, t, \lambda)$ is meromorphic by parameter λ with the poles coinciding with the eigenvalues of the operator L .

We consider the operator R_λ defined in $L_2(\mathbf{H}, (0, \infty))$ by the relation

$$\begin{aligned} (R_\lambda f)(x) &= \int_0^\infty G(x, t, \lambda) f(t) dt = \\ &= - \int_0^x \Phi(x, \lambda) \left(W \left\{ \tilde{Y}, \Phi \right\} \right)^{-1} \tilde{Y}(t, \lambda) f(t) dt + \int_x^\infty Y(x, \lambda) \left(W \left\{ \tilde{\Phi}, Y \right\} \right)^{-1} \tilde{\Phi}(t, \lambda) f(t) dt \quad (3.9) \end{aligned}$$

Theorem 3.2. The operator R_λ is the resolvent of the operator L .

Since the resolvent R_λ is a meromorphic function with respect to λ , which poles coincide with the eigenvalues of the operator L , the following theorem is true.

Theorem 3.3. If the conditions (2.3)-(2.5) where $\alpha > 1$ or condition (2.10) where $0 < \alpha \leq 1$ are satisfied for the equation (2.1), then the spectrum of the operator L is real, discrete and coincides with the union of spectra of self-adjoint operators L_k , $k = \overline{1, m}$, i.e.

$$\sigma(L) = \bigcup_{k=1}^r \sigma(L_k)$$

4 The series expansion of the Greens function

Let, as before, $Y(x, \lambda)$ the operator solution of equation (2.1) satisfying the initial conditions $Y(0, \lambda) = \cos A$, $Y'(0, \lambda) = \sin A$, and $Z(x, \lambda)$ the operator solution of equation (1) satisfying the initial conditions $Z(0, \lambda) = -\sin A$, $Z'(0, \lambda) = \cos A$. Since operator $Y(x, \lambda)$ and $Z(x, \lambda)$ to form the fundamental system of solutions of the equation (2.1), we conclude that the operator solution $\Phi(x, \lambda)$ of the equation (2.1) can be written as

$$\Phi(x, \lambda) = Y(x, \lambda) A(\lambda) + Z(x, \lambda) B(\lambda)$$

where $A(\lambda) = -W \left\{ \tilde{Z}, \Phi \right\}$, $B(\lambda) = W \left\{ \tilde{Y}, \Phi \right\}$, however

$$\Phi(x, \lambda) = Z(x, \lambda) W \left\{ \tilde{Y}, \Phi \right\} - Y(x, \lambda) W \left\{ \tilde{Z}, \Phi \right\} \quad (4.1)$$

Similarly

$$\tilde{\Phi}(x, \lambda) = W \left\{ \tilde{\Phi}, Z \right\} \tilde{Y}(x, \lambda) - W \left\{ \tilde{\Phi}, Y \right\} \tilde{Z}(x, \lambda). \quad (4.2)$$

Green's function (3.8) can be rewritten as

$$\begin{aligned} G(x, t, \lambda) &= Y(x, \lambda) \left(W \left\{ \tilde{\Phi}, Y \right\} \right)^{-1} W \left\{ \tilde{\Phi}, Z \right\} \tilde{Y}(t, \lambda) + \dots = \\ &= Y(x, \lambda) W \left\{ \tilde{Z}, \Phi \right\} \left(W \left\{ \tilde{Y}, \Phi \right\} \right)^{-1} \tilde{Y}(t, \lambda) + \dots \end{aligned}$$

Here the dots means the entire function on λ . Consider a disk on the complex plane, bounded by the circle C_{R_n} of radius R_n with center at the origin such that, for sufficiently large n , there hold conditions $|\lambda_n| < R_n$ and $\lambda_{n+1} > R_n$. By integrating the function $\frac{G(x, t, \lambda)}{\lambda - z}$ for a given contour, we obtain

$$\frac{1}{2\pi i} \int_{C_{R_n}} \frac{G(x, t, \lambda)}{\lambda - z} d\lambda = G(x, t, z) + \sum_{j=1}^n \text{Res}_{\lambda_j} \left\{ \frac{1}{\lambda - z} Y(x, \lambda) W \left\{ \tilde{Z}, \Phi \right\} \left(W \left\{ \tilde{Y}, \Phi \right\} \right)^{-1} \tilde{Y}(t, \lambda) \right\}.$$

By letting n to infinity, we conclude that

$$G(x, t, z) = - \sum_{j=1}^{\infty} \text{Res}_{\lambda_j} \left\{ \frac{1}{\lambda - z} Y(x, \lambda) W \left\{ \tilde{Z}, \Phi \right\} \left(W \left\{ \tilde{Y}, \Phi \right\} \right)^{-1} \tilde{Y}(t, \lambda) \right\}. \quad (4.3)$$

Lemma 4.1. *If the operators $A(\lambda)$ and $C(\lambda)$ are the entire functions and the operator $B(\lambda)$ has a pole of order r at the point λ_0 , then residue of the operator $A(\lambda) B(\lambda) C(\lambda)$ at λ_0 can be calculated according to the formula:*

$$\begin{aligned} \text{Res}_{\lambda_0} \{A(\lambda) B(\lambda) C(\lambda)\} = \\ = \sum_{k=0}^{r-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} A(\lambda) \Big|_{\lambda=\lambda_0} \sum_{l=0}^{r-(k+1)} \text{Res}_{\lambda_j} \left\{ B(\lambda) (\lambda - \lambda_j)^{k+l} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} C(\lambda) \Big|_{\lambda=\lambda_0} \end{aligned} \quad (4.4)$$

Proof. We write the expansion in the series of the operator functions $A(\lambda)$, $B(\lambda)$, $C(\lambda)$:

$$A(\lambda) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k A_k, \quad \text{where } A_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} A(\lambda) \Big|_{\lambda=\lambda_0},$$

similarly $C(\lambda)$, and finally, $B(\lambda) = \sum_{k=-r}^{\infty} (\lambda - \lambda_0)^k B_k$, where

$$B_{-k} = \text{Res}_{\lambda_0} \left\{ (\lambda - \lambda_0)^{k-1} B(\lambda) \right\}, \quad \text{if } k = \overline{1, r}, \quad \text{and } B_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} B(\lambda) \Big|_{\lambda=\lambda_0}, \quad \text{if } k \geq 0.$$

Multiplying the series, we find the coefficient at $(\lambda - \lambda_0)^{-1}$:

$$\sum_{k=0}^{r-1} A_k \sum_{l=0}^{r-(k+1)} B_{-(k+l+1)} C_l.$$

Hence, formula (4.4). □

Corollary 4.2. *If the $A(\lambda) = I$, that*

$$\text{Res}_{\lambda_0} \{B(\lambda) C(\lambda)\} = \sum_{l=0}^{r-1} \text{Res}_{\lambda_0} \left\{ B(\lambda) (\lambda - \lambda_j)^l \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} C(\lambda) \Big|_{\lambda=\lambda_0}. \quad (4.5)$$

By Lemma 4.1, formula (4.3) takes the form

$$G(x, t, z) = - \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} \left(\frac{1}{\lambda - z} Y(x, \lambda) W \left\{ \tilde{Z}, \Phi \right\} \right) \Big|_{\lambda=\lambda_j} E_{jk}(t, \lambda), \quad (4.6)$$

$$E_{jk}(t, \lambda) = \sum_{l=0}^{r_j-(k+1)} \operatorname{Re} s_{\lambda_j} \left\{ \left(W \left\{ \tilde{Y}, \Phi \right\} \right)^{-1} (\lambda - \lambda_j)^{k+l} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} \tilde{Y}(t, \lambda) \Big|_{\lambda=\lambda_j}$$

Similarly as [AgM15], [RBZ10] and [BRB11] define the normalizing polynomials by formulas

$$N_j(t) = e^{-\lambda_j t} \operatorname{Re} s_{\lambda_j} \left\{ e^{\lambda t} \left(W \left\{ \tilde{Y}, \Phi \right\} \right)^{-1} W \left\{ \tilde{Y}, \Psi \right\} \right\}$$

or

$$N_j(t) = \sum_{k=0}^{r_j-1} \left(\sum_{l=0}^{r_j-(k+1)} \operatorname{Re} s_{\lambda_j} \left\{ \left(W \left\{ \tilde{Y}, \Phi \right\} \right)^{-1} (\lambda - \lambda_j)^{l+k} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} W \left\{ \tilde{Y}, \Psi \right\} \Big|_{\lambda=\lambda_j} \right) \frac{t^k}{k!}$$

Note that

$$\frac{d^k}{dt^k} (N_j(t)) \Big|_{t=0} = \sum_{l=0}^{r_j-(k+1)} \operatorname{Re} s_{\lambda_j} \left\{ \left(W \left\{ \tilde{Y}, \Phi \right\} \right)^{-1} (\lambda - \lambda_j)^{l+k} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} W \left\{ \tilde{Y}, \Psi \right\} \Big|_{\lambda=\lambda_j} \quad (4.7)$$

or

$$\frac{d^k}{dt^k} (N_j(t)) \Big|_{t=0} = \operatorname{Re} s_{\lambda_j} \left\{ \left(W \left\{ \tilde{Y}, \Phi \right\} \right)^{-1} (\lambda - \lambda_j)^k W \left\{ \tilde{Y}, \Psi \right\} \right\}$$

Lemma 4.3. Assume that $\Omega(\lambda)$ is an entire operator function, an $\Omega(\lambda)$ does not return at some point λ_0 and that the operator $\Omega^{-1}(\lambda)$ has a pole of order r at λ_0 . Then

$$\sum_{l=0}^k \operatorname{Re} s_{\lambda_0} \left\{ \Omega^{-1}(\lambda) (\lambda - \lambda_0)^{r-(k+1)+l} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} \Omega(\lambda) \Big|_{\lambda=\lambda_0} = 0, \quad k = 0, 1, \dots, r-1,$$

or

$$\sum_{l=0}^{r-(k+1)} \operatorname{Re} s_{\lambda_0} \left\{ \Omega^{-1}(\lambda) (\lambda - \lambda_0)^{k+l} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} \Omega(\lambda) \Big|_{\lambda=\lambda_0} = 0, \quad k = 0, 1, \dots, r-1. \quad (4.8)$$

The proof, as in [AgM15] and [BRB10] for $r = 1$, it follows from the expansion of the functions $\Omega(\lambda)$ and $\Omega^{-1}(\lambda)$ in the neighborhood at λ_0 :

$$\Omega(\lambda) = \sum_{l=r}^{\infty} \frac{1}{l!} \frac{d^l}{d\lambda^l} \Omega(\lambda) \Big|_{\lambda=\lambda_0} \cdot (\lambda - \lambda_0)^l,$$

$$\Omega^{-1}(\lambda) = \frac{A_{-r}}{(\lambda - \lambda_0)^r} + \frac{A_{-(r-1)}}{(\lambda - \lambda_0)^{r-1}} + \dots + \frac{A_{-1}}{\lambda - \lambda_0} + A_0 + \dots,$$

where $A_{-l} = \operatorname{Re} s_{\lambda_0} \left\{ \Omega^{-1}(\lambda) (\lambda - \lambda_0)^{l-1} \right\}$.

If $\Omega(\lambda) = W(\tilde{Y}, \Phi)$ and λ_j is the pole of multiplicity r_j , then the formula (4.8) takes the form

$$\sum_{l=0}^{r_j-(k+1)} \operatorname{Re} s_{\lambda_j} \left\{ \left(W(\tilde{Y}, \Phi) \right)^{-1} (\lambda - \lambda_0)^{k+l} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} W(\tilde{Y}, \Phi) \Big|_{\lambda=\lambda_j} = 0, \quad k = 0, 1, \dots, r-1 \quad (4.9)$$

Lemma 4.4. *The following formulas hold for all $k = 0, 1, \dots, r_j - 1$*

$$\sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} \frac{d^l}{d\lambda^l} W \{ \tilde{\Phi}, Z \} \Big|_{\lambda=\lambda_j} = \operatorname{Res}_{\lambda_j} \left\{ \left(W \{ \tilde{Y}, \Phi \} \right)^{-1} (\lambda - \lambda_j)^k \right\}, \quad (4.10)$$

$$\sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} \frac{d^l}{d\lambda^l} W \{ \tilde{\Phi}, Y \} \Big|_{\lambda=\lambda_j} = 0, \quad (4.11)$$

$$\sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^l}{d\lambda^l} W \{ \tilde{Y}, \Phi \} \Big|_{\lambda=\lambda_j} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} = 0. \quad (4.12)$$

Formula (4.11), (4.12) similar to the formula (25) [RBZ11], but they are proved for an equation with operator coefficients different way similar to the finite-dimensional case (see [KhR13]).

Theorem 4.5. *Let L' be a minimal differential operator generated by the differential expression (2.1), which coefficients satisfy conditions (2.3), (2.4), (2.5) for $\alpha > 1$ or the condition (2.10) for $0 < \alpha \leq 1$, and the boundary condition (3.1). Let also L be the extension of the operator L' generated by the requirement for functions of the domain of the operator L to belong to $L_2(H_m, (0, \infty))$. Then the Green function of the operator has the form*

$$G(x, t, z) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} \left(\frac{1}{\lambda - z} \Phi(x, \lambda) \right) \Big|_{\lambda=\lambda_j} \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} \frac{d^l}{d\lambda^l} \left(\tilde{\Phi}(t, \lambda) \right) \Big|_{\lambda=\lambda_j} \quad (4.13)$$

Proof. Transform the formula (4.6) for the Green's function $G(x, t, z)$. Consider the interior sum and substitute the values of residues in it according to (4.10):

$$\begin{aligned} & \sum_{l=0}^{r_j-(k+1)} \operatorname{Res}_{\lambda_j} \left\{ \left(W \{ \tilde{Y}, \Phi \} \right)^{-1} (\lambda - \lambda_j)^{k+l} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} \tilde{Y}(t, \lambda) \Big|_{\lambda=\lambda_j} = \\ & = \sum_{l=0}^{r_j-(k+1)} \left(\sum_{s=0}^{r_j-(k+l+1)} \frac{1}{s!} \frac{d^{k+l+s}}{dt^{k+l+s}} N_j(t) \frac{d^s}{d\lambda^s} W \{ \tilde{\Phi}, Z \} \right) \frac{1}{l!} \frac{d^l}{d\lambda^l} \tilde{Y}(t, \lambda). \end{aligned}$$

Making the change $l + s = u$, the last sum takes the form

$$\sum_{l=0}^{r_j-(k+1)} \left(\sum_{u=l}^{r_j-(k+1)} \frac{1}{(u-l)!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \frac{d^{u-l}}{d\lambda^{u-l}} W \{ \tilde{\Phi}, Z \} \right) \frac{1}{l!} \frac{d^l}{d\lambda^l} \tilde{Y}(t, \lambda).$$

By changing summation limits, we obtain

$$\sum_{u=0}^{r_j-(k+1)} \frac{1}{u!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \left(\sum_{l=0}^u C_u^l \frac{d^{u-l}}{d\lambda^{u-l}} W \{ \tilde{\Phi}, Z \} \frac{d^l}{d\lambda^l} \tilde{Y}(t, \lambda) \right) =$$

$$= \sum_{u=0}^{r_j-(k+1)} \frac{1}{u!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \frac{d^u}{d\lambda^u} \left(W \left\{ \tilde{\Phi}, Z \right\} \tilde{Y}(t, \lambda) \right). \quad (4.14)$$

Using the formula (4.2) and proving that

$$\sum_{u=0}^{r_j-(k+1)} \frac{1}{u!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \frac{d^u}{d\lambda^u} \left(W \left\{ \tilde{\Phi}, Y \right\} \tilde{Z}(t, \lambda) \right) = 0, \quad (4.15)$$

the right side of formula (4.14) takes the form

$$\sum_{u=0}^{r_j-(k+1)} \frac{1}{u!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \frac{d^u}{d\lambda^u} \left(\tilde{\Phi}(t, \lambda) \right).$$

By rewriting the left-hand side of formula (4.14) in the form

$$\sum_{u=0}^{r_j-(k+1)} \frac{1}{u!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \sum_{s=0}^u C_u^s \frac{d^{u-s}}{d\lambda^{u-s}} \left(W \left\{ \tilde{\Phi}, Y \right\} \right) \frac{d^s}{d\lambda^s} \tilde{Z}(t, \lambda)$$

and by changing the summation limits, we get

$$\sum_{s=0}^{r_j-(k+1)} \frac{1}{s!} \left(\sum_{u=s}^{r_j-(k+1)} \frac{1}{(u-s)!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \frac{d^{u-s}}{d\lambda^{u-s}} \left(W \left\{ \tilde{\Phi}, Y \right\} \right) \right) \frac{d^s}{d\lambda^s} \tilde{Z}(t, \lambda)$$

Then, making the change $u - s = m$, we obtain

$$\sum_{s=0}^{r_j-(k+1)} \frac{1}{s!} \left(\sum_{m=0}^{r_j-(k+s+1)} \frac{1}{m!} \frac{d^{k+s+m}}{dt^{k+s+m}} N_j(t) \frac{d^m}{d\lambda^m} \left(W \left\{ \tilde{\Phi}, Y \right\} \right) \right) \frac{d^s}{d\lambda^s} \tilde{Z}(t, \lambda).$$

In view of (4.11), the interior sum is equal to 0, which implies (4.15). Hence,

$$\begin{aligned} & \sum_{l=0}^{r_j-(k+1)} \operatorname{Re} s_{\lambda_j} \left\{ \left(W \left\{ \tilde{Y}, \Phi \right\} \right)^{-1} (\lambda - \lambda_j)^{k+l} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} \tilde{Y}(t, \lambda) \Big|_{\lambda=\lambda_j} = \\ & = \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} \frac{d^l}{d\lambda^l} \left(\tilde{\Phi}(t, \lambda) \right) \Big|_{\lambda=\lambda_j} \end{aligned}$$

and the right-hand side of (4.6) takes the form

$$- \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} \left(\frac{1}{\lambda - z} Y(x, \lambda) W \left\{ \tilde{Z}, \Phi \right\} \right) \Big|_{\lambda=\lambda_j} \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} \frac{d^l}{d\lambda^l} \left(\tilde{\Phi}(t, \lambda) \right) \Big|_{\lambda=\lambda_j}$$

Finally, by interchanging the summation limits respect k and l and by using formulas (4.1) and (4.12), we obtain to the formula (4.13) for the Green's function. $\square$

If all eigenvalues λ_j of the operator L are simple, i.e. poles λ_j operator $(W\{\tilde{Y}, \Phi\})^{-1}$ are simple, then the operator $N_j(t)$ is constant and can be calculated by the formula

$$N_j = \operatorname{Res}_{\lambda_j} \left\{ \left(W\{\tilde{Y}, \Phi\} \right)^{-1} \right\} W\{\tilde{Y}, \Psi\} \Big|_{\lambda=\lambda_j}$$

Then

$$\operatorname{Res}_{\lambda_j} \left\{ \left(W\{\tilde{Y}, \Phi\} \right)^{-1} \right\} = N_j W\{\tilde{\Phi}, Z\} \Big|_{\lambda=\lambda_j},$$

therefore in the case if all the eigenvalues the operator L are simple, formula (57) is considerably simplified

$$\begin{aligned} G(x, t, z) &= - \sum_{j=1}^{\infty} \frac{1}{\lambda_j - z} Y(x, \lambda_j) W\{\tilde{Z}, \Phi\} \Big|_{\lambda=\lambda_j} N_j W\{\tilde{\Phi}, Z\} \Big|_{\lambda=\lambda_j} \tilde{Y}(t, \lambda_j) = \\ &= \sum_{j=1}^{\infty} \frac{1}{\lambda_j - z} \Phi(x, \lambda_j) N_j \tilde{\Phi}(t, \lambda_j) \end{aligned}$$

This formula is equivalent to (4.13) for the Green's function that was obtained in [BRB10] in the case of a potential decreasing at infinity and such that its first moment is bounded.

5 Conclusions

In this paper we obtain series expansion of the Green's function of a non-self-adjoint operator with block-triangular operator potential, increasing at infinity.

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