

Analytical Solution of the Steady-state Isothermal Melt Spinning model

Abstract

In this paper, we consider an isothermal melt spinning process modeled by a system of coupled non-linear ordinary differential equations. We use an approach based on a numerical method to develop an analytical solution of the nonlinear system of equations and compared with numerical solution.

Keywords: melt spinning process; isothermal, steady-state, shooting method.

1 Introduction

At industrial scales, melt spinning is the most economic and convenient method for polymer fiber manufacturing. In the process of melt spinning, molten polymer is thrust out of the spinneret to form thin, cylindrical fibers. The drum placed at a distance from the spinneret is used to collect the fibers at a set take up speed. The take up speed is considerably higher than extruding speed. The process is shown in the figure 1. The extrusion speed v_0 is much smaller than the take up speed v_L so that the fiber filament considerably becomes stretched in length and decreases its diameter between the take up point and the spinneret (die). The actual process is to cool the producing fiber in between these two points such that the fiber becomes solid at the take up point. Moreover, at the spinneret, the initial area A_0 and velocity v_0 are specified. The take up speed v_L is also specified at the take up point.

There are different mathematical models with different level of demands available in the literature that describe the process of melt spinning, e.g. see [2, 3, 4, 8, 12]. In this paper, we consider a steady-state model of an isothermal melt spinning process consists of coupled non-linear ordinary differential equations [8] of first and second order.

In [2, 8] numerical solutions of the mathematical models have been determined and used only to stability and optimal control of the melt spinning processes. In these research articles, the analytical solution of the model is not determined. Our approach is to find the analytical solution of the steady-state isothermal model of the melt spinning process as given in [2, 8, 12]. It is not possible to determine the analytical solution with the given initial and boundary conditions attached to the mathematical model. While solving the second order ordinary differential equation appears in the model, we are unable to determine all the constants of integration. To remove this difficulty, we use a numerical approach *Shooting Method* to convert the boundary value problem to a set of initial value problems (Detailed procedure is given in subsection 2.2), has been used in [2, 8] to compute the numerical solutions of the boundary value problems.

The model has been converted into a system of initial problems and then we use some existing

integrating techniques to develop the analytical solution of the model.

The structure of the paper is as follows: In section 2, we give a brief description of the mathematical model of the isothermal melt spinning process. Analytical solution is developed in section 3 and plotted in comparison to numerical solution in section 4. We conclude the results in section 5.

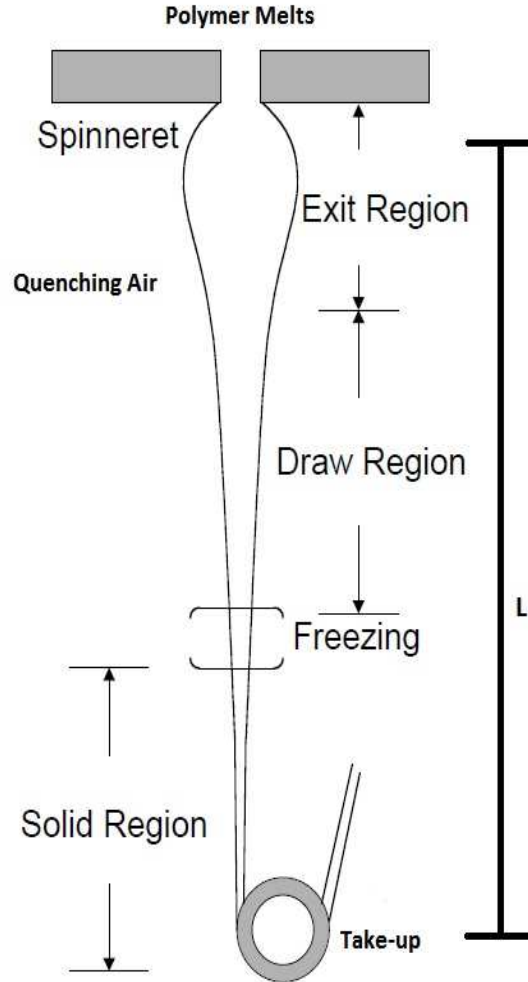


Figure 1: Melt Spinning Process.

2 Isothermal Melt Spinning Model

We consider a mathematical model for an isothermal melt spinning process [8] given as

$$\frac{\partial A}{\partial t} = -\frac{\partial}{\partial x}(Av), \quad (1a)$$

$$\frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial x} + \rho^{-1}\frac{1}{A}\frac{\partial}{\partial x}(A\omega), \quad (1b)$$

along with the initial and boundary conditions:

$$A(x=0, t) = A_0, \quad v(x=0, t) = v_0, \quad v(x=L, t) = v_L, \quad \text{for all } t \geq 0, \quad (1c)$$

where L is the total length of the spinline, t is the time and x denotes the coordinate along the spinline. System (1) gives us the velocity v and the cross-sectional area A of the fiber. The polymer density ρ is considered to be constant for isothermal process.

The constitutive model (1) needs a relation of axial stress ω to the viscosity η given by

$$\omega = 3\eta \frac{dv}{dx}. \quad (2)$$

2.1 Dimensionless Form

To convert the simplified model (1) into dimensionless form, we introduce the dimensionless quantities

$$\tilde{t} = \frac{tv_0}{L}, \quad \tilde{v} = \frac{v}{v_0}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{A} = \frac{A}{A_0}, \quad \tilde{\omega} = \frac{\omega L}{\eta v_0}.$$

Omitting the tilde notation, we reach at the following system of dimensionless differential equations

$$\frac{\partial A}{\partial t} = -\frac{\partial}{\partial x}(Av), \quad (3a)$$

$$\frac{\partial v}{\partial t} = -v \frac{\partial v}{\partial x} + \frac{3}{Re} \frac{1}{A} \frac{\partial}{\partial x} \left(\eta A \frac{\partial v}{\partial x} \right), \quad (3b)$$

with the conditions

$$A(x=0, t) = 1, \quad v(x=0, t) = 1, \quad v(x=1, t) = d, \quad \text{for all } t \geq 0, \quad (3c)$$

where $Re = \frac{\rho L v_0}{\eta}$ is the dimensionless parameter known as Reynolds number and $d = \frac{v_L}{v_0} > 1$ denotes the draw ratio, is an operating parameter of the melt spinning process. For the isothermal Newtonian flow of the melt spinning, the viscosity η of the melt polymer throughout the process remains constant and is taken as 1.

The steady state form of the model (3) is given as

$$\frac{d}{dx}(Av) = 0, \quad (4a)$$

$$-v \frac{dv}{dx} + \frac{3}{Re} \frac{1}{A} \frac{d}{dx} \left(A \frac{dv}{dx} \right) = 0, \quad (4b)$$

subject to the conditions

$$A(x=0) = 1, \quad v(x=0) = 1, \quad v(x=1) = d. \quad (4c)$$

The system (4) is a system of coupled nonlinear ordinary differential equations. We are interested to develop analytical solution of this model.

2.2 Shooting Method

We can solve boundary value problems analytically by different methods available in literature. The solution given by these methods contains constants of integration that can be easily found by using boundary conditions. However, there exist some problems (e.g problem (4)) where we are unable to find constants of integration with the given boundary conditions. In such case, we may take help of numerical methods that transform the boundary value problem into an initial

value problem. *Shooting method* is the numerical technique, used to transform a boundary value problem

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}) \quad y(a) = \alpha, \quad y(b) = \beta,$$

to an initial value problem

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}) \quad y(a) = \alpha, \quad \frac{dy}{dx}(a) = w, \quad (5a)$$

where the number w is simply a guess and sometimes it is called an integrator. We apply one of the step-by-step numerical techniques like Euler's method or the MATLAB ODE solver *ode45* (based on Runge-Kutta method) to the second order differential equation (5a) to find an approximation w_i for the value of $y(b)$.

For the first guess w_1 , let us assume

$$w_1 = \frac{dy}{dz}(a) \approx \frac{y(b) - y(a)}{b - a}.$$

Let us consider N segments between the two boundaries, $z = a$ and $z = b$ then, $h = \frac{b - a}{N}$. We use Euler's method with w_1 to find an approximation β_1 for $y(b)$. If β_1 agrees with the given value $y(b)$ to some pre-assigned tolerance, we stop; otherwise starting with a different guess w_2 , defined as

$$w_2 = 2w_1,$$

to obtain a second approximation β_2 for $y(b)$. If β_2 agrees with the value $y(b)$ to pre-assigned tolerance, we stop; otherwise this time we use the secant method to find a new guess

$$w_3 = \frac{dy}{dz}(a) \approx w_2 - e(w_2) \left(\frac{w_1 - w_2}{e(w_1) - e(w_2)} \right),$$

where

$$e(w) = \beta_i - y(b),$$

$i = 1, 2, \dots$ be the error in the result of the shooting method.

Repeating the Euler's method with w_3 to obtain an approximation β_3 for $y(b)$. If β_3 agrees with the given value, we stop; otherwise, starting with a new guess w_4 to obtain an approximation β_4 for $y(b)$. This method can be continued in a trial and error manner and we get $w_5, w_6, \dots$ in a systematic way. In this method, the aim is the choice of the initial value of w which agrees with the given value $y(b) = \beta$.

Parameter	Symbol	Approximate Value	Units
Density of the polymer	ρ	0.001	kg/m^3
Feeding speed	v_0	10	m/s
Take up speed	v_L	50	m/s
Length of the spinline	L	1	m
Viscosity of the polymer	η	1	kg/ms

Table 1: Summary of parameteric values appearing in the steady state model (4).

3 Analytical Solution

To find the analytical solution of the steady state model (4), we proceed as follows:

By using (4c), solution of the continuity equation (4a) in terms of state variable v is given as

$$A = \frac{1}{v}, \quad v > 0, \quad (6)$$

where A and v are functions of $x \in \Omega$. This solution show a constant flow rate of the melt polymer. Now, the equation (4b) can be put into the form

$$\frac{d^2v}{dx^2} = \frac{Re}{3}v \frac{dv}{dx} + \frac{1}{v} \left(\frac{dv}{dx} \right)^2. \quad (7a)$$

This is a non-linear ordinary differential equation along with the conditions

$$v(x=0) = 1, \quad v(x=1) = v_d. \quad (7b)$$

We transform the boundary value problem (7) into two initial value problems by setting

$$\frac{dv}{dx} = \frac{1}{3}\omega.$$

This implies that

$$\frac{d^2v}{dx^2} = \frac{1}{9}\omega \frac{d\omega}{dv}.$$

Then, with the above two substitutions, the initial value problem corresponding to (7) is defined as.

$$\frac{dv}{dx} = \frac{1}{3}\omega, \quad (8a)$$

$$\frac{d\omega}{dv} - \frac{1}{v}\omega = Rev. \quad (8b)$$

with the conditions

$$v(x=0) = 1, \quad \omega(x=0) = \omega_0. \quad (8c)$$

where ω_0 is approximated value guessed by shooting the value of v at 1 i.e. $v(1) = v_d$. Detailed procedure to find the numerical solution of the above type of problems (i.e. problem (8)) by *shooting method* is given in section 4.

After some basic calculus, the solution of the equation (8b) is determined to be

$$\omega = Rev^2 + C_1v, \quad (9)$$

where C_1 is a constant of integration, still to be determined.

The solution (9) can also be written as:

$$3 \frac{dv}{dx} = Rev^2 + C_1v. \quad (10)$$

Separating variables and then integrating, we have

$$\int \frac{1}{\frac{Re}{3}v^2 + \frac{1}{3}C_1v} dv = x + c_2,$$

where C_2 is another constant of integration. Solving the left hand integral, we reach at

$$\frac{3}{C_1} \ln \left| \frac{v}{v + \frac{1}{Re} C_1} \right| = x + C_2. \quad (11)$$

Using (2), (10) and then applying (4c), we obtain

$$C_1 = \omega(0) - Re, \quad (12)$$

where $\omega(0)$ is the numerical value guessed by shooting the boundary $v(x = 1) = d$. In view of (4c), the equation (11) gives us

$$C_2 = -\frac{3}{C_1} \ln \left| 1 + \frac{1}{Re} C_1 \right|. \quad (13)$$

Now equation (11) yield us

$$v = \frac{C_1 \exp \left[\frac{1}{3} C_1 x + C_1 C_2 \right]}{Re \left[1 - \exp \left(\frac{1}{3} (C_1 x + C_1 C_2) \right) \right]}, \quad (14)$$

where v is the analytical form of steady state solution of the melt spinning model (4) along with C_1 and C_2 given in (12)-(13). For future reference, we denote it by v_s . Thus, the solutions A and v together constitute the analytical solution of the steady state model (4). Both the solutions are plotted in the figure 2.

4 Comparison of the Solutions

In this section, we compare both the numerical and analytical solutions of the model (4). To find the numerical solution, we used both of the numerical techniques: Eulers method and Matlab ODE solver *ode45*. The values of the parameters appearing in the steady state equations for the isothermal model (4) are given in the table 1. Figure 2 shows the numerical and analytical solutions of the steady-state model (4) for velocity v and cross-sectional area A of the fiber along the spinline. Both the solutions are very close to each-other. The error between the numerical and the analytical solutions for velocity v and cross-sectional area A as shown in figure 3 is multiple of 10^{-7} and 10^{-5} respectively, which is negligibly small.

5 Conclusions

In this work we have used a numerical approach to find analytical solution of a boundary value problem where one cannot find constants of integrations with the given boundary conditions. The approach is to convert a boundary value problem into initial value problems and then to develop the analytical solution of the resulting problems with the usual methods. To explain the approach, we have developed an analytical solution of the steady state model of the isothermal melt spinning process with the help of *shooting method*. The obtained analytical solutions are also plotted.

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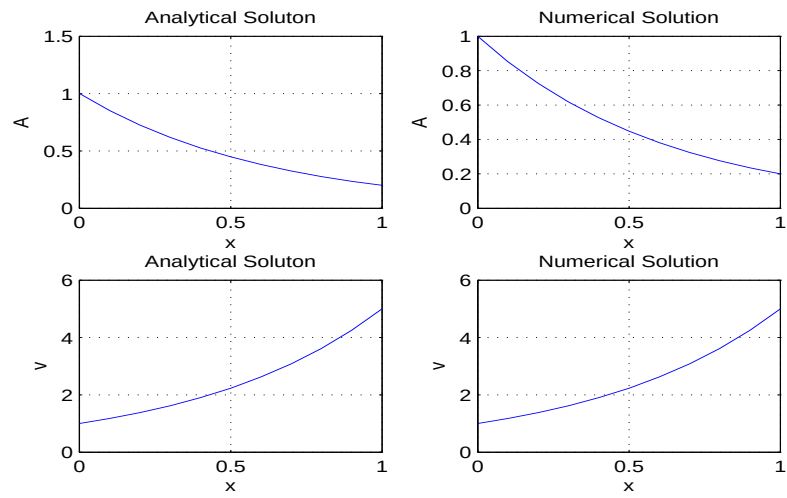


Figure 2: Steady-state exact and numerical solutions for velocity v and cross-sectional area A with $h = 0.01$.

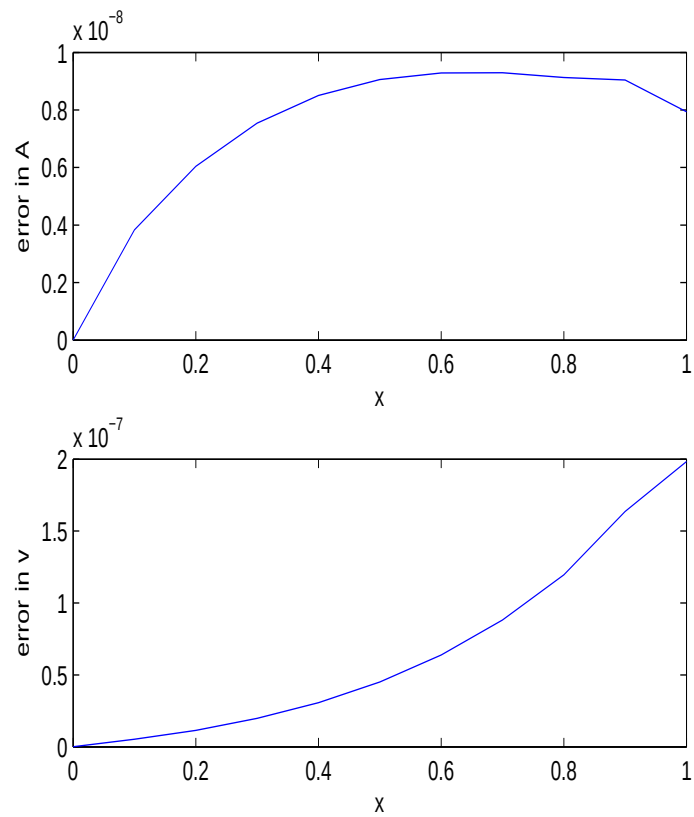


Figure 3: Errors between exact and numerical Steady-state solutions for velocity v and cross-sectional area A with $h = 0.01$.

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