

SOME NEW THREE STEP ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATION USING STEFFENSEN'S AND HALLEY METHOD

Abstract : In this paper , we introduce the comparative study of new three step iterative methods for finding the zeros of the nonlinear equation $f(x) = 0$. The new method based on the Steffensen's method and Halley method with using predictor – corrector technique . It is established that the new method (NTSM-1) has convergence order sixth and second new method (NTSM-2) has convergence order seventh . Numerical tests show that the new methods is comparable with the well known existing methods and gives better results .

Key word : Non linear equations , Iterative methods , Three step , Convergence analysis , Halley method , Steffensen's method .

1. Introduction : Numerical analysis is the area of mathematics and computer sciences that creates , analyzes and implements algorithms for solving numerically the problems of continuous mathematics . Such problems originate generally from real – world applications of algebra , geometry and calculus and they involve variables which vary continuously : these problems occur throughout the natural sciences , social sciences , engineering , medicine and business . New three step iterative methods for finding the approximate solutions of the nonlinear equation $f(x) = 0$ are being developed using several different techniques including Taylor series , quadrature formulas , homotopy and decomposition techniques , see [2-5 , 8, 11-20] and the references therein . The most famous of these methods is the classical Newton's method (NM)[10].

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \dots\dots\dots(1)$$

The Newton's method (1) was modified by Steffensen's method who replaced the first derivative $f'(x)$ in Newton's method by forward difference approximation [10]

$$f'(x) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}$$

and obtained the famous Steffensen's method (SM) [6 , 7 , 10]

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)} \dots\dots\dots(2)$$

Newton's method and Steffensen's method are of second order converges .

For a given x_0 , compute approximation solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n) f'(x_n)}{2f^2(x_n) - f(x_n)f''(x_n)} \dots\dots\dots(3)$$

This is known as Halley's method has cubic convergence (see [12, 17]).

We use Predictor – corrector methods, we shall now discuss the application of the explicit and implicit multistep methods, for the solution of the initial value problems . We use explicit (predictor) method for predicting a value and then use the implicit (corrector) method iteratively until the convergence is obtained [9] .

2. Iterative methods : It is well known that a wide class of problems , which arise in various fields of pure and applied sciences can be formulated in terms of nonlinear equations of the type.

$$f(x) = 0 \dots\dots\dots(4)$$

Various numerical methods have been developed using the Taylor series and other techniques . In this paper , we use another series of the nonlinear function $f(x)$ which can be obtained by using the trapezoidal rule and the Fundamental Theorem of Calculus . To be more precise , we assume that α is a simple root of (4) and γ is an initial guess sufficiently close to α Now using the trapezoidal rule and fundamental theorem of calculus , one can show that the function $f(x)$ can be approximated by the series [1]

$$f(x) = f(\gamma) + \frac{x-\gamma}{2} [f'(x) + f'(\gamma)] \dots\dots\dots(5)$$

where $f'(x)$ is the differential of f .

From (4) and (5) , we have

$$x = \gamma - 2 \frac{f(\gamma)}{f'(\gamma)} - (x - \gamma) \frac{f'(x)}{f'(\gamma)} \dots\dots\dots(6)$$

Using (6) , one can suggest the following iterative method for solving the nonlinear equations (1) .

For a given initial choice x_0 , find the approximate solution x_{n+1} , by the iterative scheme [1] .

$$x_{n+1} = x_n - 2 \frac{f(x_n)}{f'(x_n)} - (x_{n+1} - x_n) \frac{f'(x_{n+1})}{f'(x_n)}$$

$$n = 0, 1, 2, 3, \dots\dots\dots(7)$$

we use the predictor – corrector technique . Using the Steffensen’s method as a predictor , Halley method and equation (6) as a corrector , we suggest and analyze the following iterative method for solving the nonlinear equation (4) and this is the main motivation of this note .[6 , 7 , 19]

Theorem 1 : For a given initial choice x_0 , find the approximate solution x_{n+1} by the iterative schemes . From equation (2) , (3) and (6)

$$\begin{aligned} a_n &= x_n - \frac{[f(x_n)]^2}{f(x_n+f(x_n))-f(x_n)} \\ b_n &= a_n - \frac{2f(a_n) f'(a_n)}{2f^2(a_n)-f(a_n)f''(a_n)} \\ x_{n+1} &= x_n - 2 \frac{f(x_n)}{f'(x_n)} - (b_n - x_n) \frac{f'(b_n)}{f'(x_n)} \\ n &= 0, 1, 2, 3, \dots \end{aligned}$$

Theorem 1 is called the NEW THREE STEP METHOD -1 (NTSM-1) and has sixth order convergence .

Theorem 2 : From equation (4) and equation (5) we can have

$$x_{n+1} = x_n - \frac{2 f(x_n)}{[f'(x_{n+1})+f'(x_n)]}$$

This is fixed point formulation enable us to suggest the following iterative method for solution the nonlinear equation .

Theorem 3 : For a given initial choice x_0 , find the approximate solution x_{n+1} by the iterative schemes . From equation (2) , (3) and theorem (2)

$$\begin{aligned} a_n &= x_n - \frac{[f(x_n)]^2}{f(x_n+f(x_n))-f(x_n)} \\ b_n &= a_n - \frac{2f(a_n) f'(a_n)}{2f^2(a_n)-f(a_n)f''(a_n)} \\ x_{n+1} &= x_n - \frac{2 f(x_n)}{[f'(b_n)+f'(x_n)]} \\ n &= 0, 1, 2, 3, \dots \end{aligned}$$

Theorem 1 is called the NEW THREE STEP METHOD -2 (NTSM-2) and has seventh order convergence .

3. Convergence analysis : let us now discuss the convergence analysis of the above theorem 1.

Theorem 3.1: let $\alpha \in I$ be a simple zero of sufficiently differential function $f : I \subseteq R \rightarrow R$ for an open interval I , if x_0 is sufficiently close to α then the three step iterative method defined by theorem 1 sixth order convergence .

Proof :Let α be a simple zero of f . Than by expanding $f(x_n)$ and $f'(x_n)$ about α we have

$$f(x_n) = e_n c_1 + e_n^2 c_2 + e_n^3 c_3 + \dots \dots \dots (8)$$

$$f'(x_n) = 1 + \frac{2c_2}{c_1} e_n + \frac{3c_3}{c_1} e_n^2 + \frac{4c_4}{c_1} e_n^3 + \dots \dots \dots (9)$$

$$\text{Where } c_k = \frac{1}{k!} f^{(k)}(\alpha) \quad k=1, 2, 3, \dots \dots \dots$$

and $e_n = x_n - \alpha$

from (8) , we have

$$[f(x_n)]^2 = c_1^2 e_n^2 + 2 c_1 c_2 e_n^3 + c_2^2 e_n^4 + \dots \dots \dots (10)$$

$$f(x_n + f(x_n)) = c_1^2 e_n + (3c_1 c_2 + c_1^2 c_2 + 2c_2^2) e_n^2 + \dots \dots \dots (11)$$

From (10) and (11) , we have

$$\frac{[f(x_n)]^2}{f(x_n + f(x_n))} = e_n - \left(\frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^2 + \dots \dots \dots (12)$$

From (12) , we have

$$a_n = \alpha + \left(\frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^2 + \dots \dots \dots (13)$$

Let us set $A = a_n - \alpha$. Then the equation (13) can be re – written in the form

$$A = \left(\frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^2 + \dots \dots \dots (14)$$

Now expanding $f(a_n)$, $f'(a_n)$, $f''(a_n)$ about α and using (13) , we have

$$f(a_n) = A c_1 + A^2 c_2 + A^3 c_3 + \dots \dots \dots (15)$$

$$f'(a_n) = c_1 + A 2 c_2 + A^2 3 c_3 + A^3 4 c_4 + \dots \dots \dots (16)$$

$$f''(a_n) = 2c_2 + A6c_3 + A^2 12c_4 + \dots \dots \dots (17)$$

Combining (13) – (17) , we have

$$b_n = \alpha + (c_2^2 - c_3) A^3 \dots \dots \dots (18)$$

Also expanding $f'(b_n)$ about α and using (18) , we have

$$f'(b_n) = c_1 \left[1 + 2(c_2^2 - c_3)A^3 \frac{c_2}{c_1} + 3(c_2^2 - c_3)^2 A^6 \frac{c_3}{c_1} + \dots \right] \dots \dots \dots (19)$$

By substituting (8) , (9) , (18) , (19) in theorem(1) and after some simple calculations , we obtain

$$x_{n+1} = \alpha + (c_2^2 - c_3) \left(\frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^6 + \dots \dots \dots (20)$$

$$e_{n+1} = (c_2^2 - c_3) \left(\frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^6 + \dots \dots \dots (21)$$

This shows that Theorem 1 is sixth order convergence .

Let us now discuss the convergence analysis of the above theorem 3.

Theorem 3.2: let $\alpha \in I$ be a simple zero of sufficiently differential function $f : I \subseteq R \rightarrow R$ for an open interval I , if x_0 is sufficiently close to α then the three step iterative method defined by theorem 3 seventh order convergence .

Proof :

From equation (19) , we have

$$f'(b_n) = 2c_1 e_n + 2 c_2 e_n^2 + 2 c_3 e_n^3 + 2 c_4 e_n^4 + \dots \dots \dots (22)$$

By substituting (8) , (9) , (22) in theorem(3) and after some simple calculations , we obtain

$$x_{n+1} = \alpha + 2(c_2^2 - c_3) \left(\frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^7 + \dots \dots \dots (23)$$

$$e_{n+1} = 2(c_2^2 - c_3) \left(\frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^7 + \dots \dots \dots (24)$$

This shows that Theorem 3 is seventh order convergence .

4. Numerical Examples : For comparisons , we have used Steffensen' s method [9] and three step predictor-corrector Newton-Halley method (PCNH) [13] defined respectively by

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}$$

and

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$y_n = w_n - \frac{2f(w_n)f'(w_n)}{2f'^2(w_n) - f(w_n)f''(w_n)},$$

$$x_{n+1} = y_n - 2 \frac{f(y_n)}{f'(y_n)} - \frac{f^2(y_n)f''(y_n)}{2f'(y_n)^3},$$

$$n = 0, 1, 2, \dots$$

All computations are performed using MATLAB . The following examples are used for numerical testing .

$$f_1(x) = 3x - \sqrt{1 + \sin(x)}$$

$$f_2(x) = x + \sin(x) - x^3$$

$$f_3(x) = e^x - 3x$$

$$f_4(x) = \cos(x) - 3x + 1$$

$$f_5(x) = xe^x - 2$$

$$f_6(x) = x \log_{10} x - 1.2$$

$$f_7(x) = x - \cos(x)$$

$$f_8(x) = 4x - e^x$$

$$f_9(x) = \sin(x) - 1 - x^3$$

$$f_{10}(x) = e^x - 1.5 - \tan^{-1}(x)$$

As for the convergence criteria , it was required that the distance of two consecutive approximations δ and also displayed is the number of iterations to approximate the zero (IT) , the approximate zero x_n and the value $f(x_n)$

Table 1 : (Numerical Examples and Comparison)

method	IT	x_n	$f(x_n)$	δ
$f_1, x_0 = 0.5$				
SM	4	0.391846907002648	-4.44089209850063e-016	1.01410879693731e-010
PCNH	3	0.391846907002648		2.50388310085725e-008
NTSM-1	2	0.391846907002648		7.53028660899213e-006
NTSM-2	2	0.391846907002648		3.78833973964160e-006
$f_2, x_0 = 1$				
SM	5	1.31716296100603	9.76996261670138e-015	3.32861537577501e-005
PCNH	6	1.31716296100603		4.68398653197255e-011
NTSM-1	3	1.31716296100603		5.34990427110637e-012
NTSM-2	3	1.31716296100603		6.47483999749454e-009
$f_3, x_0 = 0.2$				
SM	4	0.619061286735945	2.22044604925031e-016	7.0389180040209e-009
PCNH	5	0.619061286735945		6.835598753691e-011
NTSM-1	3	0.619061286735945		3.10652923940502e-007
NTSM-2	3	0.619061286735945		3.59104859182224e-008
$f_4, x_0 = 0.2$				
SM	5	0.607101648103123	-1.33226762955019e-015	9.99200722162641e-016
PCNH	4	0.607101648103123		5.16754083967896e-010
NTSM-1	3	0.607101648103123		6.95768997971413e-011
NTSM-2	3	0.607101648103123		3.65396601864632e-012
$f_5, x_0 = 1$				
SM	6	0.852605502013726	-2.44249065417534e-015	3.517610647279e-010
PCNH	4	0.852605502013726		2.43496630636386e-008
NTSM-1	4	0.852605502013726		2.68309041651094e-010
NTSM-2	3	0.852605502013726		2.88919999036352e-011
$f_6, x_0 = 2$				
SM	4	2.74064609597369	-3.10862446895044e-015	1.70161289503313e-008
PCNH	4	2.74064609597369		1.95032203720302e-008
NTSM-1	3	2.74064609597369		3.98418986691024e-009
NTSM-2	3	2.74064609597369		1.98129956885396e-010
$f_7, x_0 = 0.5$				
SM	5	0.739085133215161	6.66133814775094e-016	7.13984427136438e-013
PCNH	4	0.739085133215161		7.05561942204724e-010
NTSM-1	3	0.739085133215161		4.4899972628798e-011
NTSM-2	3	0.739085133215161		2.45103937146496e-012
$f_8, x_0 = 1.8$				
SM	6	2.15329236411035	-1.77635683940025e-015	5.27400345617934e-011
PCNH		Divergent		
NTSM-1	3	2.15329236411035		1.48246099840321e-008
NTSM-2	3	2.15329236411035		2.29886798486234e-008
$f_9, x_0 = -1$				
SM	4	-1.24905214850119	-1.99840144432528e-014	0.000227912085001725
PCNH	5	-1.24905214850119		2.53839993469285e-009
NTSM-1	3	-1.24905214850119		1.20446399520802e-008
NTSM-2	3	-1.24905214850119		1.01799013663140e-009
$f_{10}, x_0 = 1$				
SM	5	0.767653266201279	0	2.27928786955545e-012
PCNH	5	0.767653266201279		1.48392409471398e-012
NTSM-1	3	0.767653266201279		9.48948919443637e-010
NTSM-2	3	0.767653266201279		1.48009937639415e-010

Conclusion:

In this paper , we have suggested and analyzed newly developed technique is faster than Steffensen' s method (SM) and predictor-corrector Newton-Halley method (PCNH) . This method based on a Steffensen' s method and Halley method and using predictor – corrector technique . Our method can be considered as significant improvement of Steffensen's method and PCNH and can be considered as alternative method of solving nonlinear equations .

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