

Original Research Article Pseudo Lifting Modules

ABSTRACT

Let R be a ring with identity, and M is a unitary R -module. In this paper we introduce and study the concepts of pseudo lifting modules and give basic properties, examples and characterizations of these concepts.

Lifting modules have been studied recently by several authors, such as, K. Oshiro [3]. We call a sub module N of an R -module M a pseudo stable, if $f(N) \leq N$ for each R -monomorphism $f : N \rightarrow M$ [4]

Keywords: Direct Summand, Hollow Module, Lifting Module, Pseudo Lifting Module, Radical, R -Module, Semi Simple, Submodule, Z -Module.

1. INTRODUCTION

Recall that: an R -module is called lifting if for every sub module N of M there exists a sub module K of N such that $M = K \oplus K'$ and $N \cap K' \ll K'$, [6].

In this paper, we introduce and study the concept pseudo lifting modules. An R -module M is called pseudo lifting module if for every submodule N of M there exists a pseudo stable submodule K of N such that $M = K \oplus K'$ and $N \cap K' \ll K'$.

2. DEFINITIONS AND PROPERTIES

Definition 2.1: Let M be a module is called a pseudo lifting module if for every submodule N of M there exists a pseudo stable submodule K of N such that $M = K \oplus K'$ and $N \cap K' \ll K'$.

Note: A ring R is called pseudo lifting if R is pseudo lifting R -module.

2.1 Remarks and Examples:

1-Every pseudo lifting module is lifting module, but the converse is not true. This is shown in the following example:

The Z -module $M = Z_{p^\infty} \oplus Z_{p^\infty}$ is lifting Z -module because M is projective and it is well-known that every projective module is lifting [5, Theorem 4-1-16], while M is not pseudo lifting.

2-Let the Q as Z -module. Q is not pseudo lifting module.

The following lemma gives some properties of small submodules which are needed later

Lemma 2 : [5] Let N, K and L are submodules of M .

- 1- If $N \ll K$ and $K \ll L$, then $N \ll L$.
- 2- If $N \ll M$, $N \leq K$ and K is a direct summand of M then $N \ll K$.
- 3- If $N \ll M$ and $f : M \rightarrow M'$ is a submodule homomorphism, then $f(N) \ll f(M')$

2.2. Applications

The following theorem gives another characterization of pseudo lifting modules.

Theorem 3: Let M be an R -module. The following statements are equivalent:

1. M is pseudo lifting module
2. Every pseudo stable submodule N of M can be written as $N = A \oplus S$, where A is a

- direct summand of M and $S \ll M$.
3. For every pseudo stable submodule N of M , there exists a direct summand K of M such that $K \ll N$ and $N/K \ll M/K$.

Proof : $1 \rightarrow 2$:

Let M be a pseudo lifting module. There exists a pseudo stable submodule K of N such that $M = K \oplus K'$ and $N \cap K' \ll K$. So $N \cap K' \ll M$ by [lemma 2]. Now, $N = N \cap M$

$$= N \cap (K \oplus K') = K \oplus N \cap K'$$

By modular law. We take $A = K$ and $S = N \cap K'$. Therefore $N = A \oplus S$ with A is a pseudo stable direct summand of M and $S \ll M$.

$2 \rightarrow 3$:

Let N be a submodule of M . By (2) $N = A \oplus S$ where A is a pseudo stable direct summand of M and $S \ll M$, we are done if we can show that $N/A \ll M/N$. To see that, suppose that $N/A + L/A = M/A$. Then $(A \oplus S)/A + L/A = M/A$. Therefore, $M = A + S + L = S + L$, since $A \leq L = L$, since $S \ll M$. Hence, $N/A \ll M/A$.

$3 \rightarrow 1$:

Let N be a submodule of M , there exists a pseudo stable submodule K of N such that $M = K \oplus K'$ and $N/K \ll M/K$. We want to show that $N \cap K' \ll K'$. Suppose that $N \cap K' + B = K'$, where $B \leq K'$. Therefore, $M = K + K' = K + N \cap K' + B$. Thus

$$M/K = (K + N \cap K' + B)/K = (N \cap K + K)/K + (K + B)/K$$
Since $(N \cap K' + K)/K \leq N/K$, then $(N \cap K' + K)/K \ll M/K$.

Therefore, $(K + B)/K = M/K$. Thus, $M = K + B$. Since, $B \leq K'$ and $K \cap K' = 0$, then $B \cap K = 0$ and hence, $M = K \oplus B$. This implies $B = K'$. Hence, $N \cap K' \ll K'$.

The following theorem give another characterization of pseudo lifting modules.

Theorem 4 : An R-module M is a pseudo lifting if and only if for each submodule A of M there is a direct decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$ where M_1 is a pseudo stable submodule of M and $A + M_2$ is small in M .

Proof: $(\Rightarrow)$ Suppose that M is pseudo lifting module. Let A be a sub module of M . Thus there exists a pseudo stable direct summand K of M such that A is small in K . That is $M = K \oplus K_1$ for some sub module K_1 of M . Also, since A is small in K and K_1 is small in K_1 , then $A \oplus K_1$ is small in $M = K \oplus K_1$. Hence $A \oplus K_1$ is small sub module of M .

$(\Leftarrow)$ Let A be a sub module of M , then there is a direct decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$. Where M_1 is pseudo stable sub module of M and $A + M_2$ is small in M . To prove that M is pseudo lifting module we claim that A is small in M_1 . Let K be a non-zero sub module of M_1 hence K is a sub module of M , since $A + M_2$ in M . Then $(A + M_2) \cap K \neq (0)$. Let $k = a + m_2$ be a non-zero element of K , where $a \in A$ and $m_2 \in M_2$, thus $m_2 = k - a \in M_1 \cap M_2 = (0)$. Therefore, $k = a \in K \cap A$, then $K \cap A \neq (0)$. Hence A is small in M . Thus M is a pseudo lifting module.

The following proposition gives some properties of small submodule.

Proposition 5 : [1] Let K, L and N be submodules of an R -module M , then

- 1- If $K \ll M$ and $f : M \rightarrow M'$ is a homomorphism, then $f(K) \ll M'$.
- 2- If $K \subseteq L \subseteq N \subseteq M$ and $L \ll N$, then $K \ll M$.
- 3- If $K \ll M$ and $L \ll M$, then $K + L \ll M$.
- 4- If L is a direct summand of M and $K \subseteq L$ with $K \ll M$, then $K \ll L$.

Remark 6 : Let M be an R -module, M is pseudo lifting if for every submodule N of M , there exists a pseudo stable submodule K of N such that $M = K \oplus K'$ and $N \cap K' \ll M$.

Proof : Let N be any submodule of M , since M is pseudo lifting module, so by definition (1) there exists a submodule K of N such that $M = K \oplus K'$ and $N \cap K' \ll K'$, but $K' \subseteq M$, hence $N \cap K' \ll M$.

Recall that a module M is semi simple, if every submodule is a direct summand of M , [7].

It is clear that (0) is the only small submodule in a semi simple module.

Proposition 7: Every semi simple module is pseudo lifting module.

Proof: Let M be a semi simple module and N a pseudo stable submodule of M . Since M is semi simple, N is a direct summand of M , i.e. $M = N \oplus T$, $N \cap T = 0 \ll M$. Hence, M is pseudo lifting.

Recall that the Jacobson Radical of a module M (notation $J(M)$) is the intersection of all maximal submodules of M .

Note that the sum of all submodule of a module M is equal to $J(M)$. [2, p.55].

Proposition 8: Let M be a module with $J(M) = 0$, if M is Artinian, then M is semi simple and hence, a pseudo lifting module.

Proof: Let N be a pseudo stable submodule of M . Set $S = \{L : M = N + L, N \ll M\}$. It is clear that $S \neq \emptyset$, since $M \in S$. Now M is Artinian implies that S has a minimal element say K . Let $M = U + (N \cap K)$, where $U \ll M$, $M = N + K = N + N \cap K = N + U \cap K$ by the minimality of K , we have $U \cap K = K$, i.e., $K \leq U$. Hence, $U = M$. This implies that $N \cap K \ll M$. Thus, $N \cap K \leq J(M) = 0$. i.e. $N \cap K = 0$.

Therefore, $M = N \oplus K$ and thus M is semi simple.

A module M is said to be hollow, if every proper submodule is small in M .

Proposition 9: Every hollow module is pseudo lifting module.

Proof : Let N be a pseudo stable submodule of M , then $N = M \oplus 0$ and by Theorem (3), M is pseudo lifting.

The following proposition shows that any direct summand of pseudo lifting module is pseudo lifting.

Proposition 10: Any direct summand of a pseudo lifting module is pseudo lifting.

Proof: Let M be a pseudo lifting module. Suppose that $M = M_1 \oplus M_2$.

We want to show that M_1 is pseudo lifting, let N be a submodule of M_1 , so $N \leq M$.

By Theorem (3), $N = A \oplus S$, where A is a pseudo stable direct summand of M and $S \ll M$.

By Lemma (2) $S \ll M_1$. Now $M = A \oplus T$, where $T \leq M$, since A is pseudo stable direct summand of M . We are done if we can show that A is a pseudo stable direct summand of M_1 .

Now $M_1 = M_1 \cap M = M_1 \cap (A \oplus T) = A \oplus (M_1 \cap T)$, by the modular law. Thus, A is a pseudo stable direct summand of N . By Theorem (3), M_1 is pseudo lifting.

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