

On Solving Comrade Linear systems Via Transformations

Abstract

Recently, a new symbolic algorithm for solving comrade linear systems is given by Karawia[A. A. Karawia, Symbolic algorithm for solving comrade linear systems based on a modified Stair-Diagonal Approach, Appl. Math. Lett. 26 (2013) 913-918]. This paper introduce different numerical and symbolic algorithms for solving comrade linear systems via transformations. Our symbolic algorithm removes the cases where the numerical algorithms fail. The computational cost of these algorithms is given. It gives better absolute error than the other methods. The algorithm is implementable to the Computer Algebra System (CAS) such as MAPLE and MATLAB. Some illustrative examples are presented for the sake of illustration.

Keywords:Comrade matrices; Tridiagonal matrices; Determinants; Computer algebra systems (CAS).

AMS Subject Classification:15A15; 15A23; 33F10; F.2.1; G.1.0; 68W30; 11Y05.

1 Introduction

Comrade linear system of size n takes the form:

$$CX = Y, \tag{1.1}$$

where

$$C = \begin{pmatrix} -\frac{\beta_1}{\alpha_1} & \frac{1}{\alpha_1} & 0 & 0 & \dots & \dots & \dots & 0 & 0 \\ \frac{\gamma_2}{\alpha_2} & -\frac{\beta_2}{\alpha_2} & \frac{1}{\alpha_2} & 0 & \dots & \dots & \dots & 0 & 0 \\ 0 & \frac{\gamma_3}{\alpha_3} & -\frac{\beta_3}{\alpha_3} & \frac{1}{\alpha_3} & 0 & \dots & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & \vdots & \frac{\gamma_{n-2}}{\alpha_{n-2}} & -\frac{\beta_{n-2}}{\alpha_{n-2}} & \frac{1}{\alpha_{n-2}} & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & \frac{\gamma_{n-1}}{\alpha_{n-1}} & -\frac{\beta_{n-1}}{\alpha_{n-1}} & \frac{1}{\alpha_{n-1}} \\ -\frac{a_n}{\alpha_n} & -\frac{a_{n-1}}{\alpha_n} & -\frac{a_{n-2}}{\alpha_n} & \dots & \dots & \frac{-a_4}{\alpha_n} & \frac{-a_3}{\alpha_n} & \frac{\gamma_{n-a_2}}{\alpha_n} & \frac{-\beta_{n-a_1}}{\alpha_n} \end{pmatrix}, n \geq 3, \quad (1.2)$$

where $X = (x_1, x_2, \dots, x_n)^t$, and $Y = (y_1, y_2, \dots, y_n)^t$.

Matrix C is called comrade matrix and it can be stored in $4n - 4$ memory locations. Comrade matrix is a generalization of the companion matrix and is associated with a polynomial expressed as a linear combination of an arbitrary orthogonal basis. This matrix appears frequently in many areas of science and engineering, for example in linear multi-variable systems theory [1], computing the greatest common divisor of polynomials [2] and division of generalized polynomials [3].

Comrade linear system is widely used in solving problems in many areas of science, for example solving differential equations using finite differences, parallel computing, and telecommunication system analysis[4-6]. Finding the solution of such a linear system is usually required in these fields. This problem has been investigated by many authors (see for instance,[7-10]).In [7], the author has introduced an efficient numerical and symbolic algorithms for solving comrade linear system using LU decomposition. Two algorithms are presented for solving the comrade linear systems based on the use of conventional fast tridiagonal solvers and an efficient way of evaluating the determinant of the comrade matrix is given[8]. The author in [9] has described a symbolic computational algorithm for solving comrade linear systems based on a modified Stair-Diagonal Approach. Recently, the authors have derived a novel elementary algorithm for solving comrade linear systems based on matrix reordering and the ShermanMorrisonWoodbury formula[10]. In this paper, we introduce more efficient algorithms based on transformations. Our proposed algorithms are completely different than the algorithms in[9].

The paper is organized as follows. In Section 2, new algorithms for solving comrade linear systems are given. Some illustrative examples are introduced in Section 4. In Section 5, Conclusions of the work are given.

2 Main Results

In this section, we shall concentrate on the construction of new algorithms for computing the solution of comrade linear system. For this goal, it is suitable to give five vectors

$\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_{n-1})$, $\boldsymbol{\psi} = (\psi_1, \psi_2, \dots, \psi_{n-1})$, $\boldsymbol{\phi} = (\phi_1, \phi_2, \dots, \phi_{n-1})$, and $\boldsymbol{Z} = (z_1, z_2, \dots, z_n)$, where

$$\mu_i = \begin{cases} -\frac{\beta_1}{\alpha_1} & i = 1, \\ \frac{-\beta_i - \gamma_i \phi_{i-1}}{\alpha_i} & i = 2, 3, \dots, n-1, \\ -\sigma_{n-1} \phi_{n-1} - \frac{a_1 + \beta_n}{\alpha_n} & i = n, \end{cases} \quad (2.1)$$

$$\phi_i = \frac{1}{\alpha_i \mu_i}, \quad i = 1, 2, \dots, n-1, \quad (2.2)$$

$$\sigma_i = \begin{cases} -\frac{a_n}{\alpha_n} & i = 1, \\ -\sigma_{i-1} \phi_{i-1} - \frac{a_{n-i+1}}{\alpha_n} & i = 2, 3, \dots, n-2, \\ -\sigma_{n-2} \phi_{n-2} + \frac{\gamma_{n-1} a_2}{\alpha_n} & i = n-1, \end{cases} \quad (2.3)$$

$$z_i = \begin{cases} \frac{y_1}{\mu_1} & i = 1, \\ \frac{1}{\mu_i} (y_i - \frac{\gamma_i}{\alpha_i} z_{i-1}) & i = 2, 3, \dots, n-1, \\ \frac{1}{\mu_n} (-\sigma_{n-1} z_{n-1} + \psi_{n-1}) & i = n, \end{cases} \quad (2.4)$$

and

$$\psi_i = \begin{cases} y_n & i = 1, \\ -\sigma_{i-1} z_{i-1} + \psi_{i-1} & i = 2, 3, \dots, n-1. \end{cases} \quad (2.5)$$

By using those vectors together with appropriate elementary row operations, one can see that the system (1.1) may be transformed to the equivalent linear system:

$$\begin{pmatrix} 1 & \phi_1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & \phi_2 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 & \phi_3 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 0 & 1 & \phi_{n-2} & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & 0 & 1 & \phi_{n-1} \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ \vdots \\ \vdots \\ z_{n-2} \\ z_{n-1} \\ z_n \end{pmatrix} \quad (2.6)$$

By using a backward substitution, it is easy to solve the transformed system (2.6). As a result, the comrade linear system (1.1) can be solved using the following numerical algorithm:

Algorithm 2.1. Numerical algorithm for solving comrade linear system

To solve the comrade linear system (1.1), we carried out as follows:

INPUT: Order of the matrix n and the components $\alpha_i, \beta_i, \gamma_i, a_i$, $i = 1, 2, \dots, n$, ($\gamma_1 = 0$).

OUTPUT: The solution vector $X = (x_1, x_2, \dots, x_n)^t$.

Step 1: Set $\mu_1 = -\beta_1/\alpha_1$, $\phi_1 = 1/(\alpha_1 \mu_1)$, $z_1 = y_1/\mu_1$, $\sigma_1 = -a_n/\alpha_n$, and $\psi_1 = y_n$.

Step 2: Compute and simplify:

For $i = 2, 3, \dots, n-2$ do

$$\mu_i = (-\beta_i - \gamma_i \phi_{i-1})/\alpha_i$$

$$z_i = (y_i - \frac{\gamma_i}{\alpha_i} z_{i-1})/\mu_i$$

$$\begin{aligned}\phi_i &= 1/(\alpha_i \mu_i) \\ \sigma_i &= -\sigma_{i-1} \phi_{i-1} - a_{n-i+1}/\alpha_n \\ \psi_i &= -\sigma_{i-1} z_{i-1} + \psi_{i-1}\end{aligned}$$

End do

Step 3: Compute and simplify:

$$\begin{aligned}\mu_{n-1} &= (-\beta_{n-1} - \gamma_{n-1} \phi_{n-2})/\alpha_{n-1} \\ \phi_{n-1} &= 1/(\alpha_{n-1} \mu_{n-1}) \\ \sigma_{n-1} &= -\sigma_{n-2} \phi_{n-2} + (\gamma_n - a_2)/\alpha_n \\ \psi_{n-1} &= -\sigma_{n-2} z_{n-2} + \psi_{n-2} \\ z_{n-1} &= (y_{n-1} - \frac{\gamma_{n-1}}{\alpha_{n-1}} z_{n-2})/\mu_{n-1} \\ \mu_n &= -\sigma_{n-1} \phi_{n-1} - (a_1 + \beta_n)/\alpha_n \\ z_n &= (-\sigma_{n-1} z_{n-1} + \psi_{n-1})/\mu_n\end{aligned}$$

Step 4: Compute $\det(C) = \prod_{i=1}^n \mu_i$.

Step 5: If $\det(C) = 0$, then Exit and Print Message ("No solutions") end if.

Step 6: Set $x_n = z_n$.

Step 7: Compute and simplify:

For $i = n-1, n-2, \dots, 1$ do

$$x_i = z_i - \phi_i x_{i+1}$$

End do

The Algorithm 2.1, will be refer to as **COMTRANS** in the sequel. The computational cost of **COMTRANS** is $15n - 14$ operations. The sufficient conditions for its validity are $\mu_i \neq 0, i = 1, 2, \dots, n$.

The following symbolic version algorithm is developed with a view to remove the cases where the numerical algorithm **COMTRANS** fails. The parameter " s " in the following symbolic algorithms is just a symbolic character. It is a dummy argument and its actual value is zero.

Algorithm 2.2. Symbolic version algorithm for **COMTRANS** algorithm

To solve the comrade linear system (1.1), we carried out as follows:

INPUT: Order of the matrix n and the components $\alpha_i, \beta_i, \gamma_i, a_i, \quad i = 1, 2, \dots, n, (\gamma_1 = 0)$.

OUTPUT: The solution vector $X = (x_1, x_2, \dots, x_n)^t$.

Step 1: Set $\mu_1 = -\beta_1/\alpha_1$. If $\mu_1 = 0$ then $\mu_1 = s$ end if. $\phi_1 = 1/(\alpha_1 \mu_1)$, $z_1 = y_1/\mu_1$, $\sigma_1 = -a_n/\alpha_n$, and $\psi_1 = y_n$.

Step 2: Compute and simplify:

For $i = 2, 3, \dots, n-2$ do

$$\mu_i = (-\beta_i - \gamma_i \phi_{i-1})/\alpha_i. \text{ If } \mu_i = 0 \text{ then } \mu_i = s \text{ end if.}$$

$$z_i = (y_i - \frac{\gamma_i}{\alpha_i} z_{i-1})/\mu_i$$

$$\phi_i = 1/(\alpha_i \mu_i)$$

$$\sigma_i = -\sigma_{i-1} \phi_{i-1} - a_{n-i+1}/\alpha_n$$

$$\psi_i = -\sigma_{i-1} z_{i-1} + \psi_{i-1}$$

End do

Step 3: Compute and simplify:

$$\mu_{n-1} = (-\beta_{n-1} - \gamma_{n-1} \phi_{n-2})/\alpha_{n-1}. \text{ If } \mu_{n-1} = 0 \text{ then } \mu_{n-1} = s \text{ end if.}$$

$$\begin{aligned}
\phi_{n-1} &= 1/(\alpha_{n-1}\mu_{n-1}) \\
\sigma_{n-1} &= -\sigma_{n-2}\phi_{n-2} + (\gamma_n - a_2)/\alpha_n \\
\psi_{n-1} &= -\sigma_{n-2}z_{n-2} + \psi_{n-2} \\
z_{n-1} &= (y_{n-1} - \frac{\gamma_{n-1}}{\alpha_{n-1}}z_{n-2})/\mu_{n-1} \\
\mu_n &= -\sigma_{n-1}\phi_{n-1} - (a_1 + \beta_n)/\alpha_n. \text{ If } \mu_n = 0 \text{ then } \mu_n = s \text{ end if.} \\
z_n &= (-\sigma_{n-1}z_{n-1} + \psi_{n-1})/\mu_n
\end{aligned}$$

Step 4: Compute $\det(C) = \left(\prod_{i=1}^n \mu_i \right)_{s=0}$.

Step 5: If $\det(C) = 0$, then Exit and Print Message ("No solutions") end if.

Step 6: Set $x_n = z_n$.

Step 7: Compute and simplify:

For $i = n - 1, n - 2, \dots, 1$ do

$$x_i = z_i - \phi_i x_{i+1}$$

End do

Step 8: Substitute $s = 0$ in all expressions of the solution vector $X = (x_1, x_2, \dots, x_n)^t$.

The Algorithm 2.1, will be refer to as **SCOMTRANS** in the sequel.

3 ILLUSTRATIVE EXAMPLES

In this section, we give three examples for the sake of illustration. All experiments carried out in MATLAB 2014a with an Intel(R) Core(TM) i7-4700MQ CPU@2.40GHz 2.40 GHz.

Example 3.1. Case I: $\mu_i \neq 0, \quad i = 1, 2, \dots, n$.

Solve the following comrade linear system of size 7[9]

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{1} & -\frac{3}{-1} & \frac{1}{-1} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & -\frac{4}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2} & -\frac{6}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{7} & -\frac{5}{7} & \frac{1}{7} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{4}{3} & -\frac{3}{3} & \frac{4-1}{3} & \frac{-3-1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \\ \frac{3}{3} \\ \frac{1}{-1} \\ \frac{-6}{1} \\ \frac{-13}{7} \\ \frac{-50}{3} \end{pmatrix} \quad (3.1)$$

Solution: We have

$n = 7, \alpha = (2, 3, -1, 5, 2, 7, 3)^t, \beta = (1, 2, 3, 4, 6, 5, 3)^t, \gamma = (0, 3, 1, 2, 3, 2, 4)^t, a = (1, 1, 3, 4, 2, 1, 1)^t$, and $Y = (1/2, 2/3, 3/1, -1/1, -6/1, -13/7, -50/3)^t$.

Applying the **COMTRANS** algorithm, it yields

$X = \text{Comtrans\_I}(\alpha, \beta, \gamma, a, Y, n) = (1, 2, 3, 4, 5, 6, 7)^t$ and $\det(C) = -0.9286$.

Example 3.2. Case II: At least one of $\mu_i = 0, \quad i = 1, 2, \dots, n$.

Solve the following comrade linear system of size 4

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{3}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{3}{3} & \frac{1}{-1} \\ -\frac{4}{5} & -\frac{2}{5} & -\frac{3-2}{5} & \frac{-4-3}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{3}{3} \\ \frac{-18}{5} \end{pmatrix} \quad (3.2)$$

Solution: We have

$n = 4$, $\alpha = (2, 3, -1, 5)^t$, $\beta = (1, 3, 3, 4)^t$, $\gamma = (0, 3, -1, -3)^t$, $a = (3, 2, 2, 4)^t$, and $Y = (0, 1/3, 3/1, -1/5)^t$.

The numerical algorithm **COMTRANS** fails to solve the comrade linear system (4.2) since $\mu_2 = 0$.

Applying the **SCOMTRANS** algorithm, it yields

$$X = \text{SCOMTRANS}(\alpha, \beta, \gamma, a, Y, n) = \left(-\frac{1}{6s-1}, -\frac{1}{6s-1}, \frac{9s-1}{6s-1}, \frac{9s-1}{6s-1}\right)_{s=0} = (1, 1, 1, 1)^t$$

Example 3.3. With a view to show the efficiency and competitiveness of the proposed algorithm described in this paper, we now consider an n -by- n comrade linear system $Cx = y$ takes the form [9]

$$\begin{pmatrix} \frac{1}{2} & 1 & 0 & \cdots & \cdots & 0 \\ -\frac{1}{2} & \frac{1}{2} & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\frac{1}{2} & \frac{1}{2} & 1 \\ -1 & -1 & \cdots & -1 & -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ \vdots \\ \vdots \\ 1 \\ -n \end{pmatrix} \quad (3.3)$$

The exact solution of linear system (3.3) is $x = (1, 1, \dots, 1)^t$. We give some comparisons between our proposed algorithm and the algorithms in literature to compute $\bar{x}$. The absolute errors $\varepsilon = \|x - \bar{x}\|_\infty$, and CPU times with these algorithms are shown in Table 1.

Table 1

Errors and CPU times in seconds for our proposed algorithm and **KCOM**, **ECLS**, Sogabe[8], and **SCMSDA** algorithms for different values of n .

Algorithms	n	100	500	1000	5000
COMTRANS	$\ x - \bar{x}\ _\infty$	7.7716×10^{-16}	7.7716×10^{-16}	7.7716×10^{-16}	7.7716×10^{-16}
	$CPUtime(s)$	0.0065	0.0076	0.0086	0.0341
KCOM	$\ x - \bar{x}\ _\infty$	1.1102×10^{-15}	1.1102×10^{-15}	1.1102×10^{-15}	1.1102×10^{-15}
	$CPUtime(s)$	0.0055	0.0066	0.0071	0.0280
ECLS	$\ x - \bar{x}\ _\infty$	2.8422×10^{-14}	1.1369×10^{-13}	1.1369×10^{-13}	1.8190×10^{-12}
	$CPUtime(s)$	0.0048	0.0080	0.0086	0.0117
Sogabe[8]	$\ x - \bar{x}\ _\infty$	2.8422×10^{-14}	1.7053×10^{-13}	9.0949×10^{-13}	3.0923×10^{-10}
	$CPUtime(s)$	0.0081	0.0433	0.0476	0.0525
SCMSDA	$\ x - \bar{x}\ _\infty$	1.4211×10^{-13}	2.8422×10^{-13}	8.6402×10^{-11}	3.1041×10^{-9}
	$CPUtime(s)$	0.0261	0.0843	0.1506	0.6437

The obtained results show that the algorithm **COMTRANS** gives better absolute error than the other algorithms used in comparisons for large values of n . Moreover, the obtained results indicate that the value of the running times for **KCOM** and **ECLS** algorithms are small in comparisons with other algorithms for large sizes.

4 CONCLUSIONS

In this work, new algorithms, **COMTRANS** and **SCOMTRANS**, have been developed for computing the solving comrade linear systems via transformations. The symbolic algorithm, **SCOMTRANS** is reliable, computationally efficient and will not fail. Three numerical examples were introduced to explain the effectiveness of our algorithms and competitiveness with other algorithms in literature.

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