

# Oscillation criteria for second-order nonlinear delay dynamic equations on time scales

**Abstract:** In this paper, we are concerned with a class of second-order nonlinear delay dynamic equations on time scales. By using the generalized Riccati technique and the integral averaging technique of Grace-type, four new oscillation criteria are obtained for every solutions of the equations to be oscillatory. Our results not only extend and improve some known theorems, but also unify the oscillation of the second-order nonlinear delay differential equation and the second-order nonlinear delay difference equation on time scales.

**Keywords:** oscillation criterion, dynamic equations, time scale

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## 1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [1], in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of

this new theory, see [2-4]. A time scale  $\mathbb{T}$  is an arbitrary closed subset of the reals, and when this time scale is equal to the reals or to the integers the cases represent the classical theories of differential and difference equations. Not only does the new theory of so-called “dynamic equations” unify the theories of differential equations and difference equations, but also the theory extends these classical cases to cases “in between”, e.g., to so-called  $q$ -difference equations when  $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$  or  $\mathbb{T} = \overline{q^{\mathbb{Z}}} := q^{\mathbb{Z}} \cup \{0\}$  and can be applied on different types of time scales like  $\mathbb{T} = h\mathbb{N}$ ,  $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$  and  $\mathbb{T} = \mathbb{T}_n = \{t_n = \sum_{k=1}^n \frac{1}{k}, n \in \mathbb{N}_0\}$  and the space of the harmonic numbers.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to Bohner [5], Erbe [6,7], and [8-15]. In this paper we deal with the oscillatory behavior of all solutions of nonlinear second-order delay dynamic equation

$$(a(t)(x^\Delta(t))^\gamma)^\Delta + q(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0, \quad (1.1)$$

subject to the hypotheses:

(H<sub>1</sub>)  $\mathbb{T}$  is a time scale (i.e., a nonempty closed subset of the real numbers  $\mathbb{R}$ ) which is unbounded above, and  $t_0 \in \mathbb{T}$  with  $t_0 > 0$ . We define the time scale interval of the form  $[t_0, \infty)_{\mathbb{T}}$  by  $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$ .

(H<sub>2</sub>)  $\gamma \geq 1$  is the ratio of two positive odd integers;

(H<sub>3</sub>)  $a, q$  are positive real-valued right-dense continuous functions on an arbitrary time scale  $\mathbb{T}$ .

(H<sub>4</sub>)  $\tau \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$  is a strictly increasing and differentiable function such that  $\tau(t) \leq t$ ,  $\tau(t) \rightarrow \infty$  as  $t \rightarrow \infty$  and  $\tilde{\mathbb{T}} := \tau(\mathbb{T}) \subset \mathbb{T}$ .

(H<sub>5</sub>)  $f \in C(\mathbb{R}, \mathbb{R})$  is a continuous function such that satisfies for some positive constant  $L$ ,

$$\frac{f(x)}{x^\gamma} \geq L \quad \text{for all } x \neq 0.$$

By a solution of (1.1), we mean a nontrivial real-valued function  $x$  satisfying (1.1) for  $t \in \mathbb{T}$ . We recall that a solution  $x$  of equation (1.1) is said to be oscillatory on  $[t_0, \infty)_{\mathbb{T}}$  in case it is neither eventually positive nor eventually negative; otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory in case all of its solutions are oscillatory. Our attention is restricted on those solutions

$x$  of (1.1) which  $x$  is not the eventually identically zero. Since  $a(t) > 0$ , we shall consider both the cases

$$\int_{t_0}^{\infty} \left( \frac{1}{a(t)} \right)^{\frac{1}{\gamma}} \Delta t = \infty, \quad (1.2)$$

and

$$\int_{t_0}^{\infty} \left( \frac{1}{a(t)} \right)^{\frac{1}{\gamma}} \Delta t < \infty. \quad (1.3)$$

It is easy to see that (1.1) can be transformed into a second-order nonlinear delay dynamic equation

$$(a(t)x^{\Delta}(t))^{\Delta} + q(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0, \quad (1.4)$$

where  $\gamma = 1$ . In (1.1), if  $f(x) = x^{\gamma}$ ,  $\tau(t) = t$ , then (1.1) is simplified to an equation

$$(a(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + q(t)x^{\gamma}(t) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \quad (1.5)$$

In (1.4), if  $a(t) = 1$ , then (1.4) is simplified to an equation

$$x^{\Delta\Delta}(t) + q(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \quad (1.6)$$

In (1.6), if  $f(x) = x$ , then (1.6) is simplified to an equation

$$x^{\Delta\Delta}(t) + q(t)x(\tau(t)) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \quad (1.7)$$

In 2005, Agarwal, Bohner and Saker [12] considered the linear delay dynamic equations (1.7), Sahiner [13] considered the nonlinear delay dynamic equations (1.6), and established some sufficient conditions for oscillation of (1.7) and (1.6). In 2007, Erbe, Peterson and Saker [14] considered the general nonlinear delay dynamic equations (1.4), setting to obtain some new oscillation criteria which improve the results given by Sahiner [13]. In 2005, Saker [15] and in 2009, Grace, Bohner and Agarwal [10] considered the half-linear dynamic equations (1.5), and established some sufficient conditions for oscillation of (1.5). In 2009, Sun, Han and Zhang [8] extended and improved the results of [12-14] to (1.1), meanwhile obtained some oscillatory criteria of (1.1). On this basis, we continue to discuss the oscillation of solutions of the equation (1.1), by using the generalized Riccati transformation and the inequality technique, we obtain some new oscillation criteria of the equation (1.1), our results extend and improve some known results.

The paper is organized as follows: In Sect. 2 we present basic definitions, and theory of calculus on time scale. In Sect. 3, we give several lemmas. In Sect. 4, we intend to use the generalized Riccati transformation and the inequality technique, to obtain some sufficient conditions for oscillation of all solutions of equation (1.1).

## 2 Some preliminaries

On the time scale  $\mathbb{T}$  we define the forward and the backward jump operators by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

A point  $t \in \mathbb{T}$  is said to be left-dense if  $\rho(t) = t$ , right-dense if  $\sigma(t) = t$ , left-scattered if  $\rho(t) < t$  and right-scattered if  $\sigma(t) > t$ . The graininess  $\mu$  of the time scale is defined by  $\mu(t) = \sigma(t) - t$ . For a function  $f : \mathbb{T} \rightarrow \mathbb{R}$ , the (delta) derivative is defined by <sup>[9]</sup>

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if  $f$  is continuous at  $t$  and  $t$  is right-scattered. If  $t$  is not right-scattered then derivative is defined by

$$f^\Delta(t) = \lim_{s \rightarrow t^+} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s \rightarrow t^+} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists. A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit at all left-dense points. The set of rd-continuous functions  $f : \mathbb{T} \rightarrow \mathbb{R}$  is denoted by  $C_{rd}(\mathbb{T}, \mathbb{R})$ . The derivative  $f^\Delta$  of  $f$  and the shift  $f^\sigma$  of  $f$  are related by the formula

$$f^\sigma = f + \mu f^\Delta \quad \text{where} \quad f^\sigma = f \circ \sigma.$$

We will make use of the following product and quotient rules for the derivative of the product  $fg$  and the quotient  $f/g$  of two differentiable functions  $f$  and  $g$

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)), \quad (2.1)$$

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}. \quad (2.2)$$

For  $b, c \in \mathbb{T}$  and a differentiable function  $f$ , the Cauchy integral of  $f^\Delta$  is defined by

$$\int_b^c f^\Delta(t) \Delta t = f(c) - f(b).$$

The integration by parts formula reads

$$\int_b^c f^\Delta(t) g(t) \Delta t = f(c)g(c) - f(b)g(b) - \int_b^c f^\sigma(t) g^\Delta(t) \Delta t \quad (2.3)$$

and infinite integrals are defined by

$$\int_b^\infty f(s) \Delta s = \lim_{t \rightarrow \infty} \int_b^t f(s) \Delta s.$$

Note that in the case  $\mathbb{T} = \mathbb{R}$ , we have

$$\sigma(t) = \rho(t) = t, \quad \mu(t) = 0, \quad f^\Delta(t) = f'(t), \quad \int_b^c f(t) \Delta t = \int_b^c f(t) dt,$$

and in the case  $\mathbb{T} = \mathbb{Z}$ , we have

$$\sigma(t) = t + 1, \quad \rho(t) = t - 1, \quad \mu(t) = 1, \quad f^\Delta(t) = \Delta f(t) = f(t + 1) - f(t),$$

and (if  $b < c$ )

$$\int_b^c f(t) \Delta t = \sum_{t=b}^{c-1} f(t).$$

For more details, see [3,4].

### 3 Several lemmas

**Lemma 3.1** (Bohner et al. [3, Theorem 1.93]) Assume that  $v : \mathbb{T} \rightarrow \mathbb{R}$  is strictly increasing and  $\tilde{\mathbb{T}} := v(\mathbb{T})$  is a time scale. Let  $w : \tilde{\mathbb{T}} \rightarrow \mathbb{R}$ . If  $v^\Delta(t)$  and  $w^{\tilde{\Delta}}(v(t))$  exist for  $t \in \mathbb{T}^k$ , where

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Then

$$(w \circ v)^\Delta = (w^{\tilde{\Delta}} \circ v) v^\Delta. \quad (3.1)$$

**Lemma 3.2** (Bohner et al. [3, Theorem 1.90]) Assume that  $x$  is delta-differentiable and eventually positive or eventually negative, then

$$((x(t))^\gamma)^\Delta = \gamma \int_0^1 [hx(\sigma(t)) + (1-h)x(t)]^{\gamma-1} x^\Delta(t) dh. \quad (3.2)$$

**Lemma 3.3** (Sun et al. [8, Lemma 2.1]) Assume  $(H_1)$ – $(H_5)$  and (1.2). Let  $x$  be an eventually positive solution of (1.1). Then there exists  $t_1 \in [t_0, \infty)_{\mathbb{T}}$  such that

$$x^\Delta(t) > 0, \quad (a(t)(x^\Delta(t))^\gamma)^\Delta < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \quad (3.3)$$

## 4 Main Results

**Theorem 4.1** Assume that the conditions  $(H_1)$ – $(H_5)$  and (1.2) hold, if there exists a positive nondecreasing  $\Delta$ –differentiable function  $\delta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$  such that for every  $T \in [t_0, \infty)_{\mathbb{T}}$

$$\limsup_{t \rightarrow \infty} \int_T^t [L\delta(s)q(s) - \eta^\gamma(\tau(s))\delta^\Delta(s)] \Delta s = \infty, \quad (4.1)$$

where

$$\eta(\tau(t)) = \left( \int_T^t \left( \frac{1}{a(\tau(s))} \right)^{\frac{1}{\gamma}} \tau^\Delta(s) \Delta s \right)^{-1}. \quad (4.2)$$

Then (1.1) is oscillatory on  $[t_0, \infty)_{\mathbb{T}}$ .

**Proof** Suppose to the contrary that  $x$  is a nonoscillatory solution of (1.1) on  $[t_0, \infty)_{\mathbb{T}}$ . We may assume without loss of generality that  $x(t) > 0$  and  $x(\tau(t)) > 0$  for all  $t \in [t_1, \infty)_{\mathbb{T}}$ ,  $t_1 \in [t_0, \infty)_{\mathbb{T}}$ . We shall consider only this case, since the proof when  $x$  is eventually negative is similar. By Lemma 3.3 we have  $x^\Delta(t) > 0$  for all  $t \in [t_2, \infty)_{\mathbb{T}}$ ,  $t_2 \in [t_1, \infty)_{\mathbb{T}}$ , and by Lemma 3.1 and  $(H_4)$ , there exists  $T \in [t_2, \infty)_{\mathbb{T}}$  such that  $(x(\tau(t)))^\Delta > 0$  for all  $t \in [T, \infty)_{\mathbb{T}}$ . Using (3.2) and (3.3), we have

$$\begin{aligned} ((x(\tau(t)))^\gamma)^\Delta &= \gamma \int_0^1 [h(x(\tau(t)))^\sigma + (1-h)x(\tau(t))]^{\gamma-1} (x(\tau(t)))^\Delta dh \\ &\geq \gamma \int_0^1 [h(x(\tau(t))) + (1-h)x(\tau(t))]^{\gamma-1} (x(\tau(t)))^\Delta dh \\ &= \gamma (x(\tau(t)))^{\gamma-1} (x(\tau(t)))^\Delta. \end{aligned}$$

Let  $v = \tau, w = x$  in Lemma 3.1, we know that  $\mathbb{T}$  is unbounded above by  $(H_1)$ , which implies  $\mathbb{T}^k = \mathbb{T}$ .

Further, as  $\tilde{\mathbb{T}} = \tau(\mathbb{T}) = \mathbb{T}$  by  $(H_4)$ , we can get from Lemma 3.1 that

$$(x(\tau(t)))^\Delta = x^\Delta(\tau(t))\tau^\Delta(t).$$

Thus

$$((x(\tau(t)))^\gamma)^\Delta \geq \gamma(x(\tau(t)))^{\gamma-1}x^\Delta(\tau(t))\tau^\Delta(t) > 0. \quad (4.3)$$

Define the function  $W(t)$  by

$$W(t) = \delta(t) \frac{a(t)(x^\Delta(t))^\gamma}{(x(\tau(t)))^\gamma}, \quad t \in [T, \infty)_{\mathbb{T}}. \quad (4.4)$$

Then on  $[T, \infty)_{\mathbb{T}}$ , we have  $W(t) > 0$ , by (2.1), (2.2) and (1.1), we obtain

$$\begin{aligned} W^\Delta(t) &= \frac{\delta(t)}{(x(\tau(t)))^\gamma} (a(t)(x^\Delta(t))^\gamma)^\Delta + a(\sigma(t))(x^\Delta(\sigma(t)))^\gamma \frac{(x(\tau(t)))^\gamma \delta^\Delta(t) - \delta(t)((x(\tau(t)))^\gamma)^\Delta}{(x(\tau(t)))^\gamma (x(\tau(\sigma(t))))^\gamma} \\ &\leq -Lq(t)\delta(t) + \frac{a(\sigma(t))(x^\Delta(\sigma(t)))^\gamma \delta^\Delta(t)}{(x(\tau(\sigma(t))))^\gamma} - \frac{\delta(t)a(\sigma(t))(x^\Delta(\sigma(t)))^\gamma ((x(\tau(t)))^\gamma)^\Delta}{(x(\tau(t)))^\gamma (x(\tau(\sigma(t))))^\gamma} \end{aligned} \quad (4.5)$$

$$\begin{aligned} &\leq -Lq(t)\delta(t) + \frac{a(\sigma(t))(x^\Delta(\sigma(t)))^\gamma \delta^\Delta(t)}{(x(\tau(\sigma(t))))^\gamma} \\ &\leq -Lq(t)\delta(t) + \frac{a(\tau(t))(x^\Delta(\tau(t)))^\gamma \delta^\Delta(t)}{(x(\tau(t)))^\gamma} = -Lq(t)\delta(t) + a(\tau(t))\delta^\Delta(t) \left( \frac{x^\Delta(\tau(t))}{x(\tau(t))} \right)^\gamma. \end{aligned} \quad (4.6)$$

Now

$$\begin{aligned} x(\tau(t)) &= x(\tau(T)) + \int_T^t x^\Delta(\tau(s))\tau^\Delta(s)\Delta s \\ &= x(\tau(T)) + \int_T^t \left( \frac{1}{a(\tau(s))} \right)^{\frac{1}{\gamma}} (a(\tau(s)))^{\frac{1}{\gamma}} x^\Delta(\tau(s))\tau^\Delta(s)\Delta s \\ &\geq \left( \int_T^t \left( \frac{1}{a(\tau(s))} \right)^{\frac{1}{\gamma}} \tau^\Delta(s)\Delta s \right) (a(\tau(t)) (x^\Delta(\tau(t)))^\gamma)^{\frac{1}{\gamma}}, \end{aligned}$$

and thus

$$\left( \frac{x^\Delta(\tau(t))}{x(\tau(t))} \right)^\gamma \leq \frac{1}{a(\tau(t))} \left( \int_T^t \left( \frac{1}{a(\tau(s))} \right)^{\frac{1}{\gamma}} \tau^\Delta(s)\Delta s \right)^{-\gamma} = \frac{\eta^\gamma(\tau(t))}{a(\tau(t))} \quad \text{for } t \in [T, \infty)_{\mathbb{T}}. \quad (4.7)$$

Using (4.7) in (4.6), we have

$$W^\Delta(t) \leq -Lq(t)\delta(t) + \eta^\gamma(\tau(t))\delta^\Delta(t) \quad \text{on } [T, \infty)_{\mathbb{T}}. \quad (4.8)$$

Integrating (4.8) from  $T$  to  $t$ , we obtain

$$0 < W(t) \leq W(T) - \int_T^t [L\delta(s)q(s) - \eta^\gamma(\tau(s))\delta^\Delta(s)] \Delta s,$$

which gives a contradiction using (4.1). This completes the proof.  $\square$

**Remark 4.1** From Theorem 4.1, we can obtain different conditions for oscillation of all solutions of (1.1) with different choices of  $\delta(t)$ .

Now, when (1.3) holds, we give the condition that guarantee that every solution of (1.1) oscillates.

**Theorem 4.2** Assume that the conditions  $(H_1)$ – $(H_5)$  and (1.3) hold. Let  $\delta$  is as in Theorem 4.1 and conditions (4.1), (4.2) hold, if for every  $T \in [t_0, \infty)_{\mathbb{T}}$

$$\int_T^\infty \left[ \frac{1}{a(s)} \int_T^s \theta^\gamma(u)q(u)\Delta u \right]^{\frac{1}{\gamma}} \Delta s = \infty, \quad (4.9)$$

where

$$\theta(t) = \int_t^\infty \left( \frac{1}{a(s)} \right)^{\frac{1}{\gamma}} \Delta s. \quad (4.10)$$

Then (1.1) is oscillatory on  $[t_0, \infty)_{\mathbb{T}}$ .

**Proof** Suppose to the contrary that  $x$  is a nonoscillatory solutions of (1.1) on  $[t_0, \infty)_{\mathbb{T}}$ . We may assume without loss of generality that  $x(t) > 0$  and  $x(\tau(t)) > 0$  for all  $t \in [t_1, \infty)_{\mathbb{T}}$ ,  $t_1 \in [t_0, \infty)_{\mathbb{T}}$ . We shall consider only this case, since the proof when  $x$  is eventually negative is similar. Since  $a(t)(x^\Delta(t))^\gamma$  is decreasing for all  $t \in [T, \infty)_{\mathbb{T}}$ ,  $T \in [t_1, \infty)_{\mathbb{T}}$ , it is eventually of one sign and hence  $x^\Delta(t)$  is eventually of one sign. Thus, we shall distinguish the following two cases:

(I)  $x^\Delta(t) > 0$  for  $t \geq T$ ; and

(II)  $x^\Delta(t) < 0$  for  $t \geq T$ .

*Case (I).* The proof when  $x^\Delta(t)$  is an eventually positive is similar to that of the proof of Theorem 4.1 and it hence is omitted.

*Case (II).* For  $s \geq t \geq T$ , we have

$$a(s)(-x^\Delta(s))^\gamma \geq a(t)(-x^\Delta(t))^\gamma,$$

and hence

$$-x^\Delta(s) \geq \left( \frac{a(t)}{a(s)} \right)^{\frac{1}{\gamma}} (-x^\Delta(t)). \quad (4.11)$$

Integrating (4.11) from  $t \geq T$  to  $u \geq t$  and letting  $u \rightarrow \infty$  yields

$$x(t) \geq \left[ \int_t^\infty \left( \frac{1}{a(s)} \right)^{\frac{1}{\gamma}} \Delta s \right] (a(t))^{\frac{1}{\gamma}} (-x^\Delta(t)) = -\theta(t) a^{\frac{1}{\gamma}}(t) x^\Delta(t) \quad \text{for } t \in [T, \infty)_{\mathbb{T}},$$

and thus

$$(x(t))^\gamma \geq -(\theta(t))^\gamma a(t) (x^\Delta(t))^\gamma \geq -(\theta(t))^\gamma a(T) (x^\Delta(T))^\gamma = b(\theta(t))^\gamma \quad \text{for } t \in [T, \infty)_{\mathbb{T}}, \quad (4.12)$$

with  $b = -a(T)(x^\Delta(T))^\gamma > 0$ . Using (4.12) in equation (1.1), we find

$$-(a(t)(x^\Delta(t))^\gamma)^\Delta \geq Lq(t)(x(\tau(t)))^\gamma \geq Lq(t)(x(t))^\gamma \geq bL\theta^\gamma(t)q(t) \quad \text{for } t \in [T, \infty)_{\mathbb{T}}, \quad (4.13)$$

Integrating (4.13) from  $T$  to  $t$ , we have

$$-a(t)(x^\Delta(t))^\gamma \geq -a(T)(x^\Delta(T))^\gamma + bL \int_T^t \theta^\gamma(s)q(s) \Delta s \geq bL \int_T^t \theta^\gamma(s)q(s) \Delta s,$$

so that

$$-x^\Delta(t) \geq \left[ \frac{bL}{a(t)} \int_T^t \theta^\gamma(s)q(s) \Delta s \right]^{\frac{1}{\gamma}}. \quad (4.14)$$

Integrating (4.14) from  $T$  to  $t$ , we obtain

$$\infty > x(T) \geq -x(t) + x(T) \geq \int_T^t \left[ \frac{bL}{a(s)} \int_T^s \theta^\gamma(u)q(u) \Delta u \right]^{\frac{1}{\gamma}} \Delta s \rightarrow \infty \quad \text{as } t \rightarrow \infty$$

by (4.9), a contradiction. This completes the proof.  $\square$

**Remark 4.2** Suppose that the condition (1.3) is satisfied, then Theorem 4.2 obtains the sufficient condition of oscillation for equation (1.1). The usual result is that the conditions (1.3) was established, then every solution of the equation (1.1) is either oscillatory or converges to zero on  $[t_0, \infty)_{\mathbb{T}}$ .

**Theorem 4.3** Assume that the conditions  $(H_1)$ – $(H_5)$  and (1.2) hold, if there exists a positive  $\Delta$ -differentiable function  $\delta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$  such that for every  $T \in [t_0, \infty)_{\mathbb{T}}$

$$\limsup_{t \rightarrow \infty} \int_T^t \left[ Lq(s)\delta(s) - \frac{(a(\tau(s)))^{\frac{1}{\gamma}} (\delta^\Delta(s))^2}{4\gamma\delta(s)\tau^\Delta(s)} (\eta^\sigma(s))^{\gamma-1} \right] \Delta s = \infty, \quad (4.15)$$

where  $\eta$  is as in (4.2). Then (1.1) is oscillatory on  $[t_0, \infty)_{\mathbb{T}}$ .

**Proof** Suppose to the contrary that  $x$  is a nonoscillatory solutions of (1.1) on  $[t_0, \infty)_{\mathbb{T}}$ . We may assume without loss of generality that  $x(t) > 0$  and  $x(\tau(t)) > 0$  for all  $t \in [t_1, \infty)_{\mathbb{T}}$ ,  $t_1 \in [t_0, \infty)_{\mathbb{T}}$ . We

shall consider only this case, since the proof when  $x$  is eventually negative is similar. Proceeding as in the proof of Theorem 4.1, we obtain (4.3) and (4.5). Using (4.3) in (4.5), we have on  $[T, \infty)_{\mathbb{T}}$  that

$$W^{\Delta}(t) \leq -Lq(t)\delta(t) + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)a(\sigma(t))(x^{\Delta}(\sigma(t)))^{\gamma}x^{\Delta}(\tau(t))\tau^{\Delta}(t)}{(x(\tau(\sigma(t))))^{\gamma+1}}. \quad (4.16)$$

By  $(a(t)(x^{\Delta}(t))^{\gamma})^{\Delta} < 0$ , we have

$$a(\tau(t))(x^{\Delta}(\tau(t)))^{\gamma} \geq a(\sigma(t))(x^{\Delta}(\sigma(t)))^{\gamma} \quad \text{i.e.} \quad x^{\Delta}(\tau(t)) \geq \frac{(a(\sigma(t)))^{\frac{1}{\gamma}}}{(a(\tau(t)))^{\frac{1}{\gamma}}}x^{\Delta}(\sigma(t)). \quad (4.17)$$

Substituting (4.17) in (4.16), we obtain

$$W^{\Delta}(t) \leq -Lq(t)\delta(t) + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^{\Delta}(t)}{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta(\sigma(t)))^{\frac{\gamma+1}{\gamma}}}(W(\sigma(t)))^{\frac{\gamma+1}{\gamma}} \quad \text{on } [T, \infty)_{\mathbb{T}}.$$

i.e., on  $[T, \infty)_{\mathbb{T}}$

$$W^{\Delta}(t) \leq -Lq(t)\delta(t) + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^{\Delta}(t)}{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta(\sigma(t)))^{\frac{\gamma+1}{\gamma}}}(W(\sigma(t)))^{\frac{1-\gamma}{\gamma}}(W(\sigma(t)))^2. \quad (4.18)$$

Now inequality (4.7), i.e.

$$\frac{x(\tau(t))}{x^{\Delta}(\tau(t))} \geq \frac{a^{\frac{1}{\gamma}}(\tau(t))}{\eta(\tau(t))} \quad \text{i.e.} \quad \frac{x(t)}{x^{\Delta}(t)} \geq \frac{a^{\frac{1}{\gamma}}(t)}{\eta(t)}$$

implies on  $[T, \infty)_{\mathbb{T}}$  that

$$(W(t))^{\frac{1-\gamma}{\gamma}} = (\delta(t)a(t))^{\frac{1-\gamma}{\gamma}} \left( \frac{x(t)}{x^{\Delta}(t)} \right)^{\gamma-1} \geq (\delta(t)a(t))^{\frac{1-\gamma}{\gamma}} \frac{a^{\frac{\gamma-1}{\gamma}}(t)}{\eta^{\gamma-1}(t)} = \delta^{\frac{1-\gamma}{\gamma}}(t)\eta^{1-\gamma}(t). \quad (4.19)$$

Using (4.19) in (4.18), we have on  $[T, \infty)_{\mathbb{T}}$  that

$$\begin{aligned} W^{\Delta}(t) &\leq -Lq(t)\delta(t) + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^{\Delta}(t)}{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta(\sigma(t)))^2}(\eta(\sigma(t)))^{1-\gamma}(W(\sigma(t)))^2 \\ &= -Lq(t)\delta(t) + \frac{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta^{\Delta}(t))^2}{4\gamma\delta(t)\tau^{\Delta}(t)}(\eta(\sigma(t)))^{\gamma-1} \\ &\quad - \left[ \frac{\sqrt{\gamma\delta(t)\tau^{\Delta}(t)(\eta(\sigma(t)))^{1-\gamma}}}{\delta(\sigma(t))\sqrt{(a(\tau(t)))^{\frac{1}{\gamma}}}}W(\sigma(t)) - \frac{\sqrt{(a(\tau(t)))^{\frac{1}{\gamma}}}\delta^{\Delta}(t)}{2\sqrt{\gamma\delta(t)\tau^{\Delta}(t)(\eta(\sigma(t)))^{1-\gamma}}} \right]^2 \\ &\leq - \left[ Lq(t)\delta(t) - \frac{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta^{\Delta}(t))^2}{4\gamma\delta(t)\tau^{\Delta}(t)}(\eta(\sigma(t)))^{\gamma-1} \right]. \end{aligned}$$

Integrating both sides of this inequality from  $T$  to  $t$ , taking the lim sup of the resulting inequality as  $t \rightarrow \infty$  and applying condition (4.15), we obtain a contradiction to the fact that  $W(t) > 0$  for  $t \in [T, \infty)_{\mathbb{T}}$ . This completes the proof.  $\square$

Using the same ideas as in the proof of Theorem 4.2, when (1.3) holds, we can now obtain following result.

**Theorem 4.4** Assume that the conditions  $(H_1)$ – $(H_5)$ , (1.3), (4.9) and (4.15) hold, let  $\eta$  and  $\theta$  be as in (4.2) and (4.10). Then (1.1) is oscillatory on  $[t_0, \infty)_{\mathbb{T}}$ .  $\square$

**Remark 4.3** Our results in this paper improve and extend some theorems in [15] and [10]. Especially, when  $f(x) = x^\gamma$ ,  $\tau(t) = t$ , the Theorems 4.1–4.4 can convert into Theorems 3.2, 3.1, 3.6 and 3.5 in [10]. The theorems in this paper are new even for the cases  $\mathbb{T} = \mathbb{R}$  and  $\mathbb{T} = \mathbb{Z}$ .

Here, we shall reformulate the above conditions which are sufficient for the oscillation of (1.1) when (1.2) holds on different time scales:

If  $\mathbb{T} = \mathbb{R}$ , then (1.1) becomes

$$(a(t)(x'(t))^\gamma)' + q(t)f(x(\tau(t))) = 0, \quad t \in [t_0, \infty), \quad (4.20)$$

condition (4.1) and (4.15), respectively, becomes

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left[ L\delta(s)q(s) - \delta'(s) \left( \int_{\tau(t_1)}^{\tau(s)} \left( \frac{1}{a(u)} \right)^{\frac{1}{\gamma}} du \right)^{-\gamma} \right] ds = \infty, \quad (4.21)$$

and

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left[ Lq(s)\delta(s) - \frac{(a(\tau(s)))^{\frac{1}{\gamma}}(\delta'(s))^2}{4\gamma\delta(s)\tau'(s)} \left( \int_{\tau(t_1)}^{\tau(s)} \left( \frac{1}{a(u)} \right)^{\frac{1}{\gamma}} du \right)^{1-\gamma} \right] ds = \infty, \quad (4.22)$$

conditions (4.21) and (4.22) are new. The Theorems 4.1 and 4.3 are new even for the cases  $\mathbb{T} = \mathbb{R}$ .

If  $\mathbb{T} = \mathbb{Z}$ , then (1.1) becomes

$$\Delta(a_n(\Delta x_n)^\gamma) + q_n f(x_{n-\sigma}) = 0, \quad n = 0, 1, 2, \dots, \quad (4.23)$$

condition (4.1) and (4.15), respectively, becomes

$$\limsup_{n \rightarrow \infty} \sum_{l=n_0}^n \left[ Lq_l \delta_l - \Delta \delta_l \left( \sum_{k=n_0-\sigma}^{l-\sigma-1} \left( \frac{1}{a_k} \right)^{\frac{1}{\gamma}} \right)^{-\gamma} \right] = \infty, \quad (4.24)$$

and

$$\limsup_{n \rightarrow \infty} \sum_{l=n_0}^n \left[ Lq_l \delta_l - \frac{(a_{l-\sigma})^{\frac{1}{\gamma}} (\Delta \delta_l)^2}{4\gamma \delta_l} \left( \sum_{k=n_0-\sigma}^{l-\sigma} \left( \frac{1}{a_k} \right)^{\frac{1}{\gamma}} \right)^{1-\gamma} \right] = \infty, \quad (4.25)$$

conditions (4.24) and (4.25) are new. The Theorem 4.1 and 4.3 are new even for the cases  $\mathbb{T} = \mathbb{Z}$ .

**Example** Considered the second-order nonlinear delay 2-difference equations

$$\left( t^{\frac{2}{3}} (x^\Delta(t))^{\frac{5}{3}} \right)^\Delta + t^{-\frac{8}{5}} \left( x \left( \frac{t}{2} \right) \right)^{\frac{5}{3}} \left( 1 + x^2 \left( \frac{t}{2} \right) \right) = 0, \quad t \in \overline{2\mathbb{Z}}, \quad t \geq t_0 := 2. \quad (4.26)$$

Here

$$a(t) = t^{\frac{2}{3}}, \quad q(t) = t^{-\frac{8}{5}}, \quad f(x) = x^{\frac{5}{3}} (1 + x^2), \quad \tau(t) = \frac{t}{2}, \quad \gamma = \frac{5}{3}.$$

The conditions (H<sub>1</sub>)-(H<sub>4</sub>) are clearly satisfied, (H<sub>5</sub>) holds with  $L = 1$ , next, for  $t \geq 2$  so that

$$\int_2^t \left( \frac{1}{a(s)} \right)^{\frac{1}{\gamma}} \Delta s = \int_2^t s^{-\frac{2}{5}} \Delta s = \frac{t^{\frac{3}{5}} - 2^{\frac{3}{5}}}{2^{\frac{3}{5}} - 1} \rightarrow \infty \quad \text{as } t \rightarrow \infty,$$

hence (1.2) is satisfied. Now let  $\delta(t) = t^{\frac{3}{5}}$  for all  $t > s \geq 2$ ,

$$\eta(\tau(t)) = \left( \int_2^t \left( \frac{1}{a(\tau(s))} \right)^{\frac{1}{\gamma}} \tau^\Delta(s) \Delta s \right)^{-1} = \left( \frac{1}{2^{\frac{2}{5}}} \int_2^t s^{-\frac{2}{5}} \Delta s \right)^{-1} = 2^{\frac{3}{5}} \frac{2^{\frac{3}{5}} - 1}{t^{\frac{3}{5}} - 2^{\frac{3}{5}}},$$

then

$$\int_2^t [L\delta(s)q(s) - \eta^\gamma(\tau(s))\delta^\Delta(s)] \Delta s = \int_2^t \left[ \frac{1}{s} - 2 \left( \frac{2^{\frac{3}{5}} - 1}{s^{\frac{3}{5}} - 2^{\frac{3}{5}}} \right)^{\frac{5}{3}} \left( s^{\frac{3}{5}} \right)^\Delta \right] \Delta s \rightarrow \infty \quad \text{as } t \rightarrow \infty,$$

so that (4.1) is satisfied as well. Altogether, by Theorem 4.1, the equation (4.26) is oscillatory.

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