

Original Research Article

Out-of-plane Equilibrium Points in the Photogravitational Restricted Four-Body Problem with Oblateness

ABSTRACT

The restricted four-body problem consists of an infinitesimal particle which is moving under the Newtonian gravitational attraction of three massive bodies, called primaries. The three bodies are moving in circles around their common centre of mass fixed at the origin of the coordinate system, according to the solution of Lagrange where they are always at the vertices of an equilateral triangle. The fourth body does not affect the motion of the primaries. We consider that the primary body P_1 is dominant and is a source of radiation while the other two small primaries P_2 and P_3 modelled as oblate spheroids have equal masses and oblateness coefficients. The out of equilibrium points of the problem are sought and we found that such critical points exist. These points lie in the xz –plane in symmetrical positions with respect to xy –plane. We investigate numerically the effects of radiation and oblateness on the positions of out-of-plane equilibrium points, their stability, as well as the regions allowed to motion of the infinitesimal body as determined by the zero velocity surface. It is found that radiation and oblateness have strong effects on the positions of the critical points. We examined the stability of these points and found that the out of plane equilibrium points are unstable.

Keywords: Equilibrium points, Restricted four-body problem, Radiation, Oblateness, Zero velocity curves.

1. INTRODUCTION

One of the most extensively studied problems in Celestial Mechanics is the restricted three-body problem (R3BP, for short), where one of the bodies is considered massless, so it does not affect

the motion of the other two that move in a Keplerian orbit. The restricted four-body problem (R4BP) is perhaps the simplest model after the R3BP and a natural generalization of it. It deals with the motion of an infinitesimal particle under the Newtonian gravitational attraction of three bodies, called primaries, whose trajectories are the solution of the three Newtonian body problem. In the same direction as several restricted three body problems have given much insight about real three body problems, the study of special type of four-body problem can be of help to understand the dynamical behavior of stellar systems through analytical or numerical studies. In the case of four-body problems, even the case of a restricted one can impose greater difficulties from an analytical point of view.

The classical restricted four-body problem may be generalized to include different types of effects such as radiation pressure force, Poynting-Robertson drag, oblateness coefficient, Coriolis and centrifugal forces, variation of the masses of the primaries, etc. The importance of radiation influence on celestial bodies has been recognized by many scientists, especially in connection with the formation of concentrations of interplanetary and interstellar dust or grains in planetary and binary star systems, as well as the perturbations on artificial satellites (see, e.g., Kalvouridis [1], Bewick et al. [2] and references therein). A modification of the restricted problem of four bodies exists when consider that some or all the primaries exert light pressure on the infinitesimal fourth body. This is the so called photogravitational problem of four bodies.

The existence of out-of-plane equilibrium points in the R3BP was first studied by Radzievskii [3] who primarily studied the cases of Sun-Planet-particle and Galaxy Kernel-Sun particle. He found two equilibrium points, L_6 and L_7 , on the Oxy plane symmetrical with respect to the orbital plane. Many authors, following Radzievskii [3,4] model, studied the existence and the stability of out-of-plane equilibrium points in the R3BP with or without oblateness of primaries

(Perezhogin [5]; Kunitsyn and Perezhogin [6]; Mignard [7]; Simmons et al. [8]; Ragos and Zagouras [9,10]; Singh [11]). Recently Douskos and Markellos [12] investigated the out-of-plane equilibrium points in the R3BP with oblateness. They used the three-dimensional equations of motion written by Sharma and SubbaRao [13] and expanded the equations of motion in power series solutions about the oblateness coefficient of the second primary.

It is known that in the general problem of three bodies there is a particular solution in which the bodies are placed at the vertices of an equilateral triangle, each moving in a Keplerian orbit. This is well known, and was first studied by Lagrange in 1772. He found a solution where the three bodies remain at constant distances from each other while they revolved around their common center of mass. There has been recently an increased interest for this model because of some observational evidence. In this paper we study a R4BP in which the three primaries are in the Lagrange equilateral triangle configuration. This model was used, among others, by Pedersen [14], Brumberg [15], Simo [16], Majorona [17], Alvarez-Ramirez and Vidal [18], Baltagiannis and Papadakis [19,20], Ceccaroni and Biggs [21], Burgos-Garcia and Delgado [22], Papadouris and Papadakis [23,24], Kumari and Kushvah [25], Singh and Vincent [26], etc. The model has been used for practical applications by some researchers, among others, Ceccaroni and Biggs, [21], Baltagiannis and Papadakis [20], and references therein. However, to the present authors' knowledge the equilibrium points out of the orbital plane of the primaries for this model has remained an open problem to date.

Some studies have considered some or all the primaries as source of radiation in the R4BP (Lagrange configuration). Papadouris and Papadakis [23] investigated the problem when only the first primary body radiates. They studied the existence, location and stability of the equilibrium points, on and out of the orbital plane for two cases depending on the masses of the primaries,

namely two equal masses and all the primary bodies with equal masses. They observed that existence and stability of the equilibrium points depends on the mass parameters of the primaries and the radiation factor. Besides, closed regions where the motion of the infinitesimal body can be trapped are given. In Papadouris and Papadakis [24], the authors studied the simplest symmetric periodic solutions of the problem for the case of two equal masses. They studied the effect of radiation on the distribution of the periodic orbits, their stability, as well as the evolution of the families as the radiation parameter varies. Poincare surface of section of the problem as the dominant primary radiates are illustrated. Series of horizontal-and-vertical-critical periodic orbits by varying the mass m_3 and typical critical asymmetric solutions of the problem were also given. Using a similar form of their equations of motion, Singh and Vincent [26] studied the out-of-plane equilibrium points of the problem by taking all primaries as radiation sources with two of the bodies having the same radiation and mass value. They noted that radiation factors have noticeable effects on the locations of the critical points and the zero velocity curves. The stability of these points is found to be unstable.

In considering the first two bigger primaries as oblate spheroids, Kumari and Kushvah [25] studied the stability regions of equilibrium points in the R4BP in such configuration. They established eight equilibrium points, two collinear and six non-collinear and observed that the stability regions of the equilibrium points expanded due to the presence of oblateness coefficients and various values of Jacobi constant, C . The allowed regions of motion of the infinitesimal body as well as the regions of the basins of attraction for the equilibrium points were given.

In the present paper we deal with the equilibrium points which exist out of the orbital plane in the case when two of the small primaries modelled as oblate spheroids are of equal mass and

oblateness coefficients and the dominant primary body is a source of radiation. Also, the infinitesimal body is assumed to have no influence on the motion of the primaries. This model could be used to examine the existence of a dust particle in the Sun, Jupiter, Saturn, Spacecraft system.

Specifically, the paper is organized as follows: Section 2 provides the equations of motion of the considered model-problem. In Section 3, the existence and location of the out-of-plane equilibrium points are investigated while Section 4 is devoted to the surfaces and curves of zero velocity. The regions of allowed motion as determined by the zero velocity surface and corresponding equipotential curves as well as the positions of out of plane points as the radiation and oblateness parameters varies are given. Section 5 established their stability; while Section 6 discusses the obtained results and conclusion of the paper.

2. EQUATIONS OF MOTION

Consider three primary bodies P_1, P_2, P_3 , called hereafter the primaries, of masses m_1, m_2, m_3 , respectively, with $m_1 \gg m_2 = m_3$ moving in circles around their center of mass fixed at the origin of the coordinates. These masses always lie at the vertices of equilateral triangle with the dominant body P_1 being on the negative x -axis at the origin of time. A massless particle is moving under the Newtonian gravitational attraction of the primaries and does not affect the motion of the three bodies. The motion of the system is referred to axes rotating with uniform angular velocity. The mutual distances of the three primaries remain unchanged with respect to time. This configuration is well known to be stable if the masses satisfy the condition of Gascheau's inequality (see for details Gascheau [27] and Baltagiannis and Papadakis [19]). This system is also dimensionless, i.e., the units of measure of length and time are taken so that the sum of the masses and the distance between the primaries is unity, and, also, the Gaussian

constant of gravitation G is 1. The equilateral configuration is possible for all distributions of the masses, whilst the fourth body of negligible mass moves in the same plane. The factors characterizing the radiation pressure of the dominant primary P_1 and the oblateness coefficients of the two small primaries (P_2 and P_3) are also taken into account. The notation $q_1 = 1 - \beta_1$ as related to Schuerman [28] is the reduction factor for the mass of the dominant body where β_1 stand for the ratio of the magnitude of radiation (F_r) to gravitational (F_g) force due to the body. It is clear that: If $q_1 = 1$, the radiation pressure has no effect. If $0 < q_1 < 1$, gravitation force exceeds radiation. If $q_1 = 0$, the radiation force balances the gravitational one while for $q_1 < 0$, the gravity is strengthened by radiation, a case which will be considered in our study. The perturbed mean motion n is given by: $n^2 = 1 + \frac{3}{2}(A_2 + A_3)$, where $A_i = \frac{R_E^2 - R_P^2}{5R^2}$, $i = 2, 3$ are the oblateness coefficients of oblate bodies P_2 and P_3 respectively with R_E and R_P as the equatorial and polar radii respectively and R is separation between the primaries. In general we have $0 \leq A_i \ll 1$; A_i is small in the solar system, for details see Sharma and SubbaRao [13].

Let the coordinates of the infinitesimal mass be (x, y) , then the coordinates (x_i, y_i) of the primaries are: $x_1 = -\mu\sqrt{3}$, $y_1 = 0$, $x_2 = x_3 = \frac{\sqrt{3}}{2}(1 - 2\mu)$, $y_2 = -y_3 = -\frac{1}{2}$, relative to the rotating frame of reference $Oxyz$, with $\mu = \frac{m_2}{m_1 + m_2 + m_3} = \frac{m_3}{m_1 + m_2 + m_3}$ is the mass parameter where $\mu \in (0, 1/2)$.

The differential equations of motion in three dimensions in the dimensionless variables and the barycentric–synodic coordinate system are written as (Douskos and Markellos [12]; Papadouris and Papadakis [23]):

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned} \quad (1)$$

where

$$\Omega(x, y, z) = \frac{n^2}{2}(x^2 + y^2) + \frac{q_1(1-2\mu)}{r_1} + \frac{\mu}{r_2} + \frac{\mu}{r_3} + \frac{\mu A_2}{2r_2^3}(1 - \frac{3z^2}{r_2^2}) + \frac{\mu A_3}{2r_3^3}(1 - \frac{3z^2}{r_3^2}), \quad (2)$$

and

$$\begin{aligned} r_1^2 &= (x + \sqrt{3}\mu)^2 + y^2 + z^2, \\ r_2^2 &= (x - \frac{\sqrt{3}}{2}(1-2\mu))^2 + (y + \frac{1}{2})^2 + z^2, \\ r_3^2 &= (x - \frac{\sqrt{3}}{2}(1-2\mu))^2 + (y - \frac{1}{2})^2 + z^2. \end{aligned}$$

where r_1 , r_2 and r_3 are the distances of the infinitesimal body from the primaries, Ω is the photogravitational potential, dots denote time derivatives, the suffixes x , y and z indicate the partial derivatives of Ω with respect to x , y and z respectively.

System (1) admits the Jacobian integral

$$V^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\Omega(x, y, z) - C \quad (3)$$

153 where C is the Jacobi Integral Constant. The quantity on the right-hand side of (3) should be
 154 larger than or equal to zero for every value of time and the given initial conditions. If the velocity
 155 V is put equal to zero, the algebraic equation

$$156 \quad 2\Omega(x, y, z) = C \quad (4)$$

157 will define, by taking various values of the integration constant C (through the initial conditions),
 158 a family of surfaces where it is possible for the infinitesimal mass to move on one side and
 159 impossible for it to move on the other.

160 We remark that whenever $q_1 = 1$ and / or $A_2 = A_3 = 0$, the gravitational case of Baltagianis
 161 and Papadakis [19] is recovered. In the case when only the dominant primary body radiates and
 162 the other two small primaries are not oblate spheroids ($A_2 = A_3 = 0$), the equations of motion
 163 fully coincide with that given by Papadouris and Papadakis [23].

164 3. EXISTENCE AND LOCATION OF THE OUT-OF-PLANE EQUILIBRIUM POINTS

165 The position of the out-of-plane equilibrium points can be examined solving (1) by setting

$$166 \quad \dot{x} = \dot{y} = \dot{z} = \ddot{x} = \ddot{y} = \ddot{z} = 0 \quad (5)$$

167 and considering $z \neq 0$. That is, they are the solutions of the equations

$$168 \quad n^2 x - \frac{q_1(1-2\mu)(x+\sqrt{3}\mu)}{r_1^3} - \frac{\mu(x-\frac{\sqrt{3}}{2}(1-2\mu))}{r_2^3} - \frac{\mu(x-\frac{\sqrt{3}}{2}(1-2\mu))}{r_3^3} - \frac{3\mu A_2(x-\frac{\sqrt{3}}{2}(1-2\mu))(1-\frac{3z^2}{r_2^2})}{2r_2^5}$$

$$169 \quad - \frac{3\mu A_3(x-\frac{\sqrt{3}}{2}(1-2\mu))(1-\frac{3z^2}{r_3^2})}{2r_3^5} + \frac{3\mu A_2 z^2(x-\frac{\sqrt{3}}{2}(1-2\mu))}{r_2^7} + \frac{3\mu A_3 z^2(x-\frac{\sqrt{3}}{2}(1-2\mu))}{r_3^7} = 0, \quad (6)$$

$$\begin{aligned}
 & n^2 y - \frac{q_1(1-2\mu)y}{r_1^3} - \frac{\mu(y+\frac{1}{2})}{r_2^3} - \frac{\mu(y-\frac{1}{2})}{r_3^3} - \frac{3\mu A_2(y+\frac{1}{2})(1-\frac{3z^2}{r_2^2})}{2r_2^5} - \frac{3\mu A_3(y-\frac{1}{2})(1-\frac{3z^2}{r_3^2})}{2r_3^5} + \frac{3\mu A_2 z^2(y+\frac{1}{2})}{r_2^7} \\
 & + \frac{3\mu A_3 z^2(y-\frac{1}{2})}{r_3^7} = 0, \tag{7}
 \end{aligned}$$

$$\begin{aligned}
 & z \left(\frac{q_1(1-2\mu)}{r_1^3} + \frac{\mu}{r_2^3} + \frac{\mu}{r_3^3} - \frac{3\mu A_2(\frac{z^2}{r_2^4} - \frac{1}{r_2^2})}{2r_2^3} - \frac{3\mu A_3(\frac{z^2}{r_3^4} - \frac{1}{r_3^2})}{2r_3^3} + \frac{3\mu A_2(1-\frac{3z^2}{r_2^2})}{2r_2^5} + \frac{3\mu A_3(1-\frac{3z^2}{r_3^2})}{2r_3^5} \right) = 0 \tag{8}
 \end{aligned}$$

with

$$r_1^2 = (x + \sqrt{3}\mu)^2 + y^2 + z^2,$$

$$r_2^2 = (x - \frac{\sqrt{3}}{2}(1-2\mu))^2 + (y + \frac{1}{2})^2 + z^2,$$

$$r_3^2 = (x - \frac{\sqrt{3}}{2}(1-2\mu))^2 + (y - \frac{1}{2})^2 + z^2$$

If $y = 0$, (7) is fulfilled (since $A_2 = A_3$) and we solve (6) and (8) for $y = 0$ and $z \neq 0$. This results in the following equations

$$\begin{aligned}
 & n^2 x_0 - \frac{q_1(1-2\mu)(x_0 + \sqrt{3}\mu)}{r_{10}^3} - \frac{2\mu(x_0 - \frac{\sqrt{3}}{2}(1-2\mu))}{r_{20}^3} - \frac{3\mu A_2(x_0 - \frac{\sqrt{3}}{2}(1-2\mu))(1-\frac{3z_0^2}{r_{20}^2})}{r_{20}^5} + \\
 & \frac{6\mu A_2 z_0^2(x_0 - \frac{\sqrt{3}}{2}(1-2\mu))}{r_{20}^7} = 0, \tag{9}
 \end{aligned}$$

$$\begin{aligned}
 & \frac{q_1(1-2\mu)}{r_{10}^3} + \frac{2\mu}{r_{20}^3} - \frac{6\mu A_2(\frac{z_0^2}{r_{20}^4} - \frac{1}{r_{20}^2})}{r_{20}^3} + \frac{3\mu A_2(1-\frac{3z_0^2}{r_{20}^2})}{r_{20}^5} = 0, \tag{10}
 \end{aligned}$$

183 where

184 $n^2 = 1 + 3A_2,$

185 $r_{10}^2 = (x_0 + \sqrt{3}\mu)^2 + z_0^2,$

186 $r_{20}^2 = r_{30}^2 = (x_0 - \frac{\sqrt{3}}{2}(1 - 2\mu))^2 + \frac{1}{4} + z_0^2, A_2 = A_3$

187 and the subscript ‘0’ is used to denote the equilibrium values.

188 The positions of the out-of-plane equilibrium points are the solutions of (9) and (10). It is worth
 189 mentioning here that out-of-plane equilibrium points do not exist for any combination of the
 190 parameters of this model-problem (the mass parameter μ , radiation factor q_1 and oblateness
 191 coefficient A_2 of the primaries). An interesting result is that there are combinations of the
 192 parameters of the problem for which the critical points may exist.

193 In the present work, we have considered the two small equal primaries with masses
 194 $m_2 = m_3 = \mu = 0.0190$ and a dominant primary body with mass $m_1 = 1 - 2\mu = 0.962$.

195 We note here that in the previous studies a necessary condition in order to exist critical points
 196 out-of-plane is to consider negative values for the radiation factors (see for details Radzievskii
 197 [3], Simmons et al. [8], Papadouris and Papadakis [23] and Singh and Vincent [26]). So,
 198 following the Radzievskii [3,4] assumptions, we will investigate in this section the existence and
 199 location of out-of-plane equilibrium points in the case where $-1 \leq q_1 \leq 0$ and $0 \leq A_2 \ll 1$.

200 Now for $\mu = 0.0190$, there are intervals of q_1 of the form $-1 \leq q_1 \leq 0$ and $0 \leq A_2 \ll 1$ for which
 201 there exist two out-of-the plane equilibrium points, which are denoted by L_1^z and L_2^z . Their
 202 positions studied via numerical computation using the software package ‘Mathematica’ are

located in the (x, z) plane in symmetrical positions with respect to the (x, y) plane. Tables 1–3 and Figs. 1–3 present the positions of the out-of-plane equilibrium points as A_2 varies, for fixed values of q_1 . Evidently, there correspond to cases when we set $q_1 = -0.03, -0.01$ and -0.001 of varying oblateness of the two small primaries. We have observed that as the radiation and oblateness parameters increases the positions of the out-of-plane equilibrium points are affected. As has been already mentioned out-of-plane equilibrium points may exist for other values of mass parameter μ satisfying the condition of Gascheau (inequality) in order to the Lagrange central configuration to be linearly stable.

Table 1. Numerical computations of the out-of-plane equilibrium points for $\mu = 0.0190$, $q_1 = -0.03$ and varying oblateness coefficients

A_2	x_0	$\pm z_0$
0.0000	-0.00244885	2.16893
0.0015	-0.00242942	2.17089
0.0030	-0.00241019	2.17285
0.0045	-0.00239115	2.17482
0.0060	-0.00237231	2.17679
0.0075	-0.00235365	2.17877
0.0090	-0.00233518	2.18075
0.0105	-0.00231689	2.18273
0.0120	-0.00229878	2.18472
0.0135	-0.00228085	2.18672

0.0150	-0.00226310	2.18871
0.0165	-0.00224552	2.19072
0.0180	-0.00222811	2.19272
0.0195	-0.00221087	2.19474

213

214 **Table 2.** Numerical computations of the out-of-plane equilibrium points for $\mu = 0.0190$,

215 $q_1 = -0.01$ and varying oblateness coefficients

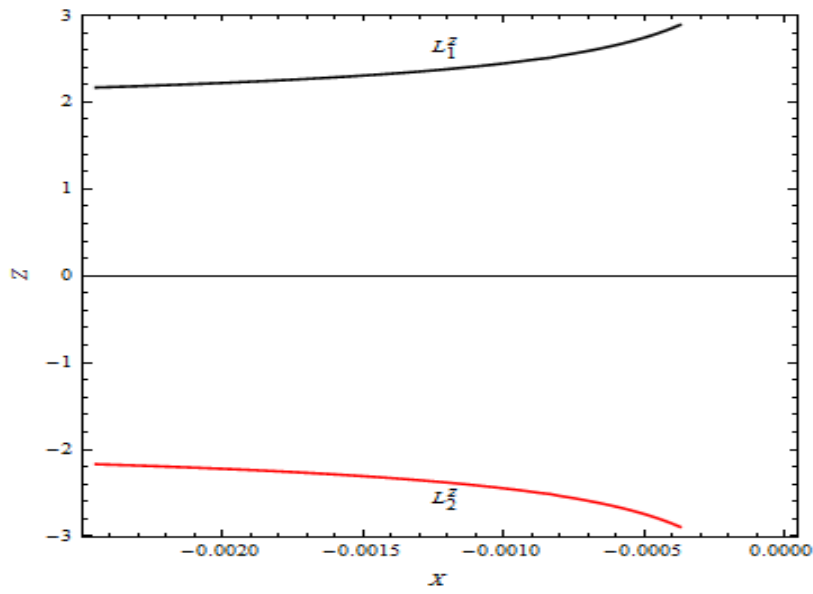
A_2	x_0	$\pm z_0$
0.0000	-0.0159853	0.804573
0.0015	-0.0159104	0.803895
0.0030	-0.0158362	0.803216
0.0045	-0.0157627	0.802536
0.0060	-0.0156899	0.801856
0.0075	-0.0156178	0.801174
0.0090	-0.0155463	0.800491
0.0105	-0.0154755	0.799808
0.0120	-0.0154054	0.799123
0.0135	-0.0153359	0.798437
0.0150	-0.0152670	0.797751
0.0165	-0.0151988	0.797063
0.0180	-0.0151312	0.796375
0.0195	-0.0150642	0.795685

216

Table 3. Numerical computations of the out-of-plane equilibrium points for $\mu = 0.0190$,
 $q_1 = -0.001$ and varying oblateness coefficients

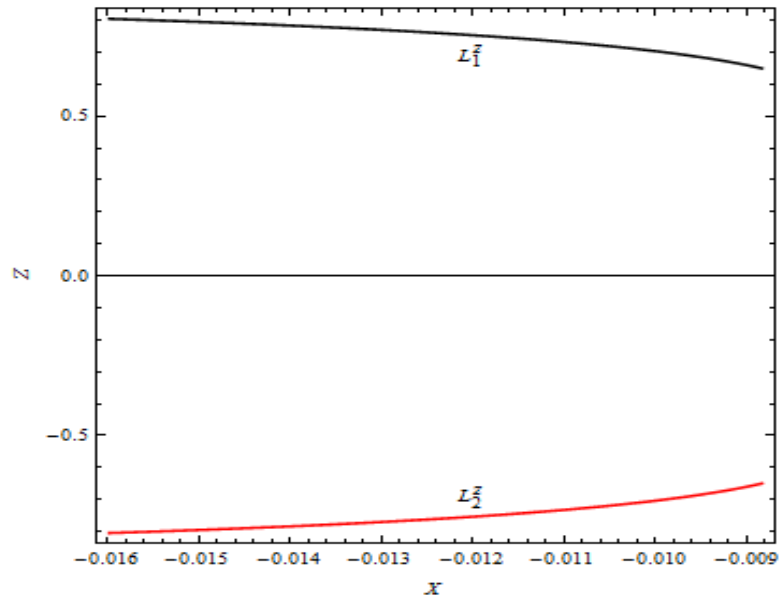
A_2	x_0	$\pm z_0$
0.0000	-0.0290377	0.306112
0.0015	-0.0289624	0.305499
0.0030	-0.0288880	0.304889
0.0045	-0.0288144	0.304282
0.0060	-0.0287416	0.303680
0.0075	-0.0286695	0.303080
0.0090	-0.0285982	0.302484
0.0105	-0.0285276	0.301892
0.0120	-0.0284578	0.301303
0.0135	-0.0283887	0.300717
0.0150	-0.0283204	0.300134
0.0165	-0.0282527	0.299555
0.0180	-0.0281857	0.298979
0.0195	-0.0281194	0.298406

219



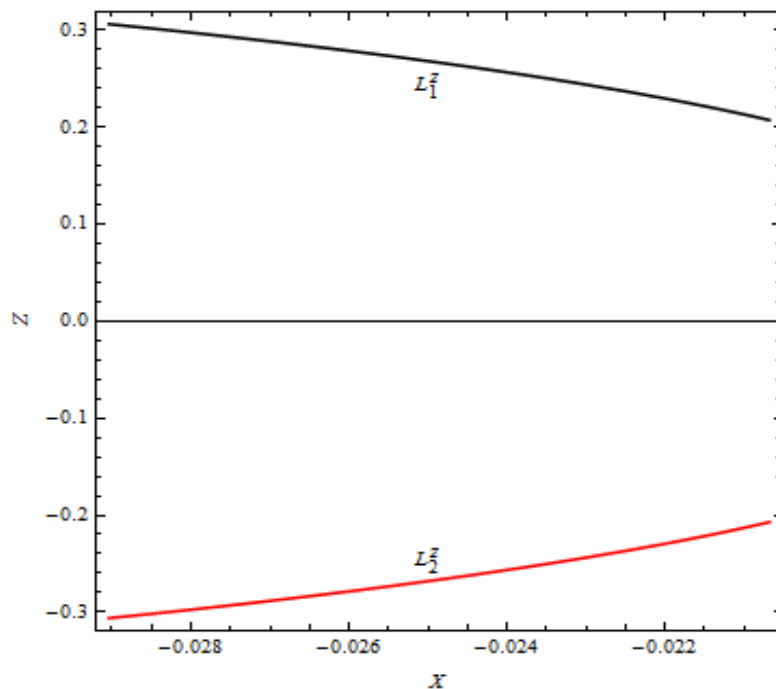
220

221 **Figure 1.** Position of L_1^z and L_2^z in the $(X - Z)$ plane as a function of A_2 in the interval
 222 $0 \leq A_2 < 1$, for $q_1 = -0.03, \mu = 0.019$



223

224 **Figure 2.** Position of L_1^z and L_2^z in the $(X - Z)$ plane as a function of A_2 in the interval
 225 $0 \leq A_2 < 1$, for $q_1 = -0.01, \mu = 0.019$



226
 227 **Figure 3.** Position of L_1^z and L_2^z in the $(X - Z)$ plane as a function of A_2 in the interval
 228 $0 \leq A_2 < 1$, for $q_1 = -0.001, \mu = 0.019$

229 From Fig. 1 it can be seen that when $\mu = 0.0190, q_1 = -0.03$ and $0 \leq A_2 < 1$, the curves
 230 (positions of out-of-plane points) increases. The results in Table 1 are the numerical evidence of
 231 the critical points. Inspection of Fig. 2 indicates that, when $\mu = 0.0190, q_1 = -0.01$ and
 232 $0 \leq A_2 < 1$, the curves decreases. The results in Table 2 present the numerical evidence. We
 233 observe from Fig. 3 that $\mu = 0.0190, q_1 = -0.001$ and $0 \leq A_2 < 1$, the curves are slightly lower as
 234 the critical points moves closer to the dominant primary body. A striking illustration of this can
 235 be seen in Table 3. As mentioned previously, the dominant primary body is on the negative x -
 236 axis at the origin of time. Comparing Figures 1 and 3 shows that the position of out-of- plane
 237 equilibrium points moves in an opposite direction. Hence, from Figs. 1—3, we see that for

238 $\mu = 0.0190$ and increasing values of radiation and oblateness parameters, the positions of out-of-
 239 plane equilibrium points differs.

240 **4. ZERO-VELOCITY CURVES IN THE (x, z) PLANE**

241 The usefulness of the Jacobi constant integral in clarifying certain general properties of the
 242 relative motion of a small body by the construction and investigation of zero velocity curves in
 243 every problem of celestial dynamics cannot be overemphasized. For certain initial conditions
 244 these surfaces divide the space into two regions where the infinitesimal fourth body is free to
 245 move for various values of the Jacobi constant C . In this section we present the contours of the
 246 surface (3) on the (x, z) plane, for zero velocity, which provide the zero velocity curves. In Fig.
 247 4 we plot these zero velocity curves for $\mu = 0.0190$ and various values of radiation and
 248 oblateness parameters i.e. $(q_1 = -0.03, A_2 = 0), (q_1 = -0.01, A_2 = 0.01), (q_1 = -0.001, A_2 = 0.02)$
 249 correspondingly. Large (black) dots indicate the primary bodies, while the small ones are the out-
 250 of-plane equilibrium points of the problem. From these figures, it is obvious, that the radiation
 251 and oblateness have significant effects on the structure of the regions allowed to motion of the
 252 infinitesimal fourth body. In all cases, between centre of the dominant primary P_1 and its
 253 companion out of plane equilibrium points, the zero velocity curves form small ovals of regions
 254 not allowed to motion which shrink as the radiation and oblateness parameters varies.

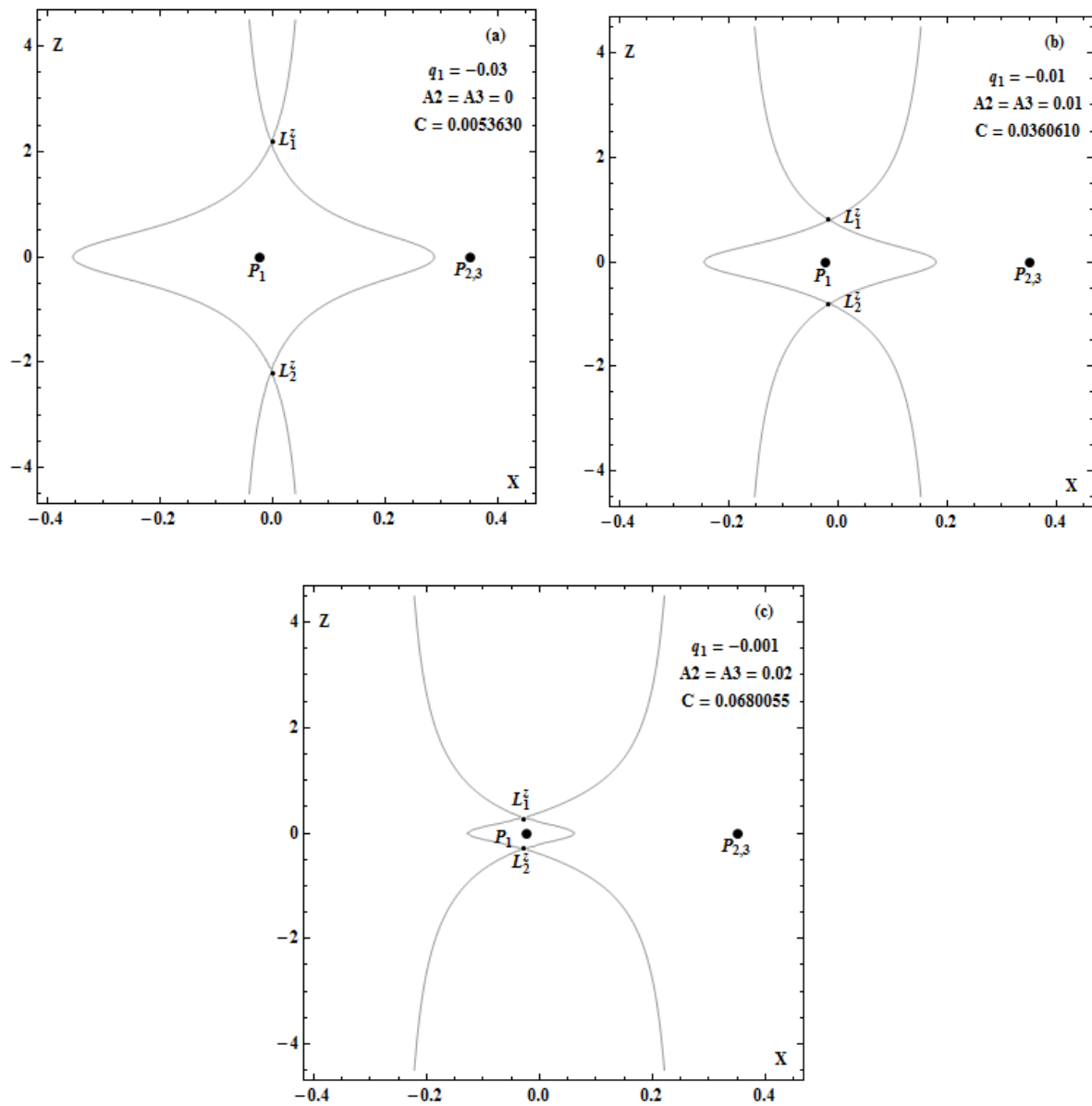


Figure 4 Zero velocity curves in the (x, z) plane for $\mu = 0.019$ and for $q_1 = -0.03$, $A_{2,3} = 0$ (Frame (a)), $q_1 = -0.01$, $A_{2,3} = 0.01$ (Frame (b)) and $q_1 = -0.001$, $A_{2,3} = 0.02$ (Frame (c))

264 5. LINEAR STABILITY OF THE OUT-OF-PLANE EQUILIBRIUM POINTS

265 In order to study the linear stability of the out of plane equilibrium points $L_{1,2}^Z$ we transfer the
 266 origin to $(x_0, 0, z_0)$ and linearize the equations of motion, obtaining:

$$\begin{aligned} 267 \quad \ddot{\xi} - 2n\dot{\eta} &= \xi(\Omega_{xx}^0) + \eta(\Omega_{xy}^0) + \zeta(\Omega_{xz}^0), \\ 268 \quad \ddot{\eta} + 2n\dot{\xi} &= \xi(\Omega_{yx}^0) + \eta(\Omega_{yy}^0) + \zeta(\Omega_{yz}^0), \\ 269 \quad \ddot{\zeta} &= \xi(\Omega_{zx}^0) + \eta(\Omega_{zy}^0) + \zeta(\Omega_{zz}^0), \end{aligned} \quad (11)$$

270 where the superscript 'o' indicates that the partial derivatives are to be evaluated at out of plane
 271 points $(x_0, 0, z_0)$

272 Explicitly, the partial derivatives of system (11) are

$$273 \quad \Omega_{xy}^0 = \Omega_{yx}^0 = \Omega_{yz}^0 = \Omega_{zy}^0 = 0,$$

$$\begin{aligned} 274 \quad \Omega_{xx}^0 &= 1 + 3A_2 - \frac{q_1(1-2\mu)}{r_{10}^3} \left[1 - \frac{3(x_0 + \sqrt{3}\mu)^2}{r_{10}^2} \right] - \frac{2\mu}{r_{20}^3} \left[1 - \frac{3A_2 z_0^2}{r_{20}^4} \right] - \frac{3A_2(r_{20}^2 - 3z_0^2)\mu}{r_{20}^7} \left[1 - \frac{5(x_0 - \frac{\sqrt{3}}{2}(1-2\mu))^2}{r_{20}^2} \right] \\ 275 \quad &+ \frac{6(x_0 - \frac{\sqrt{3}}{2}(1-2\mu))^2\mu}{r_{20}^5} \left[1 - \frac{10A_2 z_0^2}{r_{20}^4} \right], \end{aligned}$$

$$\begin{aligned} 276 \quad \Omega_{zx}^0 = \Omega_{xz}^0 &= \frac{3q_1 z_0(1-2\mu)(x_0 + \sqrt{3}\mu)}{r_{10}^5} + \frac{6z_0\mu(x_0 - \frac{\sqrt{3}}{2}(1-2\mu))}{r_{20}^5} \left[1 + \frac{2A_2}{r_{20}^2} - \frac{7A_2 z_0^2}{r_{20}^4} \right] \\ 277 \quad &+ 3A_2 z_0(x_0 - \frac{\sqrt{3}}{2}(1-2\mu))\mu \left[\frac{11r_{20}^2 - 21z_0^2}{r_{20}^9} \right], \end{aligned}$$

$$278 \quad \Omega_{yy}^0 = 1 + 3A_2 - \frac{q_1(1-2\mu)}{r_{10}^3} - \frac{2\mu}{r_{20}^3} \left(1 - \frac{3}{4r_{20}^2} \right) + \frac{6A_2\mu z_0^2}{r_{20}^7} \left(1 - \frac{5}{2r_{20}^2} \right) - \frac{3A_2(r_{20}^2 - 3z_0^2)\mu}{r_{20}^7} \left(1 - \frac{5}{4r_{20}^2} \right),$$

$$\Omega_{zz}^0 = -\frac{q_1(1-2\mu)}{r_{10}^3} \left[1 - \frac{3z_0^2}{r_{10}^2} \right] - \frac{2\mu}{r_{20}^3} \left[1 - \frac{3z_0^2}{r_{20}^2} \right] - \frac{3A_2(r_{20}^2 - 3z_0^2)\mu}{r_{20}^7} \left[1 - \frac{5z_0^2}{r_{20}^2} \right] - \frac{6A_2\mu}{r_{20}^9} [z_0^2(10z_0^2 - 11r_{20}^2) + r_{20}^4]$$

280

281 with

$$r_{10}^2 = (x_0 + \sqrt{3}\mu)^2 + z_0^2$$

282

$$r_{20}^2 = r_{30}^2 = (x_0 - \frac{\sqrt{3}}{2}(1-2\mu))^2 + \frac{1}{4} + z_0^2, \quad A_2 = A_3$$

283

284 The characteristic equation corresponding to system (11) is

$$\lambda^6 + a\lambda^4 + b\lambda^2 + c = 0 \quad (12)$$

285

286 with

$$a = 4n^2 - \Omega^0_{xx} - \Omega^0_{yy} - \Omega^0_{zz}$$

287

$$b = \Omega^0_{xx}\Omega^0_{yy} + \Omega^0_{yy}\Omega^0_{zz} + \Omega^0_{zz}\Omega^0_{xx} - 4\Omega^0_{zz} - (\Omega^0_{xz})^2$$

288

$$c = (\Omega^0_{xz})^2\Omega^0_{yy} - \Omega^0_{xx}\Omega^0_{yy}\Omega^0_{zz}$$

289

290 which is a polynomial of sixth degree in λ

291 The eigenvalues of the characteristic equation (12) determine the stability or instability of
 292 the respective equilibrium points. An equilibrium point will be stable if the characteristic
 293 equation (12) has six imaginary roots, otherwise it is unstable. We have computed the
 294 characteristic roots $\lambda_i, i=1,2,...,6$ as the radiation and oblateness parameters varies with an
 295 arbitrary small step and found no case in which all the roots are all imaginary. Hence, we
 296 conclude that the out-of-plane equilibrium points are unstable.

297

298

299

6. DISCUSSION AND CONCLUSION

In this contribution, we study the existence and stability of out-of-plane equilibrium points in restricted four-body problem formulated on the basis of Lagrangian configuration when the dominant primary is a source of radiation and the other two small primaries modeled as oblate spheroids are of equal masses and oblateness coefficients. Our result shows the existence of two out-of-plane equilibrium points. We observed that as the radiation and oblateness parameters increases, the positions of out-of-plane equilibrium points are significantly affected (see Tables 1—3 and Figs. 1—3). However, the behavior of the out-of-plane equilibrium points remain unstable despite the introduction of radiation pressure and oblateness of the dominant and small equal primaries respectively. Radiation and oblateness is seen to have significant effects on the topology of the zero-velocity curves in the (x, z) plane. In particular, between the centre of the dominant primary and its companion out-of-plane equilibrium points, the zero-velocity curves form small ovals of regions not allowed to motion which shrink as the radiation and oblateness parameters varies. Most notably, this is the first study to our knowledge to investigate the existence and stability of out-of-plane equilibrium points in the photogravitational restricted four-body problem with oblateness.

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