

SOME FIXED POINT THEOREMS IN A_b -METRIC SPACE

Abstract: *In this paper, we introduce the notion of A_b -metric space and to study its basic topological properties. We prove some fixed point theorems under different contractive type and expansive type conditions in the setting of a A_b -metric space. We also prove some fixed point results in partially ordered A_b -metric space. Our results generalize and extend various results in the existing literature.*

Keywords: A_b -metric space, contractive mapping, expansive mapping, fixed point.

1. Introduction

Most of the generalizations for metric fixed point theorems usually start from Banach contraction principle [3]. It is not easy to point out all the generalizations of this principle. The study of expansive mappings is very interesting research area of fixed point theory. In 1984, Wang et.al [36] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [7] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. In 1989, Bakhtin [2] introduced the concept of a b-metric space as a generalization of metric spaces, in which many researchers treated the fixed point theory. In 1993, Czerwik [5-6] extended many results related to the b-metric spaces. In 1994, Matthews [22] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 2013, Shukla [35] generalized both the concept of b-metric and partial metric spaces by introducing the partial b-metric spaces. Gähler [15] claimed that 2-metric space is a generalization of an ordinary metric space. He mentioned in [15] that $d(x, y, z)$ geometrically represents the area of a triangle formed

by the points $x, y, z \in X$ as its vertices. On the other hand, Ha et al. [14] and Sharma [30] found some mathematical flaws in these claims. It was demonstrated in [30] that $d(x, y, z)$ does not always represent the area of a triangle formed by the points $x, y, z \in X$. Ha et al. [18] proved that the 2-metric is not sequentially continuous in each of its arguments whereas an ordinary metric satisfies this property.

In order to carry out meaningful studies of fixed point results, Dhage [8] suggested an improvement in the basic structure of 2-metric space. In 1984, Dhage in his Ph.D. thesis [8] identified condition second as a weakness in Gähler's theory of a 2-metric space. To overcome these problems, he then introduced the concept of a D -metric space. Dhage [10] then studied topological properties of D -metric space in a series of papers. Naidu et al. [28] proved that the concepts of convergent sequences and sequential continuity are not well defined in D -metric spaces. Naidu et al. [27] pointed out some drawbacks in the idea of open balls in D -metric space. In 2003, Mustafa and Sims [26] identified condition third as a weakness in Dhage's theory of D -metric space. The tetrahedral inequality in D -metric has been replaced with the prototypical rectangular inequality adopted by Mustafa and Sims [25] in 2006 and introduced the notion of G -metric space and suggested an important generalization of metric space. Aghajani et al. [1] introduced G_b -metric space and established common fixed point of generalized weak contractive mapping in partially ordered G_b -metric spaces.

Sedghi et al. [31] have introduced D^* -metric spaces which is a probable modification of the definition of D -metric spaces introduced by Dhage [8] and proved some basic properties in D^* -metric spaces, (see [31, 33]). Every G -metric space is a D^* -metric space. The converse, however, is false in general; a D^* -metric space is not necessarily a G -metric space. Sedghi et al. [32] identified condition third of the G -metric space as a peculiar limitation but classified the symmetry condition as a common weakness of both G - and D^* -metric spaces. To overcome these difficulties, Sedghi et al. [32] introduced a new generalized metric space called an S -metric space. The S -metric space is a space with three dimensions. Sedghi et al. [32] asserted that every G -metric is an S -metric, see [32, Remarks 1.3] and [32, Remarks 2.2]. The Example 2.1 and Example 2.2 of Dung et al. [14] shows that this assertion is not correct. Moreover, the class of all S -metrics and the class of all G -metrics are distinct. Souayah et al. [29]

have introduced S_b -metric space and established some fixed point theorems. Very recently, Abbas et al. [23] introduced the notion of A -metric space, which generalization of the S -metric space.

In this paper, we introduce the notion of A_b -metric space and to study its basic topological properties. We also prove some fixed point theorems under different contractive type and expansive type conditions in the setting of a A_b -metric space. Some examples are presented to support the results proved herein. Our results generalize and extend various results in the existing literature.

2. Preliminaries

In 2015, Abbas et al. [23] introduced the notion of A -metric space.

Definition 2.1 (see [23]) Let X be a nonempty set. A mapping $A: X^n \rightarrow [0, +\infty)$ is called an A -metric on X if and only if for all $x_i, a \in X, i = 1, 2, 3, \dots, n$: the following conditions hold:

- (A1). $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \geq 0$,
- (A2). $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = \dots = x_{n-1} = x_n$,
- (A3). $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \leq A(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a)$
 $+ A(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a)$
 $+ A(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) + \dots$
 $+ A(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a)$
 $+ A(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a)]$.

The pair (X, A) is called an A -metric space.

The following is the intuitive geometric example for A -metric spaces.

Example 2.2 (see [23]) Let $X = [1, +\infty)$. Define $A: X^n \rightarrow [0, +\infty)$ by

$$A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = \sum_{i=1}^n \sum_{i < j} |x_i - x_j|$$

for all $x_i \in X, i = 1, 2, \dots, n$.

Example 2.3 (see [23]) Let $X = \mathbb{R}$. Define $A_b: X^n \rightarrow [0, +\infty)$ by

$$\begin{aligned} A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) &= \left| \sum_{i=1}^2 x_i - (n-1)x_1 \right| \\ &+ \left| \sum_{i=2}^3 x_i - (n-2)x_2 \right| + \dots \\ &+ \left| \sum_{i=n-2}^{n-3} x_i - 3x_{n-3} \right| \\ &+ \left| \sum_{i=n-1}^{n-2} x_i - 2x_{n-2} \right| \\ &+ |x_n - x_{n-1}| \end{aligned}$$

for all $x_i \in X, i = 1, 2, \dots, n$.

Lemma 2.4 (see [23]) Let (X, A) be an A -metric space. Then for all $x, y \in X$,

$$A_b(x, x, x, x, \dots, (x)_{n-1}, y) = A_b(y, y, y, y, \dots, (y)_{n-1}, x)$$

Lemma 2.5 (see [23]) Let (X, A) be an A -metric space. Then for all $x, y, z \in X$,

$$\begin{aligned} A_b(x, x, x, x, \dots, (x)_{n-1}, z) &\leq (n-1)A_b(x, x, x, x, \dots, (x)_{n-1}, y) \\ &+ A_b(z, z, z, z, \dots, (z)_{n-1}, y) \end{aligned}$$

and

$$\begin{aligned} A_b(x, x, x, x, \dots, (x)_{n-1}, z) &\leq (n-1)A_b(x, x, x, x, \dots, (x)_{n-1}, y) \\ &+ A_b(y, y, y, y, \dots, (y)_{n-1}, z) \end{aligned}$$

Lemma 2.6 (see [23]) Let (X, A) be an A -metric space. Then $(X \times X, D_A)$ is an A -metric space on $X \times X$, where D_A is given by for all $x_i, y_j \in X, i, j = 1, 2, \dots, n$:

$$\begin{aligned} D_A((x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)) \\ = A(x_1, x_2, x_3, \dots, x_n) + A(y_1, y_2, y_3, \dots, y_n). \end{aligned}$$

Definition 2.7 (see [23]) Let (X, A) be an A -metric space. Then

1. A sequence $\{x_k\}$ is called convergent to x in (X, A) if

$$\lim_{k \rightarrow +\infty} A(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$, we have

$$A(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x) \leq \epsilon$$

and we write $\lim_{k \rightarrow +\infty} x_k = x$.

2. A sequence $\{x_k\}$ is called Cauchy in (X, A) if

$$\lim_{k, m \rightarrow +\infty} A(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k, m \geq n_0$, we have

$$A(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \leq \epsilon.$$

3. (X, A) is said to be complete if every Cauchy sequence in (X, A) is a convergent.

Lemma 2.8 (see [23]) Let (X, A) be an A -metric space. If the sequence $\{x_k\}$ in X converges to x , then x is unique.

Lemma 2.9 (see [23]) Every convergent sequence in A -metric space (X, A, s) is a Cauchy sequence.

3. A_b -Metric Space

We now present the concept of an A_b -metric space and study some of its properties needed in the sequel.

Definition 3.1 Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $A_b: X^n \rightarrow [0, +\infty)$ is called an A_b -metric on X if and only if for all $x_i, a \in X, i = 1, 2, 3, \dots, n$: the following conditions hold:

$$(Ab1). A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \geq 0;$$

$$(Ab2). A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0 \text{ if and only if } x_1 = x_2 = \dots = x_{n-1} = x_n;$$

$$\begin{aligned}
 (\text{Ab3}). \quad & A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \leq s[A_b(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) \\
 & + A_b(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) \\
 & + A_b(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) + \dots \\
 & + A_b(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) \\
 & + A_b(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a)].
 \end{aligned}$$

The pair (X, A_b) is called an A_b -metric space.

Note that the class of A_b -metric spaces is larger than the class of A -metric spaces. Indeed, every A -metric space is an A_b -metric space with $s = 1$. However, the converse is not always true. Also A_b -metric space is an n -dimensional S_b -metric space. Therefore the S_b -metric are special cases of an A_b -metric with $n = 3$.

The following is the intuitive geometric example for A_b -metric spaces.

Example 3.2 Let $X = [1, +\infty)$. Define $A_b: X^n \rightarrow [0, +\infty)$ by

$$A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = \sum_{i=1}^n \sum_{i < j} |x_i - x_j|^2$$

for all $x_i \in X, i = 1, 2, \dots, n$.

Proof: Obviously $A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \geq 0$. If $x_1 = x_2 = x_3 = \dots = x_n$, then $A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0$. Thus, conditions (Ab1) and (Ab2) are satisfied.

We shall show that, for all $x_i, a \in X, i = 1, 2, \dots, n$, (Ab3) is valid. Note that

$$\begin{aligned}
 A_b(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) &= (n-1)|x_1 - a|^2, \\
 A_b(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) &= (n-1)|x_2 - a|^2, \\
 A_b(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) &= (n-1)|x_3 - a|^2 \\
 &\vdots \\
 &\vdots \\
 A_b(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) &= (n-1)|x_{n-1} - a|^2.
 \end{aligned}$$

$$A_b(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a) = (n-1)|x_n - a|^2$$

for all $x_i, a \in X, i = 1, 2, \dots, n$.

Now

$$\begin{aligned} A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) &= \sum_{i=1}^n \sum_{i < j} |x_i - x_j|^2 \\ &= \sum_{i=1}^n \sum_{i < j} |(x_i - a) - (x_j - a)|^2 \\ &\leq \{2(n-1)|x_1 - a|^2 + 2|x_2 - a|^2 + 2|x_3 - a|^2 + \dots \\ &\quad + 2|x_n - a|^2\} + \{2(n-2)|x_2 - a|^2 + 2|x_3 - a|^2 + \dots \\ &\quad + 2|x_n - a|^2\} + \{2(n-3)|x_3 - a|^2 + 2|x_4 - a|^2 + \dots \\ &\quad + 2|x_n - a|^2\} + \dots + \{2(2)|x_{n-2} - a|^2 + 2|x_{n-1} - a|^2 \\ &\quad + 2|x_n - a|^2\} + 2|x_{n-1} - a|^2 + 2|x_n - a|^2 \\ &= 2(n-1)|x_1 - a|^2 + 2(n-1)|x_2 - a|^2 + \dots \\ &\quad + 2(n-1)|x_{n-1} - a|^2 + 2(n-1)|x_n - a|^2 \\ &= 2[A_b(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) \\ &\quad + A_b(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) \\ &\quad + A_b(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) + \dots \\ &\quad + A_b(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) \\ &\quad + A_b(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a)]. \end{aligned}$$

for all $x_i, a \in X, i = 1, 2, \dots, n$. Therefore, (X, A_b) is an A_b -metric space with $s = 2 > 1$.

Example 3.3 Let $X = \mathbb{R}$. Define $A_b: X^n \rightarrow [0, +\infty)$ by

$$A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = \left| \sum_{i=1}^n x_i - (n-1)x_1 \right|^2$$

$$\begin{aligned}
 & + \left| \sum_{i=n}^3 x_i - (n-2)x_2 \right|^2 + \dots \\
 & + \left| \sum_{i=n}^{n-3} x_i - 3x_{n-3} \right|^2 \\
 & + \left| \sum_{i=n}^{n-2} x_i - 2x_{n-2} \right|^2 \\
 & + |x_n - x_{n-1}|^2
 \end{aligned}$$

for all $x_i \in X, i = 1, 2, \dots, n$.

Proof: Obviously $A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \geq 0$. If $x_1 = x_2 = x_3 = \dots = x_n$ then $A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0$. Thus, conditions (Ab1) and (Ab2) are satisfied. We shall show that, for all $x_i, a \in X, i = 1, 2, \dots, n$, (Ab3) is valid. Note that

$$\begin{aligned}
 A_b(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) &= (n-1)|a - x_1|^2, \\
 A_b(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) &= (n-1)|a - x_2|^2, \\
 A_b(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) &= (n-1)|a - x_3|^2 \\
 &\vdots \\
 &\vdots \\
 A_b(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) &= (n-1)|a - x_{n-1}|^2. \\
 A_b(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a) &= (n-1)|a - x_n|^2
 \end{aligned}$$

for all $x_i, a \in X, i = 1, 2, \dots, n$.

Now

$$\begin{aligned}
 A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) &= \left| \sum_{i=n}^2 (x_i - a) - (n-1)(x_1 - a) \right|^2 \\
 &+ \left| \sum_{i=n}^3 (x_i - a) - (n-2)(x_2 - a) \right|^2 + \dots \\
 &+ \left| \sum_{i=n}^{n-2} (x_i - a) - 3(x_{n-3} - a) \right|^2 \\
 &+ \left| \sum_{i=n}^{n-1} (x_i - a) - 2(x_{n-2} - a) \right|^2 \\
 &+ |(x_n - a) - (x_{n-1} - a)|^2
 \end{aligned}$$

$$\begin{aligned}
 &\leq 2 \sum_{i=n}^2 |x_i - a|^2 + 2(n-1)|x_1 - a|^2 \\
 &+ 2 \sum_{i=n}^3 |x_i - a|^2 + 2(n-2)|x_2 - a|^2 + \dots \\
 &+ 2 \sum_{i=n}^{n-2} |x_i - a|^2 + 2(3)|x_{n-3} - a|^2 \\
 &+ 2 \sum_{i=n}^{n-1} |x_i - a|^2 + 2(2)|x_{n-2} - a|^2 \\
 &+ 2|x_n - a|^2 + 2|x_{n-1} - a|^2 \\
 &= 2|x_n - a|^2 + 2|x_{n-1} - a|^2 + \dots + 2|x_2 - a|^2 \\
 &+ 2(n-1)|x_1 - a|^2 + 2|x_n - a|^2 + 2|x_{n-1} - a|^2 \\
 &+ \dots + 2|x_3 - a|^2 + 2(n-2)|x_2 - a|^2 + \dots \\
 &+ 2|x_n - a|^2 + 2|x_{n-1} - a|^2 + 2(3)|x_{n-3} - a|^2 \\
 &+ |x_n - a|^2 + |x_{n-1} - a|^2 + 2(2)|x_{n-2} - a|^2 \\
 &+ 2|x_n - a|^2 + 2|x_{n-1} - a|^2 \\
 &= 2(n-1)|x_n - a|^2 + 2(n-1)|x_{n-1} - a|^2 \\
 &+ 2(n-1)|x_{n-2} - a|^2 + \dots \\
 &+ 2(n-1)|x_3 - a|^2 + 2(n-1)|x_2 - a|^2 \\
 &+ 2(n-1)|x_1 - a|^2 \\
 &= 2(n-1)\{|x_1 - a|^2 + |x_2 - a|^2 + \dots + |x_n - a|^2\} \\
 &= 2[A_b(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) \\
 &+ A_b(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) \\
 &+ A_b(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) + \dots \\
 &+ A_b(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) \\
 &+ A_b(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a)].
 \end{aligned}$$

for all $x_i, a \in X, i = 1, 2, \dots, n$. Therefore, (X, A_b) is an A_b -metric space with $s = 2 > 1$.

Lemma 3.4 Let (X, A_b) be an A_b -metric space with $s \geq 1$. Then for all $x, y \in X$,

$$A_b(x, x, x, x, \dots, (x)_{n-1}, y) \leq sA_b(y, y, y, y, \dots, (y)_{n-1}, x)$$

Proof For all $x, y \in X$, applying (Ab3), we have

$$\begin{aligned} A_b(x, x, x, \dots, (x)_{n-1}, y) &\leq s[A_b(x, x, x, x, \dots, (x)_{n-1}, x) \\ &\quad + A_b(x, x, x, x, \dots, (x)_{n-1}, x) \\ &\quad + A_b(x, x, x, x, \dots, (x)_{n-1}, x) + \dots \\ &\quad + (A_b(x, x, x, x, \dots, (x)_{n-1}, x))_{n-1} \\ &\quad + A_b(y, y, y, y, \dots, (y)_{n-1}, x)] \\ &= sA_b(y, y, y, y, \dots, (y)_{n-1}, x) \end{aligned}$$

Lemma 3.5 Let (X, A_b) be an A_b -metric space with $s \geq 1$. Then for all $x, y, z \in X$,

$$\begin{aligned} A_b(x, x, x, x, \dots, (x)_{n-1}, z) &\leq s[(n-1)A_b(x, x, x, x, \dots, (x)_{n-1}, y) \\ &\quad + A_b(z, z, z, z, \dots, (z)_{n-1}, y)] \end{aligned}$$

and

$$\begin{aligned} A_b(x, x, x, x, \dots, (x)_{n-1}, z) &\leq s[(n-1)A_b(x, x, x, x, \dots, (x)_{n-1}, y) \\ &\quad + sA_b(y, y, y, y, \dots, (y)_{n-1}, z)] \end{aligned}$$

Proof For all $x, y, z \in X$, applying (Ab3), we have

$$\begin{aligned} A_b(x, x, x, \dots, (x)_{n-1}, z) &\leq s[A_b(x, x, x, x, \dots, (x)_{n-1}, y) \\ &\quad + A_b(x, x, x, x, \dots, (x)_{n-1}, y) \\ &\quad + A_b(x, x, x, x, \dots, (x)_{n-1}, y) + \dots \end{aligned}$$

$$\begin{aligned}
 & + (A_b(x, x, x, x, \dots, (x)_{n-1}, y))_{n-1} \\
 & + A_b(z, z, z, z, \dots, (z)_{n-1}, y)] \\
 & = s[(n-1)A_b(x, x, x, x, \dots, (x)_{n-1}, y) \\
 & + A_b(z, z, z, z, \dots, (z)_{n-1}, y)]
 \end{aligned}$$

which implies that

$$\begin{aligned}
 A_b(x, x, x, x, \dots, (x)_{n-1}, z) & \leq s[(n-1)A_b(x, x, x, x, \dots, (x)_{n-1}, y) \\
 & + A_b(z, z, z, z, \dots, (z)_{n-1}, y)]
 \end{aligned}$$

Using Lemma 3.4, we have

$$\begin{aligned}
 A_b(x, x, x, x, \dots, (x)_{n-1}, z) & \leq s[(n-1)A_b(x, x, x, x, \dots, (x)_{n-1}, y) \\
 & + sA_b(y, y, y, y, \dots, (y)_{n-1}, z)].
 \end{aligned}$$

Lemma 3.6 Let (X, A_b) be an A_b -metric space with $s \geq 1$. Then $(X \times X, D_{A_b})$ is an A_b -metric space with $s \geq 1$ on $X \times X$, where D_{A_b} is given by for all $x_i, y_j \in X, i, j = 1, 2, \dots, n$:

$$\begin{aligned}
 D_{A_b}((x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)) \\
 = A_b(x_1, x_2, x_3, \dots, x_n) + A_b(y_1, y_2, y_3, \dots, y_n).
 \end{aligned}$$

Proof For all $x_i, y_j \in X, i, j = 1, 2, \dots, n$; we have

$$\begin{aligned}
 & A_b(x_1, x_2, x_3, \dots, x_n) + A_b(y_1, y_2, y_3, \dots, y_n) \geq 0 \\
 \Rightarrow & D_{A_b}((x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)) \geq 0.
 \end{aligned}$$

Also

$$\begin{aligned}
 & D_{A_b}((x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)) = 0 \\
 \Leftrightarrow & A_b(x_1, x_2, x_3, \dots, x_n) + A_b(y_1, y_2, y_3, \dots, y_n) = 0
 \end{aligned}$$

$$\Leftrightarrow A_b(x_1, x_2, x_3, \dots, x_n) = 0 \text{ and } A_b(y_1, y_2, y_3, \dots, y_n) = 0$$

$$\Leftrightarrow x_1 = x_2 = x_3 = \dots = x_n \text{ and } y_1 = y_2 = y_3 = \dots = y_n$$

$$\Leftrightarrow (x_1, y_1) = (x_2, y_2) = (x_3, y_3) = \dots = (x_n, y_n).$$

Consider

$$\begin{aligned} & D_{A_b}((x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)) \\ &= A_b(x_1, x_2, x_3, \dots, x_n) + A_b(y_1, y_2, y_3, \dots, y_n) \\ &\leq s[A_b(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) + A_b(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) \\ &\quad + A_b(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) + \dots \\ &\quad + A_b(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) + A_b(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a)] \\ &\quad + s[A_b(y_1, y_1, y_1, \dots, (y_1)_{n-1}, b) + A_b(y_2, y_2, y_2, \dots, (y_2)_{n-1}, b) \\ &\quad + A_b(y_3, y_3, y_3, \dots, (y_3)_{n-1}, b) + \dots \\ &\quad + A_b(y_{n-1}, y_{n-1}, y_{n-1}, \dots, (y_{n-1})_{n-1}, b) + A_b(y_n, y_n, y_n, \dots, (y_n)_{n-1}, b)] \\ &= s[A_b(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) + A_b(y_1, y_1, y_1, \dots, (y_1)_{n-1}, b) \\ &\quad + s[A_b(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) + A_b(y_2, y_2, y_2, \dots, (y_2)_{n-1}, b)] \\ &\quad + s[A_b(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) + A_b(y_3, y_3, y_3, \dots, (y_3)_{n-1}, b)] + \dots \\ &\quad + s[A_b(x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) + A_b(y_{n-1}, y_{n-1}, \dots, (y_{n-1})_{n-1}, b)] \\ &\quad + s[A_b(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a) + A_b(y_n, y_n, y_n, \dots, (y_n)_{n-1}, b)] \\ &= s[D_{A_b}((x_1, y_1), (x_1, y_1), (x_1, y_1), \dots, (a, b)) \\ &\quad + D_{A_b}((x_2, y_2), (x_2, y_2), (x_2, y_2), \dots, (a, b)) + \dots \\ &\quad + D_{A_b}((x_n, y_n), (x_n, y_n), (x_n, y_n), \dots, (a, b))] \end{aligned}$$

Thus,

$$\begin{aligned}
 & D_{A_b}((x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)) \\
 & \leq s [D_{A_b}((x_1, y_1), (x_1, y_1), (x_1, y_1), \dots, (a, b)) \\
 & + D_{A_b}((x_2, y_2), (x_2, y_2), (x_2, y_2), \dots, (a, b)) + \dots \\
 & + D_{A_b}((x_n, y_n), (x_n, y_n), (x_n, y_n), \dots, (a, b))]
 \end{aligned}$$

Hence $(X \times X, D_{A_b})$ is an A_b -metric space with $s \geq 1$ on $X \times X$.

Remark 3.7

(a). If we put $s = 1$, then we have

$$\begin{aligned}
 D_A((x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)) &= A(x_1, x_2, x_3, \dots, x_n) \\
 &+ A(y_1, y_2, y_3, \dots, y_n).
 \end{aligned}$$

Then $(X \times X, D_A)$ is an A -metric space on $X \times X$.

(b). If we put $n = 3$, then we have

$$D_{A_b}((x_1, y_1), (x_2, y_2), (x_3, y_3)) = S_b(x_1, x_2, x_3) + S_b(y_1, y_2, y_3)$$

Then $(X \times X, D_{A_b})$ is an S_b -metric space on $X \times X$.

(c). If we put $n = 3, s = 1$, then we have

$$D_A((x_1, y_1), (x_2, y_2), (x_3, y_3)) = S(x_1, x_2, x_3) + S(y_1, y_2, y_3)$$

Then $(X \times X, D_A)$ is an S -metric space on $X \times X$.

Note also that the following implications hold.

$$\begin{array}{ccccccc}
 G\text{-metric space} & \Rightarrow & D^*\text{-metric space} & \Rightarrow & S\text{-metric space} & \Rightarrow & A\text{-metric space} \\
 \Downarrow & & & & \Downarrow & & \Downarrow \\
 G_b\text{-metric space} & \Rightarrow & & & S_b\text{-metric space} & \Rightarrow & A_b\text{-metric space}
 \end{array}$$

Definition 3.8 Let (X, A_b, s) be an A_b -metric space. Then

(1). A sequence $\{x_k\}$ is called convergent to x in (X, A_b) if

$$\lim_{k \rightarrow +\infty} A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$, we have

$$A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x) \leq \epsilon$$

and we write $\lim_{k \rightarrow +\infty} x_k = x$.

(2). A sequence $\{x_k\}$ is called Cauchy in (X, A_b) if

$$\lim_{k, m \rightarrow +\infty} A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k, m \geq n_0$, we have

$$A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \leq \epsilon.$$

(3). (X, A_b) is said to be complete if every Cauchy sequence in (X, A_b) is a convergent.

Lemma 3.9 Let (X, A_b) be an A_b -metric space with $s \geq 1$. If the sequence $\{x_k\}$ in X converges to x , then x is unique.

Proof On the contrary, assume that $\{x_k\}$ converges to x and y . Then given $\epsilon > 0$, there exist $n_1, n_2 \in \mathbb{N}$ such that for all $k \geq n_1$, we have

$$A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x) < \frac{\epsilon}{2s^2(n-1)}$$

and for all $k \geq n_2$, we have

$$A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, y) < \frac{\epsilon}{2s^2}.$$

Choose $n_0 = \max\{n_1, n_2\}$, therefore for all $k \geq n_0$, we have

$$A_b(x, x, x, x, \dots, (x)_{n-1}, y) \leq s[A_b(x, x, x, x, \dots, (x)_{n-1}, x_k)]$$

$$\begin{aligned}
 & +A_b(x, x, x, x \dots, (x)_{n-1}, x_k) \\
 & +A_b(x, x, x, x \dots, (x)_{n-1}, x_k) + \dots \\
 & +\left(A_b(x, x, x, x \dots, (x)_{n-1}, x_k)\right)_{n-1} \\
 & +A_b(y, y, y, y, \dots, (y)_{n-1}, x_k)] \\
 & = s(n-1)A_b(x, x, x, x \dots, (x)_{n-1}, x_k) \\
 & +sA_b(y, y, y, y, \dots, (y)_{n-1}, x_k)
 \end{aligned}$$

Hence from Lemma 3.4, we have

$$\begin{aligned}
 A_b(x, x, x, x \dots, (x)_{n-1}, y) & \leq s^2(n-1)A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x) \\
 & +s^2A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, y) \\
 & < s^2(n-1) \times \frac{\epsilon}{2s^2(n-1)} + s^2 \times \frac{\epsilon}{2s^2} \\
 & = \epsilon.
 \end{aligned}$$

Since ϵ is arbitrary, we have $A_b(x, x, x, x \dots, (x)_{n-1}, y) = 0$ and so $x = y$.
Establishing the uniqueness of $\{x_k\}$.

Lemma 3.10 Every convergent sequence in A_b -metric space (X, A_b) is a Cauchy sequence.

Proof Let $\{x_k\}$ be convergent in (X, A_b) . Let $\lim_{k \rightarrow +\infty} x_k = x$. Then given $\epsilon > 0$, there exist $n_1, n_2 \in \mathbb{N}$ such that for all $k \geq n_1$, we have

$$A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x) < \frac{\epsilon}{2s(n-1)}$$

and for all $m \geq n_2$, we have

$$A_b(x_m, x_m, x_m, x_m, \dots, (x_m)_{n-1}, y) < \frac{\epsilon}{2s}.$$

Choose $n_0 = \max\{n_1, n_2\}$, therefore for all $k, m \geq n_0$, we have

$$A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \leq s(n-1)A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x)$$

$$\begin{aligned}
 & +sA_b(x_m, x_m, x_m, x_m, \dots, (x_m)_{n-1}, x) \\
 & < s(n-1) \times \frac{\epsilon}{2s(n-1)} + s \times \frac{\epsilon}{2s} \\
 & = \epsilon.
 \end{aligned}$$

This implies that $\{x_k\}$ is a Cauchy sequence.

Lemma 3.11 Let (X, A_b) be an A_b -metric space with $s \geq 1$. If $\lim_{k \rightarrow +\infty} x_k = x$ and $\lim_{k \rightarrow +\infty} y_k = y$, then

$$\begin{aligned}
 \frac{1}{s^2} A_b(x, x, x, x, \dots (x)_{n-1}, y) & \leq \lim_{k \rightarrow +\infty} \inf A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, y_k) \\
 & \leq \lim_{k \rightarrow +\infty} \sup A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, y_k) \\
 & \leq s^2 A_b(x, x, x, x, \dots (x)_{n-1}, y)
 \end{aligned}$$

In particular, if $y_k = y$ is constant, then

$$\begin{aligned}
 \frac{1}{s^2} A_b(x, x, x, x, \dots (x)_{n-1}, y) & \leq \lim_{k \rightarrow +\infty} \inf A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, y) \\
 & \leq \lim_{k \rightarrow +\infty} \sup A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, y) \\
 & \leq s^2 A_b(x, x, x, x, \dots (x)_{n-1}, y)
 \end{aligned}$$

Proof Using (Ab3), we have

$$\begin{aligned}
 A_b(x, x, x, x, \dots (x)_{n-1}, y) & \leq s(n-1)A_b(x, x, x, x, \dots (x)_{n-1}, x_k) \\
 & \quad +sA_b(y, y, y, y, \dots (y)_{n-1}, x_k) \\
 & \leq s(n-1)A_b(x, x, x, x, \dots (x)_{n-1}, x_k) \\
 & \quad +s^2(n-1)A_b(y, y, y, y, \dots (y)_{n-1}, y_k) \\
 & \quad +s^2A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, y_k) \\
 & \leq s^2(n-1)A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, x) \\
 & \quad +s^3(n-1)A_b(y_k, y_k, y_k, y_k, \dots (y_k)_{n-1}, y)
 \end{aligned}$$

$$+s^2 A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, y_k)$$

On the other hand, we have

$$\begin{aligned} A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, y_k) &\leq s(n-1)A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, x) \\ &\quad +sA_b(y_k, y_k, y_k, y_k, \dots (y_k)_{n-1}, x) \\ &\leq s(n-1)A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, x) \\ &\quad +s^2(n-1)A_b(y_k, y_k, y_k, y_k, \dots (y_k)_{n-1}, y) \\ &\quad +s^2 A_b(x, x, x, x, \dots (x)_{n-1}, y) \end{aligned}$$

Taking the lower limit as $k \rightarrow +\infty$, in the first inequality and the upper limit as $k \rightarrow +\infty$ in the second inequality we obtain

$$\begin{aligned} \frac{1}{s^2} A_b(x, x, x, x, \dots (x)_{n-1}, y) &\leq \lim_{k \rightarrow +\infty} \inf A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, y_k) \\ &\leq \lim_{k \rightarrow +\infty} \sup A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, y_k) \\ &\leq s^2 A_b(x, x, x, x, \dots (x)_{n-1}, y) \end{aligned}$$

If $y_k = y$, then using (Ab3), we have

$$\begin{aligned} A_b(x, x, x, x, \dots, (x)_{n-1}, y) &\leq s(n-1)A_b(x, x, x, x, \dots, (x)_{n-1}, x_k) \\ &\quad +sA_b(y, y, y, y, \dots, (y)_{n-1}, x_k) \\ &\leq s^2(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x) \\ &\quad +s^2 A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, y) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, y) &\leq s(n-1)A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x) \\ &\quad +sA_b(y, y, y, y, \dots, (y)_{n-1}, x) \\ &\leq s(n-1)A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x) \end{aligned}$$

$$+s^2 A_b(x, x, x, \dots, (x)_{n-1}, y)$$

Taking the lower limit as $k \rightarrow +\infty$, in the first inequality and the upper limit as $k \rightarrow +\infty$ in the second inequality we obtain

$$\begin{aligned} \frac{1}{s^2} A_b(x, x, x, x, \dots (x)_{n-1}, y) &\leq \lim_{k \rightarrow +\infty} \inf A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, y) \\ &\leq \lim_{k \rightarrow +\infty} \sup A_b(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, y) \\ &\leq s^2 A_b(x, x, x, x, \dots (x)_{n-1}, y) \end{aligned}$$

Definition 3.12 The A_b -metric space (X, A_b) is said to be bounded if there exists a constant $r > 0$ such that $A_b(x, x, x, \dots, x, y) \leq r$ for all $x, y \in X$. Otherwise, X is unbounded.

Definition 3.13 Given a point x_0 in A_b -metric space (X, A_b) and a positive real number r , the set $B(x_0, r) = \{y \in X : A_b(y, y, y, \dots, y, x_0) < r\}$ is called an open ball centered at x_0 with radius r .

The set $\overline{B(x_0, r)} = \{y \in X : A_b(y, y, y, \dots, y, x_0) \leq r\}$ is called a closed ball centered at x_0 with radius r .

Definition 3.14 A subset G in A_b -metric space (X, A_b) is said to be an open set if for each $x \in G$ there exists an $r > 0$ such that $B(x, r) \subset G$. A subset $F \subset X$ is called closed if $X \setminus F$ is open.

Definition 3.15 The A_b -metric space (X, A_b) is said to be bounded if there exists a constant $r > 0$ such that $A_b(x, x, x, \dots, x, y) \leq r$ for all $x, y \in X$. Otherwise, X is unbounded.

Definition 3.16 Let (X, A_b) be an A_b -metric space with $s \geq 1$. A map $f: X \rightarrow X$ is said to be contraction if there exists a constant $\lambda \in [0, 1)$ such that

$$A_b(fx^1, fx^2, fx^3, \dots, fx^n) \leq \lambda A_b(x^1, x^2, x^3, \dots, x^n)$$

for all $x^1, x^2, x^3, \dots, x^n \in X$. In case

$$A_b(fx^1, fx^2, fx^3, \dots, fx^n) < A_b(x^1, x^2, x^3, \dots, x^n)$$

for all $x^1, x^2, x^3, \dots, x^n \in X$, T is called contractive mapping.

Definition 3.17 Let (X, A_b) be an A_b -metric space with $s \geq 1$. A map $f: X \rightarrow X$ is said to be expansive mapping if there exists $\lambda > 1$ such that

$$A_b(fx^1, fx^2, fx^3, \dots, fx^n) \geq \lambda A_b(x^1, x^2, x^3, \dots, x^n)$$

for all $x^1, x^2, x^3, \dots, x^n \in X$.

4. Fixed Point Theorems for Contractive mapping

We begin with a simple but a useful lemma.

Lemma 4.1 Let (X, A_b) be an A_b -metric space with $s \geq 1$ and $\{x_k\}$ be a sequence in (X, A_b) such that

$$A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \leq \lambda A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) \quad (1)$$

where $\lambda \in [0, \frac{1}{s^2})$ and $k = 1, 2, \dots$. Then $\{x_k\}$ is a Cauchy sequence in (X, A_b, s) .

Proof For $k = 1, 2, \dots$, we get by induction

$$\begin{aligned} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) &\leq \lambda A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) \\ &\leq \lambda^2 A_b(x_{k-2}, x_{k-2}, x_{k-2}, \dots, (x_{k-2})_{n-1}, x_{k-1}) \\ &\vdots \\ &\leq \lambda^k A_b(x_0, x_0, x_0, \dots, (x_0)_{n-1}, x_1) \end{aligned} \quad (2)$$

Let $m > k$. It follows that

$$\begin{aligned} &A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \\ &\leq s[(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &\quad + A_b(x_m, x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+1})] \\ &\leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &\quad + s^2 A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_m) \\ &\leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &\quad + s^3 [(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+2}) \\ &\quad + A_b(x_m, x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+2})] \\ &\leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \end{aligned}$$

$$\begin{aligned}
& +s^3(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+2}) \\
& +s^4A_b(x_{k+2}, x_{k+2}, x_{k+2}, \dots, (x_{k+2})_{n-1}, x_m)] \\
& \leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
& +s^3(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+2}) \\
& +s^5[(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+3}) \\
& +A_b(x_m, x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+3})] \\
& \leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
& +s^3(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+2}) \\
& +s^5(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+3}) \\
& +s^7(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+4}) + \dots \\
& +s^{2m-2k-3}(n-1)A_b(x_{m-2}, x_{m-2}, x_{m-2}, \dots, (x_{m-2})_{n-1}, x_{m-1}) \\
& +s^{2m-2k-2}A_b(x_{m-1}, x_{m-1}, x_{m-1}, \dots, (x_{m-1})_{n-1}, x_m) \\
& \leq (n-1)[s\lambda^k + s^3\lambda^{k+1} + s^5\lambda^{k+2} + s^7\lambda^{k+3} + \dots + s^{2m-2k-3}\lambda^{m-2} \\
& +s^{2m-2k-2}\lambda^{m-1}] \times A_b(x_0, x_0, x_0, x_0, \dots, (x_0)_{n-1}, x_1) \\
& = (n-1)s\lambda^k[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \dots + s^{2m-2k-4}\lambda^{m-k-2} \\
& +s^{2m-2k-3}\lambda^{m-k-1}] \times A_b(x_0, x_0, x_0, x_0, \dots, (x_0)_{n-1}, x_1) \\
& \leq (n-1)s\lambda^k[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \dots] \\
& \times A_b(x_0, x_0, x_0, x_0, \dots, (x_0)_{n-1}, x_1) \\
& \leq (n-1)\frac{s\lambda^k}{1-\lambda s^2}A_b(x_0, x_0, x_0, x_0, \dots, (x_0)_{n-1}, x_1) \tag{3}
\end{aligned}$$

Since $\lambda s^2 < 1$. Assume that $A_b(x_0, x_0, x_0, x_0, \dots, (x_0)_{n-1}, x_1) > 0$. By taking limit as $k, m \rightarrow +\infty$ in above inequality we get

$$\lim_{k, m \rightarrow +\infty} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) = 0.$$

Therefore, $\{x_k\}$ is a Cauchy sequence in X . Also, if $A_b(x_0, x_0, x_0, x_0, \dots, (x_0)_{n-1}, x_1) = 0$, then $A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) = 0$ for all $m > k$ and hence $\{x_k\}$ is a Cauchy sequence in X .

Theorem 4.2 Let (X, A_b) be a complete A_b -metric space with $s \geq 1$ and $f: X \rightarrow X$ be a continuous mapping satisfy

$$A_b(fx^1, fx^2, fx^3, \dots, fx^n) \leq \psi[A_b(x^1, x^2, x^3, \dots, x^n)] \quad (4)$$

for all $x^1, x^2, x^3, \dots, x^n \in X$, where $\psi: [0, +\infty) \rightarrow [0, +\infty)$ is an increasing function such that $\lim_{k \rightarrow \infty} \psi^k(t) = 0$ for each fixed $t > 0$. Then f has a unique fixed point in X .

Proof Let $x \in X$ and $\epsilon > 0$. Let m be a natural number such that $\psi^m(\epsilon) < \frac{\epsilon}{2s^2(n-1)}$.

Let $F = f^m$ and $x_k = F^k(x)$ for $k \in \mathbb{N}$. Then for all $x, y \in X$ and $\varphi = \psi^m$, we have

$$\begin{aligned} A_b(Fx, Fx, Fx, \dots, (Fx)_{n-1}, Fy) &\leq \psi^m(A_b(x, x, x, \dots, (x)_{n-1}, y)) \\ &= \varphi(A_b(x, x, x, \dots, (x)_{n-1}, y)) \end{aligned} \quad (5)$$

Hence, for each $k \in \mathbb{N}$, $A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_k) \rightarrow 0$ as $k \rightarrow \infty$. Therefore, let k be such that

$$A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) < \frac{\epsilon}{2s^2(n-1)} \quad (6)$$

Let's define the ball $B(x_k, \epsilon)$ such that for every $z \in B(x_k, \epsilon) = \{y \in X \mid A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, y)\}$. Note that $x_k \in B(x_k, \epsilon)$, therefore $B(x_k, \epsilon) \neq \emptyset$. Hence, for all $z \in B(x_k, \epsilon)$ we have

$$\begin{aligned} A_b(Fx_k, Fx_k, Fx_k, \dots, (Fx_k)_{n-1}, Fz) &\leq \varphi(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, z)) \\ &\leq \varphi(\epsilon) = \psi^m(\epsilon) < \frac{\epsilon}{2s^2(n-1)} < \frac{\epsilon}{2s^2} \end{aligned} \quad (7)$$

Thus,

$$\begin{aligned} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, Fz) &\leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &\quad + sA_b(Fz, Fz, Fz, \dots, (Fz)_{n-1}, x_{k+1}) \\ &\leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &\quad + s^2A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, Fz) \\ &< s^2(n-1) \times \frac{\epsilon}{2s^2(n-1)} + s^2 \times \frac{\epsilon}{2s^2} = \epsilon. \end{aligned} \quad (8)$$

Hence, F maps $B(x_k, \epsilon)$ to it self. Since $x_k \in B(x_k, \epsilon)$, we have $Fx_k \in B(x_k, \epsilon)$. By repeating this process we get

$$F^l x_k \in B(x_k, \epsilon) \text{ for all } l \in \mathbb{N}.$$

That is, $x_q \in B(x_k, \epsilon)$ for all $q \geq k$. Hence

$$A_b(x_l, x_l, x_l, \dots, (x_l)_{n-1}, x_q) < \epsilon \text{ for all } q, l > k. \quad (9)$$

Therefore $\{x_k\}$ is a Cauchy sequence and by the completeness of X , there exists $u \in X$ such that $x_k \rightarrow u$ as $k \rightarrow \infty$. Moreover, $u = \lim_{k \rightarrow \infty} x_{k+1} = \lim_{k \rightarrow \infty} x_k = F(u)$. Thus, F has u as a fixed point.

Now we prove the uniqueness of the fixed point for F . Since $\varphi(t) = \psi^m(t) < t$ for any $t > 0$, let u and v be two fixed points of F .

$$\begin{aligned} A_b(u, u, u, u, \dots, (u)_{n-1}, v) &\leq A_b(Fu, Fu, Fu, Fu, \dots, (Fu)_{n-1}, Fv) \\ &\leq \psi^m[A_b(u, u, u, u, \dots, (u)_{n-1}, v)] \\ &= \varphi[A_b(u, u, u, u, \dots, (u)_{n-1}, v)] \\ &\leq A_b(u, u, u, u, \dots, (u)_{n-1}, v) \end{aligned}$$

This implies that $A_b(u, u, u, u, \dots, (u)_{n-1}, v) = 0 \Rightarrow u = v$ and hence, F has unique fixed point in X .

Theorem 4.3 Let $(X, \leq)$ be a partially ordered set and suppose that there exists a A_b -metric A_b on X such that (X, A_b, s) is a complete A_b -metric space. Let $f: X \rightarrow X$ be an continuous non-decreasing mapping with respect to $\leq$ such that there exists an element $x_0 \in X$ with $x_0 \leq f x_0$. Suppose also that for all $x^1, x^2, x^3, \dots, x^n \in X$ with $x^1 \leq x^2 \leq x^3 \leq \dots \leq x^n$;

$$\begin{aligned} A_b(fx^1, fx^2, fx^3, \dots, fx^n) \\ \leq \alpha \prod_{i=1}^n A_b(x^i, x^i, x^i, \dots, fx^i) ([A_b(x^1, x^2, x^3, \dots, x^n)]^{n-1})^{-1} \\ + \beta A_b(x^1, x^2, x^3, \dots, x^n) \end{aligned} \quad (10)$$

where $\alpha + s^2\beta < 1$. Then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Starting with the given $x_0 \in X$, put $x_k = f^k x_0 = f x_{k-1}$. Since $x_0 \leq f x_0$ and f is an increasing function, we obtain by induction that

$$x_0 \leq f x_0 \leq f^2 x_0 \leq \dots \dots \dots \leq f^k x_0 \leq f^{k+1} x_0 \leq \dots \dots \dots \quad (11)$$

Considering the sequence and using (10), we have

$$\begin{aligned}
 & A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
 &= A_b(fx_{k-1}, fx_{k-1}, fx_{k-1}, \dots, (fx_{k-1})_{n-1}, fx_k) \\
 &\leq \alpha \{A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, fx_{k-1}) \\
 &\quad \times A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, fx_{k-1}) \times \dots \\
 &\quad \times (A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, fx_{k-1}))_{n-1} \\
 &\quad \times A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, fx_k)\} \\
 &\quad \times ([A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k)]^{n-1})^{-1} \\
 &\quad + \beta A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) \\
 &= \alpha (A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k))^{n-1} \\
 &\quad \times A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
 &\quad \times ([A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k)]^{n-1})^{-1} \\
 &\quad + \beta A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k)
 \end{aligned}$$

The last inequality gives

$$\begin{aligned}
 & A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
 &\leq \lambda A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k)
 \end{aligned} \tag{12}$$

for all $k \in \mathbb{N} \cup \{0\}$ where $\lambda = \frac{\beta}{1-\alpha} < \frac{1}{s^2}$. By Lemma 4.1, $\{x_k\}$ is a Cauchy sequence in X . Since X is a complete A_b -metric space, there exists $x^* \in X$ such that $\lim_{k \rightarrow +\infty} x_k \rightarrow x^*$. If f is A_b -continuous, then

$$fx^* = f(\lim_{k \rightarrow +\infty} x_k) = \lim_{k \rightarrow +\infty} fx_k = \lim_{k \rightarrow +\infty} x_{k+1} = x^* \tag{13}$$

This implies that x^* is a fixed point of f .

Finally, suppose that the set of fixed points of f is well ordered. Assume, to the contrary, that u and v are two distinct fixed points of f .

$$\begin{aligned}
 & A_b(fu, fu, fu, fu, \dots, (fu)_{n-1}, fv) \\
 &\leq \alpha \{A_b(u, u, u, u, \dots, (u)_{n-1}, fu) \\
 &\quad \times A_b(u, u, u, u, \dots, (u)_{n-1}, fu) \times \dots \\
 &\quad \times (A_b(u, u, u, u, \dots, (u)_{n-1}, fu))_{n-1}
 \end{aligned}$$

$$\begin{aligned} & \times A_b(v, v, v, v, \dots, (v)_{n-1}, fv)\} \\ & \times ([u, u, u, u, \dots, (u)_{n-1}, v]^{n-1})^{-1} \\ & + \beta A_b(u, u, u, u, \dots, (u)_{n-1}, v) \end{aligned}$$

Thus, we get

$$A_b(u, u, u, u, \dots, (u)_{n-1}, v) \leq \beta A_b(u, u, u, u, \dots, (u)_{n-1}, v)$$

a contradiction. Hence, f has a unique fixed point. The converse is trivial.

5. Results under Geraghty-type conditions

In 1973, Geraghty [17] proved a fixed point result, generalizing the Banach contraction principle. Several authors proved later various results using Geraghty-type conditions. Fixed point results of this kind in b -metric spaces were obtained by Dukić et al. in [11].

Following [11], for a real number $s \geq 1$ let $\mathcal{F}_s$ denote the class of all functions $\beta : [0, \infty) \rightarrow [0, \frac{1}{s})$ satisfying the following condition:

$$\beta(t_k) \rightarrow \frac{1}{s} \text{ as } k \rightarrow \infty \text{ implies } t_k \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Now, we have the following fixed point theorem in A_b -metric space.

Theorem 5.1 Let $(X, \leq)$ be a partially ordered set and suppose that there exists a A_b -metric A_b on X such that (X, A_b, s) is a A_b -complete A_b -metric space. Let $f: X \rightarrow X$ be a non-decreasing mapping with respect to $\leq$ such that there exists an element $x_0 \in X$ with $x_0 \leq fx_0$. Suppose also that for all $x^1, x^2, x^3, \dots, x^n \in X$ with $x^1 \leq x^2 \leq x^3 \leq \dots \leq x^n$,

$$sA_b(fx^1, fx^2, fx^3, \dots, fx^n) \leq \beta(A_b(x^1, x^2, x^3, \dots, x^n))M(x^1, x^2, x^3, \dots, x^n) \quad (14)$$

where

$$M(x^1, x^2, x^3, \dots, x^n) = \max \left\{ A_b(x^1, x^2, x^3, \dots, x^n), \frac{\prod_{i=1}^n A_b(x^i, x^i, x^i, \dots, fx^i)}{1 + (A_b(x^1, x^2, x^3, \dots, x^n))^{n-1}} \right\}$$

and $\beta \in \mathcal{F}_s$. Then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Starting with the given $x_0 \in X$, put $x_k = f^k x_0 = f x_{k-1}$. Since $x_0 \leq f x_0$ and f is an increasing function, we obtain by induction that

$$x_0 \leq f x_0 \leq f^2 x_0 \leq \cdots \leq f^k x_0 \leq f^{k+1} x_0 \leq \cdots$$

Considering the sequence and using (14), we have

$$\begin{aligned} sA_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &= sA_b(fx_{k-1}, fx_{k-1}, fx_{k-1}, \dots, (fx_{k-1})_{n-1}, fx_k) \\ &\leq \beta(A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k)) \\ &\quad \times M(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) \\ &\leq \frac{1}{s} A_b(x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) \\ &\leq A_b(x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) \end{aligned} \quad (15)$$

because

$$\begin{aligned} M(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) \\ &= \max\{A_b(x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k), \\ &\quad \frac{(A_b(x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k))^{n-1} A_b(x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})}{1 + (A_b(x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k))^{n-1}}\} \\ &= A_b(x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) \end{aligned}$$

Therefore, the sequence $\{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})\}$ is decreasing. Then there exists $r \geq 0$ such that

$$\lim_{k \rightarrow +\infty} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) = r. \quad (16)$$

We claim $r = 0$.

Suppose that $r > 0$. Then letting $k \rightarrow +\infty$, from (15) we have

$$\frac{1}{s} r \leq sr \leq \lim_{k \rightarrow +\infty} \beta(A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k)) r \leq r.$$

So we have $\lim_{k \rightarrow +\infty} \beta(A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k)) \geq \frac{1}{s}$ and since $\beta \in \mathcal{F}_s$, we deduce that $\lim_{k \rightarrow +\infty} A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) = 0$, which is contradiction. Hence,

$$\lim_{k \rightarrow +\infty} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) = r = 0. \quad (17)$$

Now

$$A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)$$

$$\begin{aligned}
&\leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
&+ sA_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+1}) \\
&\leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
&+ s^2A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_m) \\
&\leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
&+ s^3(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1}) \\
&+ s^3A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1}) \\
&\leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
&+ s^2(n-1)\beta(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)) \\
&\quad \times M(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \\
&+ s^3A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1})
\end{aligned} \tag{18}$$

Letting $k, m \rightarrow +\infty$ in the above inequality and applying (17), we have

$$\begin{aligned}
&\lim_{k, m \rightarrow +\infty} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \\
&\leq s^2(n-1) \lim_{k, m \rightarrow +\infty} (A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)) \\
&\quad \times \lim_{k, m \rightarrow +\infty} M(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)
\end{aligned} \tag{19}$$

Here

$$\begin{aligned}
&A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \\
&\leq M(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \\
&= \max\{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m), \\
&\quad \frac{(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_k))^{n-1} \times A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_m)}{1 + (A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m))^{n-1}}\}
\end{aligned}$$

Letting $k, m \rightarrow +\infty$ in the above inequality, we get

$$\begin{aligned}
&\lim_{k, m \rightarrow +\infty} M(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \\
&= \lim_{k, m \rightarrow +\infty} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)
\end{aligned} \tag{20}$$

Hence from (19), we have

$$\lim_{k, m \rightarrow +\infty} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)$$

$$\begin{aligned} &\leq s^2(n-1) \lim_{k,m \rightarrow +\infty} \beta(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)) \\ &\quad \times \lim_{k,m \rightarrow +\infty} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \end{aligned} \quad (21)$$

Now we claim that $\lim_{k,m \rightarrow +\infty} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) = 0$. If, to the contrary, $\lim_{k,m \rightarrow +\infty} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \neq 0$, then we get

$$\lim_{k,m \rightarrow +\infty} \beta(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)) \geq \frac{1}{s^2(n-1)} \geq \frac{1}{s^2}$$

which is contradiction. Consequently, $\{x_k\}$ is a A_b -Cauchy sequence in X . Since (X, A_b, s) is A_b -complete, the sequence $\{x_k\}$ A_b -converges to some $z \in X$, that is, $\lim_{k \rightarrow +\infty} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^*) = 0$. Now we show that x^* is a fixed point of f . If f is A_b -continuous, then

$$fx^* = f(\lim_{k \rightarrow +\infty} x_k) = \lim_{k \rightarrow +\infty} fx_k = \lim_{k \rightarrow +\infty} x_{k+1} = x^*$$

This implies that x^* is a fixed point of f .

Finally, suppose that the set of fixed points of f is well ordered. Assume, to the contrary, that u and v are two distinct fixed points of f .

$$\begin{aligned} &sA_b(u, u, u, \dots, (u)_{n-1}, v) \\ &= sA_b(fu, fu, fu, fu, \dots, (fu)_{n-1}, fv) \\ &\leq \beta(A_b(u, u, u, \dots, (u)_{n-1}, v))M(u, u, u, \dots, (u)_{n-1}, v) \\ &= \beta(A_b(u, u, u, \dots, (u)_{n-1}, v))A_b(u, u, u, \dots, (u)_{n-1}, v) \end{aligned} \quad (22)$$

because

$$\begin{aligned} M(u, u, u, \dots, (u)_{n-1}, v) &= \max\{A_b(u, u, u, \dots, (u)_{n-1}, v), \\ &\quad \frac{(A_b(u, u, u, \dots, (u)_{n-1}, fu))^{n-1} \times A_b(v, v, v, \dots, (v)_{n-1}, fv)}{1 + (A_b(u, u, u, \dots, (u)_{n-1}, v))^{n-1}}\} \\ &= \max\{A_b(u, u, u, \dots, (u)_{n-1}, v), 0\} \\ &= A_b(u, u, u, \dots, (u)_{n-1}, v) \end{aligned}$$

Then we get

$$\beta(A_b(u, u, u, \dots, (u)_{n-1}, v)) \geq \frac{1}{s}$$

a contradiction. Hence, f has a unique fixed point. The converse is trivial.

Note that the continuity of f in Theorem 5.1 can be replaced by certain property of the space itself.

Theorem 5.2 Under the hypotheses of Theorem 5.1, without the A_b -continuity assumption on f , assume that whenever $\{x_k\}$ is a non-decreasing sequence in X such that $x_k \rightarrow x^*$, one has $x_k \leq x^*$ for all $k \in \mathbb{N}$. Then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Repeating the proof of Theorem 5.1, we construct an increasing sequence $\{x_k\}$ in X such that $x_k \rightarrow x^* \in X$. Using the assumption on X we have $x_k \leq x^*$. Now, we show that $x^* = fx^*$. By (5.1) and Lemma 3.11, we have

$$\begin{aligned} & s^2 \left[\frac{1}{s^2} A_b(x^*, x^*, x^*, x^*, \dots, (x^*)_{n-1}, fx^*) \right] \\ & \leq s^2 \lim_{k \rightarrow +\infty} \sup A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, fx^*) \\ & = s^2 \lim_{k \rightarrow +\infty} \sup A_b(fx_k, fx_k, fx_k, \dots, (fx_k)_{n-1}, fx^*) \\ & \leq s \lim_{k \rightarrow +\infty} \sup \beta(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^*)) \\ & \lim_{k \rightarrow +\infty} \sup M(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^*) \end{aligned} \quad (23)$$

where

$$\begin{aligned} & \lim_{k \rightarrow +\infty} M(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^*) \\ & = \lim_{k \rightarrow +\infty} \max\{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^*), \\ & \quad \frac{(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, fx_k))^{n-1} \times A_b(x^*, x^*, x^*, \dots, (x^*)_{n-1}, fx^*)}{1 + (A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^*))^{n-1}}\} \\ & = \lim_{k \rightarrow +\infty} \max\{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^*), \\ & \quad \frac{(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}))^{n-1} \times A_b(x^*, x^*, x^*, \dots, (x^*)_{n-1}, fx^*)}{1 + (A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^*))^{n-1}}\} \\ & = 0 \end{aligned}$$

Therefore we deduce that $A_b(x^*, x^*, x^*, x^*, \dots, (x^*)_{n-1}, fx^*) = 0$, hence $fx^* = x^*$. The proof of uniqueness is the same as in Theorem 1.

6. Fixed Point Theorems for Expansive Mapping

In this section, first we prove some fixed point theorem satisfying expansive condition by considering surjective self-mapping in the context of A_b - metric space

Theorem 6.1 Let (X, A_b, s) be a complete A_b -metric space with the coefficient $s \geq 1$. Assume that the mapping $T: X \rightarrow X$ is surjection and satisfies

$$A_b(Tx^1, Tx^2, Tx^3, \dots, Tx^{n-1}, Tx^n) \geq \lambda A_b(x^1, x^2, x^3, \dots, x^{n-1}, x^n) \quad (24)$$

$\forall x^1, x^2, x^3, \dots, x^{n-1}, x^n \in X$, where $\lambda > s^2$. Then T has a fixed point.

Proof Let $x_0 \in X$, since T is surjection on X , then there exists $x_1 \in X$ such that $x_0 = Tx_1$. By continuing this process, we get

$$x_k = Tx_{k+1}, \forall k \in \mathbb{N} \cup \{0\}. \quad (25)$$

If $A_b(x_{m-1}, x_{m-1}, x_{m-1}, \dots, x_{m-1}, x_m) = 0$ for some m , then $x_{m-1} = x_m$ and $x_m \in T^{-1}(x_{m-1})$ implies $Tx_m = x_{m-1} = x_m$ and so x_m is a fixed point of T . Without loss of generality, we can suppose that $A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, x_{k-1}, x_k) > 0$, that is, $x_k \neq x_{k-1}$ for every k . Consider from (24), we have

$$\begin{aligned} A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, x_k) &= A_b(Tx_k, Tx_k, Tx_k, \dots, Tx_{k+1}) \\ &\geq \lambda A_b(x_k, x_k, x_k, \dots, x_{k+1}) \end{aligned}$$

and so

$$\begin{aligned} A_b(x_k, x_k, x_k, \dots, x_{k+1}) &\leq \frac{1}{\lambda} A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, x_k) \\ &= h A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, x_k) \end{aligned} \quad (26)$$

for all $k \in \mathbb{N}$, where $h = \frac{1}{\lambda} < \frac{1}{s^2}$. By Lemma 4.1, $\{x_k\}$ is a Cauchy sequence in X . Since X is a complete A_b -metric space, there exists $x^* \in X$ such that $x_n \rightarrow x^*$ as $k \rightarrow +\infty$. Now since T is surjective map. So there exists a point p in X such that $x^* = Tp$. Consider from (24), we have

$$\begin{aligned} A_b(x_k, x_k, x_k, \dots, x_k, x^*) &= A_b(Tx_{k+1}, Tx_{k+1}, Tx_{k+1}, \dots, Tx_{k+1}, Tp) \\ &\geq \lambda A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, p) \end{aligned}$$

Taking limit as $k \rightarrow +\infty$ in the above inequality, we get

$$0 = \lim_{k \rightarrow +\infty} A_b(x_k, x_k, x_k, \dots, x_k, x^*) \geq \lambda \lim_{n \rightarrow \infty} A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, x_{k+1}, p)$$

which implies that

$$\lim_{n \rightarrow +\infty} A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, x_{k+1}, p) = 0. \quad (27)$$

Thus $x_{n+1} \rightarrow p$ as $k \rightarrow +\infty$. By Lemma 3.9, we get $x^* = p$. Hence x^* is a fixed point of T .

Finally, assume $x^* = y^*$ is also another fixed point of T . From (24), we get

$$\begin{aligned} A_b(x^*, x^*, x^*, x^*, \dots, (x^*)_{n-1}, y^*) &= A_b(Tx^*, Tx^*, Tx^*, Tx^*, \dots, (Tx^*)_{n-1}, y^*) \\ &\geq \lambda A_b(x^*, x^*, x^*, x^*, \dots, (x^*)_{n-1}, y^*) \end{aligned}$$

This is true only when $A_b(x^*, x^*, x^*, x^*, \dots, (x^*)_{n-1}, y^*) = 0$. So $x^* = y^*$. Hence T has a unique fixed point in X .

Corollary 6.2 Let (X, A_b) be a complete A_b -metric space with the coefficient $s \geq 1$ and $T: X \rightarrow X$ be a surjection. Suppose that there exist a positive integer k and a real number $\lambda > s^2$ such that

$$A_b(T^k(x^1), T^k(x^2), \dots, T^k(x^{n-1}), T^k(x^n)) \geq \lambda A_b(x^1, x^2, \dots, x^{n-1}, x^n) \quad (28)$$

$\forall x^1, x^2, x^3, \dots, x^{n-1}, x^n \in X$. Then T has a fixed point.

Proof From Theorem 6.1, T^k has a fixed point x^* . But $T^k(Tx^*) = T(T^k x^*) = Tx^*$, So Tx^* is also a fixed point of T^k . Hence $Tx^* = x^*$, x^* is a fixed point of T . Since the fixed point of T is also fixed point of T^k , the fixed point of T is unique.

Now, motivated by the work in Jain et al. [15-16], we give the following.

Let $\Psi_{\mathcal{B}}^L$ denote the class of those function $\mathcal{B}: (0, \infty) \rightarrow (L^2, \infty)$ which satisfy the condition $\mathcal{B}(t_k) \rightarrow (L^2)^+ \Rightarrow k \rightarrow 0$, as $k \rightarrow \infty$, where $L > 0$.

Theorem 6.3 Let (X, A_b) be a complete A_b -metric space with $s \geq 1$. Assume that the mapping $T: X \rightarrow X$ is surjection and satisfies

$$A_b(Tx^1, Tx^2, Tx^3, \dots, Tx^n) \geq \mathcal{B}(A_b(x^1, x^2, x^3, \dots, x^n)) A_b(x^1, x^2, x^3, \dots, x^n) \quad (29)$$

$\forall x^1, x^2, x^3, \dots, x^n \in X$, where $\mathcal{B} \in \Psi_{\mathcal{B}}^s$. Then T has a fixed point.

Proof Pick $x_0 \in X$. Since T is surjective, choose $x_1 \in X$ such that $Tx_1 = x_0$. Inductively, we can define a sequence $\{x_k\} \subset X$ such that $x_k = Tx_{k+1}$, $\forall n \in \mathbb{N} \cup \{0\}$. If $A_b(x_{m-1}, x_{m-1}, x_{m-1}, \dots, x_{m-1}, x_m) = 0$ for some m , then $x_{m-1} = x_m$ and $x_m \in T^{-1}(x_{m-1})$ implies $Tx_m = x_{m-1} = x_m$ and so x_m is a fixed point of T .

Without loss of generality, we can suppose that $A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, x_{k-1}, x_n) > 0$, that is, $x_k \neq x_{k-1}$ for every k . Consider

$$\begin{aligned} A_b(x_{k-1}, x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) \\ = A_b(Tx_k, Tx_k, Tx_k, \dots, (Tx_k)_{n-1}, Tx_{k+1}) \end{aligned}$$

Now by (29) and definition of the sequence

$$\begin{aligned} A_b(x_{k-1}, x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) \\ = A_b(Tx_k, Tx_k, Tx_k, Tx_k, \dots, (Tx_k)_{n-1}, Tx_{k+1}) \\ \geq B(A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})) \\ \times A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ \geq s^2 A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ \geq A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \end{aligned} \quad (30)$$

Thus the sequence $\{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})\}$ is a decreasing sequence in $\mathbb{R}^+$ and so there exists $\epsilon \geq 0$ such that

$$\lim_{n \rightarrow \infty} A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) = \delta \quad (31)$$

Let us prove that $\delta = 0$. Suppose to the contrary that $\delta > 0$. By (30) we can deduce that

$$\begin{aligned} s^2 \frac{A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k)}{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})} &\geq \frac{A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k)}{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})} \\ &\geq B(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})) \geq s^2 \end{aligned} \quad (32)$$

By taking limit as $k \rightarrow +\infty$ in the above inequality, we have

$$\lim_{k \rightarrow +\infty} B(A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})) = s^2 \quad (33)$$

Hence by definition of B , we have

$$\delta = \lim_{k \rightarrow +\infty} A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) = 0 \quad (34)$$

which is a contradiction. Hence $\epsilon = 0$. Now, we shall show that for $k, m \in \mathbb{N}$,

$$\lim_{k, m \rightarrow +\infty} \sup A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) = 0 \quad (35)$$

Suppose to the contrary that $\lim_{k, m \rightarrow +\infty} \sup A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) > 0$.

By (29), we have

$$\begin{aligned}
 & A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \\
 &= A_b(Tx_{k+1}, Tx_{k+1}, Tx_{k+1}, \dots, (Tx_{k+1})_{n-1}, Tx_{m+1}) \\
 &\geq \mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1})) \\
 &\quad \times A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1})
 \end{aligned}$$

That is,

$$\begin{aligned}
 & \frac{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)}{\mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1}))} \\
 & \geq A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1}) \quad (36)
 \end{aligned}$$

From (Ab3), we have

$$\begin{aligned}
 A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) &\leq s[A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
 &\quad + A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
 &\quad + A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) + \dots \\
 &\quad + (A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}))_{n-1} \\
 &\quad + A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+1})] \\
 &= s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
 &\quad + sA_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+1}) \\
 &\leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
 &\quad + s^2[A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1}) \\
 &\quad + A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1}) \\
 &\quad + A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1}) + \dots \\
 &\quad + (A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1}))_{n-1} \\
 &\quad + A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1})] \\
 &= s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
 &\quad + s^2(n-1)A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1}) \\
 &\quad + s^2A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1}) \\
 &\leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})
 \end{aligned}$$

$$\begin{aligned}
 & +s^2(n-1)A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1}) \\
 & +s^2 \frac{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)}{\mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1}))} \quad (37)
 \end{aligned}$$

The last inequality gives,

$$\begin{aligned}
 & \left(1 - \frac{s^2}{\mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1}))}\right) A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) \\
 & \leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
 & +s^2(n-1)A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1})
 \end{aligned}$$

That is,

$$\begin{aligned}
 & \left(1 - \frac{s^2}{\mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1}))}\right) \\
 & \leq s(n-1) \frac{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})}{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)} \\
 & +s^2(n-1) \frac{A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1})}{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)}
 \end{aligned}$$

Thus,

$$\begin{aligned}
 & \left(1 - \frac{s^2}{\mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1}))}\right)^{-1} \\
 & \leq \frac{1}{(n-1)} \left(\frac{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)}{sA_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) + s^2 A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1})} \right) \quad (38)
 \end{aligned}$$

By taking limit as $k, m \rightarrow +\infty$ in the above inequality, since

$$\lim_{k, m \rightarrow +\infty} \sup A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) > 0$$

and

$$\begin{aligned}
 \epsilon = 0 & = \lim_{k \rightarrow +\infty} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\
 & = \lim_{m \rightarrow +\infty} A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1}),
 \end{aligned}$$

then we obtain

$$\lim_{k, m \rightarrow +\infty} \left(1 - \frac{s^2}{\mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1}))}\right)^{-1} = +\infty \quad (39)$$

which implies that

$$\lim_{k, m \rightarrow +\infty} \sup \mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1})) = (s^2)^+ \quad (41)$$

and so by definition of $\mathcal{B}$, we have

$$\lim_{k,m \rightarrow +\infty} \sup A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1}) = 0$$

which is a contradiction. Hence,

$$\lim_{k,m \rightarrow +\infty} \sup A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) = 0. \quad (42)$$

and so, $\{x_k\}$ is a Cauchy sequence. Since X is a complete A_b -metric space, the sequence $\{x_k\}$ in X converges to $x^* \in X$. so that

$$\lim_{k \rightarrow +\infty} A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^*) = 0. \quad (43)$$

As T is surjective, so there exists $p \in X$ such that $x^* = Tp$. Let us prove that $x^* = p$. Suppose to the contrary that $x^* \neq p$. Then by (29), we have

$$\begin{aligned} & A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^*) \\ &= A_b(Tx_{k+1}, Tx_{k+1}, Tx_{k+1}, \dots, (Tx_{k+1})_{n-1}, Tp) \\ &\geq \mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, p)) \\ &\quad \times A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, p) \end{aligned} \quad (44)$$

By Taking limit as $k \rightarrow +\infty$ in the above inequality and applying Lemma 3.11, we obtain

$$\begin{aligned} 0 &= \lim_{k \rightarrow +\infty} A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^*) \\ &\geq \lim_{k \rightarrow +\infty} \mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, p)) \\ &\quad \times \lim_{k \rightarrow \infty} A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, p) \\ &\geq \frac{1}{s} \lim_{k \rightarrow +\infty} \mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, p)) \\ &\quad \times \lim_{k \rightarrow \infty} A_b(x, x, x, \dots, (x)_{n-1}, p) \end{aligned} \quad (45)$$

and hence,

$$\lim_{k \rightarrow +\infty} \mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, p)) = 0$$

which is a contradiction. Indeed,

$$\lim_{n \rightarrow +\infty} \mathcal{B}(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, p)) \geq s^2.$$

Since $\mathcal{B}(t) > s^2$ for all $t \in [0, +\infty)$, therefore $x^* = p$. Hence $x^* = Tp = Tx^*$.

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