

A Metric for Data Set Comparison Via QR-Factorization

Type of Article

Abstract

Aims/ objectives: A new method is introduced for comparison of matrix data sets based on QR-factorizations. The method consists of a metric that measures the degree of similarity between the bases of any two matrices, with measure values in the interval $[0, 1]$.

Study design: Derivation and Implementation

Place and Duration of Study: Department of Mathematics, College of Arts and Sciences, North Carolina A&T State University, between June 2010 and July 2015.

Methodology: The metric is presented initially in a general framework, for any $m \times n$ matrix. The proposed measure is then discussed specifically in reference to matrices representing theoretical and physical data sets. This method has direct applications for projects involving particle travel, and image recognition, comparison, and registration, particularly in instances involving intermodal data acquisition.

Results: The measure is shown to satisfy the requirements for a metric. The metric is successfully used to calculate rotation angle between two data sets that differ by angle, θ . The metric is also successfully used to identify an unknown sample in a data base in a single blind test.

Conclusion: The metric works well in assessing differences between data sets that differ by small perturbations. The data sets should be aligned prior to comparison. There is potential for using the metric to determine rotation angles that optimize alignment in a minimum distance sense. More work is necessary to determine if the method can be applied to $3 - D$ data sets that differ by rotations around two different axes.

Keywords: *QR-decomposition; QR-factorization; basis comparison; trajectory comparison; matrix comparison; similarity measure*

2010 Mathematics Subject Classification: 15A23, 62E15, 60A25, 70 – 04

1 Introduction

Object recognition is one of the fundamental problems of computer vision, and shape is recognized as one of the most powerful object descriptors ([Pan (2012)], [Selvarajah (2011)]); and as the digital image storage and retrieval industry has grown, so has the need for shape matching methods ([Belorgie (2002)], [Zhang (2004)]). While many methods have been developed, most are job-specific with advantages as well as disadvantages [Nawrat (2008)], and the need for more general methods remains. Existing methods proceed in two main steps: shape representation, followed by recognition ([Selvarajah (2011)]), the latter consisting of some measure of similarity between a known template and the unknown shape normally dictated by the representation. However, while the multi-variate classic theory of shape ([Goodall (1993)]) is built on the analysis of matrix representations, no universal shape comparison metric appears to exist across representations; the purpose of this paper is to describe one such measure based on QR-factorizations of matrix representations.

Where matrix comparisons are used, whether it be in shape recognition or other general matrix data setting, ([Fowlkes (1983)], [Sebastian (2003)]), such comparisons normally use L-norms to quantify direct matrix disparities, such as in linear least squares fitting problems ([Arun (1987)], [Pratt (1987)]), where the typical matrix comparison is of the form

$$\|A_2 - A_1\| = \epsilon. \quad (1.1)$$

For data that represents an Eulerian flow fields, for example, this measures the difference in intensity values, while in Lagrangian trajectory data, it captures the difference in physical distance ([Jones (2004)]). Such L -norm measures therefore ultimately yield comparisons only from a physical distance or size perspective. In such studies, the goal is usually to draw conclusions about the inherent nature of a system based on these physical comparisons, and larger differences in the measurement are assumed to imply larger differences in the system as a whole. While this is certainly valid from the distance perspective, it is not true from a shape perspective. Consider a rock dropped into a lake. If one were to measure the difference in ripples at time t_1 versus time t_2 , the result would be a large error, appropriate for the time frame. Concurrently, however, the ripples themselves would still have largely the same shape, so to draw conclusions on the inherent nature of the system based on a distance argument could be misleading. This is also true for many biological shape applications, such as fingerprints or tree rings, where the shape is invariant while size grows. A different metric is therefore in order that can convey these shape similarities, rather than the physical distance between them; we introduce the Q-metric to address the issue.

Where QR-decomposition has been used in shape theory ([Diaz (2010)]), the focus has been on the shape-and-size distributions for which the R -matrix is central, along with the reduction of the original matrices so as to simplify direct comparison of their 'sizes', and not on a comparison measure. We use the property that the Q -matrix in the QR-decomposition is in the Stiefel manifold, that is, $Q^*Q = I$, where $()^*$ denotes conjugate transposition, to construct a direct measure. To motivate the method, consider the curves given in Figure (1), representing an image that has been scaled by constant, k .

While the images are different in scale, they are images of the same figure: while their sizes differ, their shape is the same. However, direct comparison of the matrices of points that create them using equation (1) results in a value that is dependent upon the scaling factor, that is,

$$\|A_2 - A_1\|_L = \|kA_1 - A_1\|_L = |k - 1|\|A_1\|. \quad (1.2)$$

The measure clearly captures the difference in size, but not the same-ness in shape: a different metric is required to measure the equality of the shapes. Realizing that equality in this context means that the same basis shape template produced both shapes, and since a matrix's QR-factorization

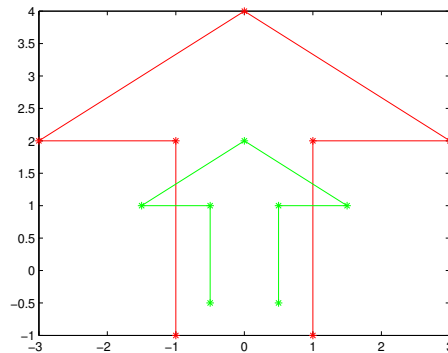


Figure 1: Images of two data sets that differ by scalar, k

produces a basis for its column space, a natural metric is suggested. We explore and exploit the connection and use it to define a measure which captures the equality of the shapes.

The rest of the paper is organized as follows: In the next section, QR -factorization is briefly reviewed, and the new metric, the Q -metric, is defined for general $n \times m$ QR -factorable matrices. We then verify the metric properties of the new measure, and its rigid transformation properties are discussed with illustrations. Applications to shape rotation and registration and to image recovery using Hispaniola Amazon palate data are presented, and a brief discussion concludes the article.

2 QR -factorization and the Q -metric

It is well-known that every matrix $A \in \mathbb{R}^{m \times n}$ ($m \geq n$) has a QR -factorization, and it is unique under given circumstances. A full discussion of QR -factorizations is beyond the scope of this article; we refer the reader to the literature for in depth analysis of the features and formation of QR -factorizations (see Stewart (1977), Stewart (1993), Sun (1991)), and recall here only a few facts from Stewart (2000). A QR -factorization decomposes a matrix $A \in \mathbb{R}^{m \times n}$ into an orthonormal matrix, Q_A , and an upper triangular matrix, R_A , such that $A = Q_A R_A$. If A is of full rank, then this factorization is unique, the columns of Q_A form an orthonormal basis for the column space of A , $S(A)$ and $Q_A^* Q_A = I$ is the projection of A onto $S(A)$: we exploit these features of the factorization to create the metric.

While matrices are widely used both as a means of shape representation and digital image storage, typical matrix comparisons using (1.1) are ill-suited for shape comparisons. The comparison in (1.1) is inferred from the measure of the difference between their representations and physical distances: there is an implicit assumption that the representations of the objects must be close in size and position in order for the objects to be similar. The same assumption underlies many shape recognition methods. We offer a viewpoint more consistent with the observation of Figure(1), requiring a metric which allows the relative representations to be different, but the shape assessment, as defined by human view, to be similar. We proceed under the perspective that two matrices A and B are the same provided that each can be perfectly projected onto the other, that is, if and only if $S(A)=S(B)$. As Q_A represents the column space ($S(A)$) of A , it is proposed that the difference between the matrices Q_A and Q_B can be used to compare the spaces $S(A)$ and $S(B)$ in the same manner that the norm of the difference between matrices A and B , $(\|B - A\|)_p$ can, albeit with an upper bound. We let A and B be two shape matrices of full rank, with guaranteed unique

QR -factorizations $A = Q_A R_A$ and $B = Q_B R_B$, and with respective column spaces denoted by $S(A)$ and $S(B)$. The expression $Q_B Q_A$ is the projection of A onto $S(B)$ and the expression $Q_A Q_B$ is the projection of B onto $S(A)$. It follows then that if $Q_B^* Q_A = I$ then $A \in S(B)$ and $B \in S(A)$. The relationship drives the formulation of the Q -metric:

$$\mu(S(A), S(B)) = \frac{1}{2} \|Q_B^* Q_A - I\| \quad (2.1)$$

which measures how different A is from $S(B)$, the column space of B , rather than how far apart A and B are from a distance perspective. It represents a projection of one basis onto another, with the expected result to be I if the two bases are equivalent. In the case when $Q_A = Q_B$, the weighted measure returns a value assessing the difference between the projection and unit vector space. In this context, any residual found from the measure can be seen as an assessment of the deviation between the space of the projection and unit vector space.

3 Properties of the Q -metric

We begin by noting that, based on the orthonormal properties of the Q -matrices in an orthogonal decomposition, $\|Q_B^* Q_A\| = 1$. It follows from the triangle inequality that

$$0 \leq \mu(S(A), S(B)) \leq 1 \quad (3.1)$$

3.1 Metric Properties

We demonstrate below that the Q -metric satisfies all metric properties, regardless of matrix norm chosen.

3.1.1 Non-negativity

$$\mu(S(A), S(B)) \geq 0 \quad (3.2)$$

The non-negativity property is satisfied by definition. For all norms, $\|\bullet\| \geq 0$, thus it follows that

$$\mu(S(A), S(B)) = \frac{1}{2} \|Q_B^* Q_A - I\| \geq 0 \quad (3.3)$$

for all matrices A and B .

3.1.2 Identity

$$\mu(S(A), S(B)) = 0 \iff S(A) = S(B)$$

A feature of the orthonormal matrices of a QR -factorization is that $Q_A^* Q_A = I$ (or $Q_A^* = Q_A^{-1}$). First, we assume that $\mu(S(A), S(B)) = 0$. It follows that

$$\mu(S(A), S(B)) = \frac{1}{2} \|Q_B^* Q_A - I\| = 0 \quad (3.5)$$

which can only be satisfied if $Q_B^* Q_A = I$. By definition, Q_B^* must be equivalent to Q_A^{-1} . Also from the properties of orthonormal matrices, $Q_A^* = Q_A^{-1}$, thus $Q_B^* = Q_A^*$ and $Q_B = Q_A$. Conversely, we can assume that $S(A) = S(B)$. Then

$$\begin{aligned}\mu(S(A), S(B)) &= \mu(S(A), S(A)) \\ &= \frac{1}{2} \|Q_A^* Q_A - I\| \\ &= \frac{1}{2} \|I - I\| \\ &= 0\end{aligned}\tag{3.6}$$

From construction of the Q matrices and uniqueness of inverses, $Q_B = Q_A$ is the only solution, verifying uniqueness of the proposed measure.

3.1.3 Symmetry

$$\mu(S(A), S(B)) = \mu(S(B), S(A))\tag{3.7}$$

We use the properties of the matrix transpose as applied to a product of matrices. Given $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times m}$,

$$(AB)^* = B^* A^*\tag{3.8}$$

where $*$ denotes the conjugate transpose. Also, relative to a general p -norm, $\|A^*\| = \|A\|$. Using these facts, it follows:

$$\begin{aligned}\mu(S(A), S(B)) &= \frac{1}{2} \|Q_B^* Q_A - I\| \\ &= \frac{1}{2} \|Q_B^* Q_A - Q_A^* Q_A\| \\ &= \frac{1}{2} \|(Q_B^* - Q_A^*) Q_A\| \\ &= \frac{1}{2} \| [Q_A^* (Q_B - Q_A)]^* \| \\ &= \frac{1}{2} \| [Q_A^* (Q_B - Q_A)] \| \\ &= \frac{1}{2} \| [Q_A^* Q_B - I]^* \| \\ &= \mu(S(B), S(A))\end{aligned}\tag{3.9}$$

3.1.4 Triangle Inequality

$$\mu(S(A), S(C)) \leq \mu(S(A), S(B)) + \mu(S(B), S(C))\tag{3.10}$$

To establish this property, we consider 5 cases in turn:

(a) Case 1: $A = B = C$

The inequality is trivially satisfied in this case since $S(A) = S(B) = S(C)$, so that $\mu(S(A), S(B)) = \mu(S(A), S(C)) = \mu(S(B), S(C)) = \mu(S(A), S(A)) = 0$ by the identity property (3.1.2).

(b) Case 2: $A = C \neq B$ In this case, $S(A) = S(C)$, so that $\mu(S(A), S(C)) = 0$ by property (3.1.2). The inequality holds since the right-hand side of the triangle inequality is non-negative from the properties of p -norms. The case can be proved directly using property (3.1.2).

$$\begin{aligned}
 \mu(S(A), S(C)) &\leq \mu(S(A), S(B)) + \mu(S(B), S(C)) \\
 \mu(S(A), S(A)) &\leq \mu(S(A), S(B)) + \mu(S(B), S(A)) \\
 0 &\leq \mu(S(A), S(B)) + \mu(S(B), S(A)) \\
 0 &\leq 2\mu(S(A), S(B))
 \end{aligned} \tag{3.11}$$

which is true for all norms and is identically equal to 0 only in the case in which $A = B$ from property (3.1.2).

(c) Case 3: $A = B \neq C$ In this case, $S(A) = S(B)$, so that $\mu(S(A), S(B)) = 0$. If we examine only the right side of the inequality that we are attempting to establish, we see:

$$\begin{aligned}
 \mu(S(A), S(C)) &\leq \mu(S(A), S(B)) + \mu(S(B), S(C)) \\
 &\leq \mu(S(A), S(A)) + \mu(S(A), S(C)) \\
 &\leq 0 + \mu(S(A), S(C)) \\
 &\leq \mu(S(A), S(C))
 \end{aligned} \tag{3.12}$$

(d) Case 4: $C = B \neq A$ Proceeding as in Case 3, we show

$$\begin{aligned}
 \mu(S(A), S(C)) &\leq \mu(S(A), S(B)) + \mu(S(B), S(C)) \\
 &\leq \mu(S(A), S(C)) + \mu(S(C), S(C)) \\
 &\leq \mu(S(A), S(C)) + 0 \\
 &\leq \mu(S(A), S(C))
 \end{aligned} \tag{3.13}$$

which establishes equality in the case.

(e) Case 5: $A \neq B \neq C$ We show that the case satisfies the property through direct calculation. For simplicity, we scale the measure by 2 to remove the $\frac{1}{2}$ scaling factor.

$$\begin{aligned}
 2\mu(S(A), S(C)) &= \|Q_C^* Q_A - I\| \\
 &= \|Q_C^* Q_B Q_B^* Q_A - I\| \\
 &= \|(Q_C^* Q_B - Q_A^* Q_B)(Q_B^* Q_A)\| \\
 &= \|(Q_C^* Q_B - Q_A^* Q_B)\| \|Q_B^* Q_A\| \\
 &= \|(Q_C^* Q_B - Q_A^* Q_B) - I + I\| \\
 &= \|(Q_C^* Q_B - I) - (Q_A^* Q_B - I)\| \leq \|(Q_C^* Q_B - I)\| + \|(Q_A^* Q_B - I)\| \\
 &\leq 2(\mu(S(A), S(B)) + \mu(S(B), S(C)))
 \end{aligned} \tag{3.14}$$

The equivalence of $\mu(S(A), S(B))$ with $\mu(S(B), S(A))$ was established in (3.1.3).

3.2 Invariance Properties and Relationship with Rigid Transformations

We present general observations about the effect of rigid transformations, namely scaling, rotation, and translation, on the Q -metric. For simplicity of exposition, we limit observations about rotation and translations to only 2×2 matrices; the general $n \times 2$ case will be demonstrated experimentally in the next section.

3.2.1 Scaling

We note that scale invariance is an intrinsic property of the measure; but for completeness, we observe that if B is a scaling of a given matrix, A , then $B \in S(A)$, so that $Q_B^* Q_A = I$, and thus

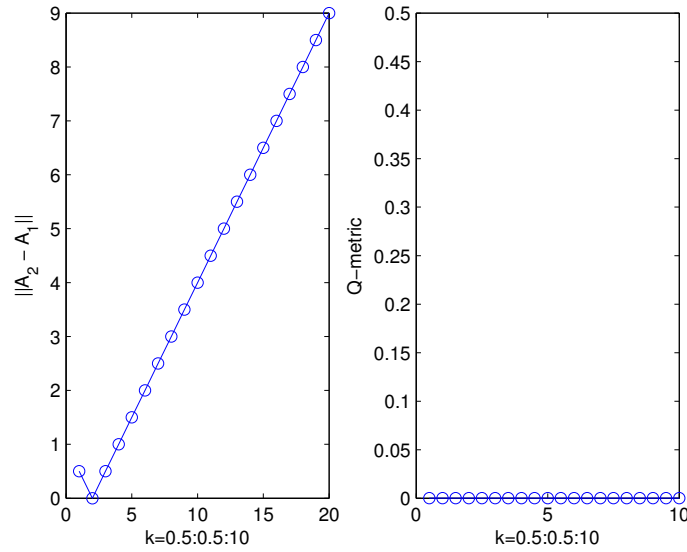


Figure 2: L-norm and Q-metric comparisons between A_2 and A_1 , where $A_2 = kA_1$

$\mu(S(A), S(B)) = \mu(S(B), S(A)) = \frac{1}{2} \|Q_B^* Q_A - I\| = 0$, and the measure is scale-invariant.

To illustrate scale invariance (assuming proper, centered alignment), additional data sets of the type in figure (1) were generated. In addition to the base image, scalings of the base image represented as $kA = k(x_i, y_i)$, $i \in [1 : n]$, were generated for constant, k .

While physically different from a distance perspective, these perturbations have the same basic shape. The results of the norm and Q-metric comparisons are summarized in figure (2).

The figure provides an example of how the Q-metric is scale invariant; in particular, while the actual difference between the representative matrices ($\|A_2 - A_1\|_2$, where $A_2 = kA_1$, increases with k , the Q-metric measure of their difference remains constant at machine zero due to the scaling component being represented in the R matrix of the decomposition rather than the Q matrix.

3.2.2 Rotation

The proposed metric is based upon the orthonormal matrices of a QR -factorization. Orthonormal matrices are by definition also rotation matrices. The Q_i matrices of a factorization should then be able to be used to identify the angle between two data sets, and by extension, the orientation for optimal alignment.

Consider orthonormal matrices Q_1 and Q_2 taken from matrices $A_1, A_2 \in \mathbb{R}^{2 \times n}$ consisting of data points (x, y) , centered at the origin. We take A_2 to be a rotation of A_1 around the origin by some angle. For a vector, $\mathbf{x}$, both Q_1 and Q_2 will rotate the vector around the origin in a counterclockwise direction, while the inverses of the matrices will rotate the vector around the origin in the clockwise direction. Assume that Q_1 rotates around the origin by an angle of α and that Q_2 rotates around the origin by an angle of β . Based on the proposed metric:

$$\mu(sp(A_1), sp(A_2)) = \frac{1}{2} \|Q_2^* Q_1 - I\|_2 \quad (3.15)$$

also, since the orthonormal matrices, Q_1 and Q_2 are also rotation matrices, they can be represented as

$$Q_1 = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

and

$$Q_2 = \begin{bmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{bmatrix}$$

If $\alpha = \beta$ then $Q_2 = Q_1$ and $Q_2^* = Q_1^{-1}$, resulting in an assessment of $\mu = 0$ from the measure. Physically, if $\alpha = \beta$, the matrices rotate a vector in one direction and then back the other, realigning the vector in its original location. If $\alpha \neq \beta$, then the Q_1 matrix will rotate a vector around the origin by α , while the Q_2^* matrix will rotate the vector back by β . The residual between the product of the rotation matrices and the identity can be exploited to calculate the difference between the rotations. If the Q_i rotation matrices are taken from two different A_i matrices, the difference can be used to calculate the rotation angle of optimal alignment. Performing the assessment assumes that either the centroids the data sets have been previously aligned, or the axis of rotation between the data sets is known.

$$\begin{aligned} Q_2^* Q_1 &= \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta) \\ -\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha - \beta) & \sin(\alpha - \beta) \\ -\sin(\alpha - \beta) & \cos(\alpha - \beta) \end{bmatrix} \end{aligned} \quad (3.20)$$

we note that $Q_1^* Q_2$ is the transpose of this matrix ($(Q_2^* Q_1)^* = Q_1^* Q_2$). To calculate the L_2 norm of the residual matrix:

$$\begin{aligned} \frac{1}{2} \|Q_2^* Q_1 - I\|_2 &= \frac{1}{2} \sqrt{\max(\lambda((Q_2^* Q_1 - I)^*(Q_2^* Q_1 - I)))} \\ \mu(sp(A_1), sp(A_2)) &= \frac{1}{2} (\sqrt{\max(\lambda(Q_1^* Q_2 Q_2^* Q_1 - Q_1^* Q_2 - Q_2^* Q_1 + I))}) \\ &= \frac{1}{2} \sqrt{\max(\lambda(2I - Q_1^* Q_2 - Q_2^* Q_1))} \\ &= \frac{1}{2} \sqrt{\max(\lambda(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2\cos(\alpha - \beta) & 0 \\ 0 & 2\cos(\alpha - \beta) \end{bmatrix}))} \\ &= \frac{1}{2} \sqrt{2\max(\lambda(\begin{bmatrix} 1 - \cos(\alpha - \beta) & 0 \\ 0 & 1 - \cos(\alpha - \beta) \end{bmatrix}))} \end{aligned} \quad (3.21)$$

Which results in one repeated eigenvalue, $\lambda = (1 - \cos(\alpha - \beta))$. Based on the definition of the metric:

$$\begin{aligned}\mu &= \frac{1}{2}(\sqrt{2(1 - \cos(\alpha - \beta))}) \\ \frac{2\mu}{\sqrt{2}} &= \sqrt{1 - \cos(\alpha - \beta)} \\ \frac{4\mu^2}{2} &= 1 - \cos(\alpha - \beta) \\ 1 - 2\mu^2 &= \cos(\alpha - \beta) \\ \cos^{-1}(1 - 2\mu^2) &= (\alpha - \beta)\end{aligned}\tag{3.22}$$

The result is a measure of the rotational difference between the two data sets. If the orientation of one of the data sets is known or assumed, the angle that results in the optimal alignment between the data sets can be calculated.

3.2.3 Translation

In its current form, the Q -metric is not translation invariant, and the problem of its normalization is not a trivial one. Based on the limitation, an alignment of the center of mass coordinate will be assumed in applications, so that comparisons are only made between representations on the same coordinate system. The practice is a standard one, found in many methods of registration. Typically, the center of mass is located and aligned *before* the registration is performed ([Fitzpatrick (19)]).

4 Applications

In this section, we present two applications of the Q -metric, first to the measure of the angle that aligns two data sets and then to digital image registration and retrieval. In the second example, data sets representing palates of the Hispaniolan Amazona parrot will be used to demonstrate image registration and identification.

4.1 Image Alignment

We consider two data A_1 and A_2 that differ only by rotation angle, θ , such that

$$A_2 = R_\theta A_1\tag{4.1}$$

where

$$R_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}\tag{4.2}$$

For testing purposes A_2 is found by rotating A_1 around the origin by an angle of 50° . The data being compared form the matrices:

$$A_1 = \begin{bmatrix} 1.3066 & 0.1585 & 2.0063 & -1.5307 & -3.35370 & -1.6892 & -0.5412 \\ -0.5412 & 2.2304 & 2.9958 & 3.6955 & 0.6997 & 1.4651 & -1.3066 \end{bmatrix}\tag{4.3}$$

$$A_2 = \begin{bmatrix} 0.0000 & 2.1213 & 3.5355 & 2.8284 & -0.7071 & 0.7071 & -1.4142 \\ -1.4142 & 0.7071 & -0.7071 & 2.8284 & 3.5355 & 2.1213 & -0.0000 \end{bmatrix}\tag{4.4}$$

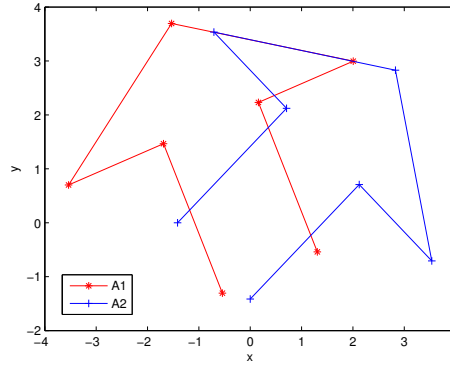


Figure 3: Comparison data sets, with aligned centroids, differing by angle, θ

We calculate the metric value by first finding the QR -factorizations of each matrix and then plugging into the formula:

$$\begin{aligned} A_1 &= Q_1 R_1 \\ A_2 &= Q_2 R_2 \\ \mu &= \frac{1}{2} \|Q_2^* Q_1 - I\| \end{aligned} \quad (4.5)$$

Based on the value of μ , we can calculate the angle between the two data sets as found in section 3.2.2.

$$\Delta\theta = \cos^{-1}(1 - 2\mu^2). \quad (4.6)$$

4.2 Image Retrieval

The Q -metric is particularly applicable to biological shape applications, such as for fingerprints, tree rings, or bird palates, where the pattern shape is effectively invariant while the size grows. We consider a data set that represents not trajectories, but fiducial landmarks as used in point based, rigid registration of images, and in particular data sets representative of the soft palates of parrots. It is believed that, while there is considerable variation in the pattern of ridges and sensory papillae in the cornified tissue of the upper palate in members of Hispaniolan Amazona, the integrated pattern of the cornified oral surface of the upper beak is likely to be as unique an identifier for parrots as fingerprints are for humans [Homberger (2008)]. In a previous study [Homberger (2008)], the palates of several parrots were digitized and stored in a database. The information was stored as a matrix data set in which the columns of the matrices contained the (x, y) coordinates for the points that were proposed to be representative of the individual parrots. In that study, a rigid registration of an unknown sample was compared to a database of previously digitized samples. The registration aligned the data in a manner that minimized the distances between corresponding points in the two data sets. The minimum values were then compared to determine the sample number of the unknown birds. The process was computationally expensive due to the need to conduct the entire registration process for all birds in the database.

We conducted a single blind test, in which an unknown image from a data set was analyzed to determine its identity. Representative landmarks (fiducial markers) were selected on the image using a prescribed criteria (which matched the criteria used for forming the database). The fiducial markers of the unknown data set were then compared to the fiducial markers of each member of the database in an effort to identify the unknown sample. An image of the unknown palate and its fiducial markers,

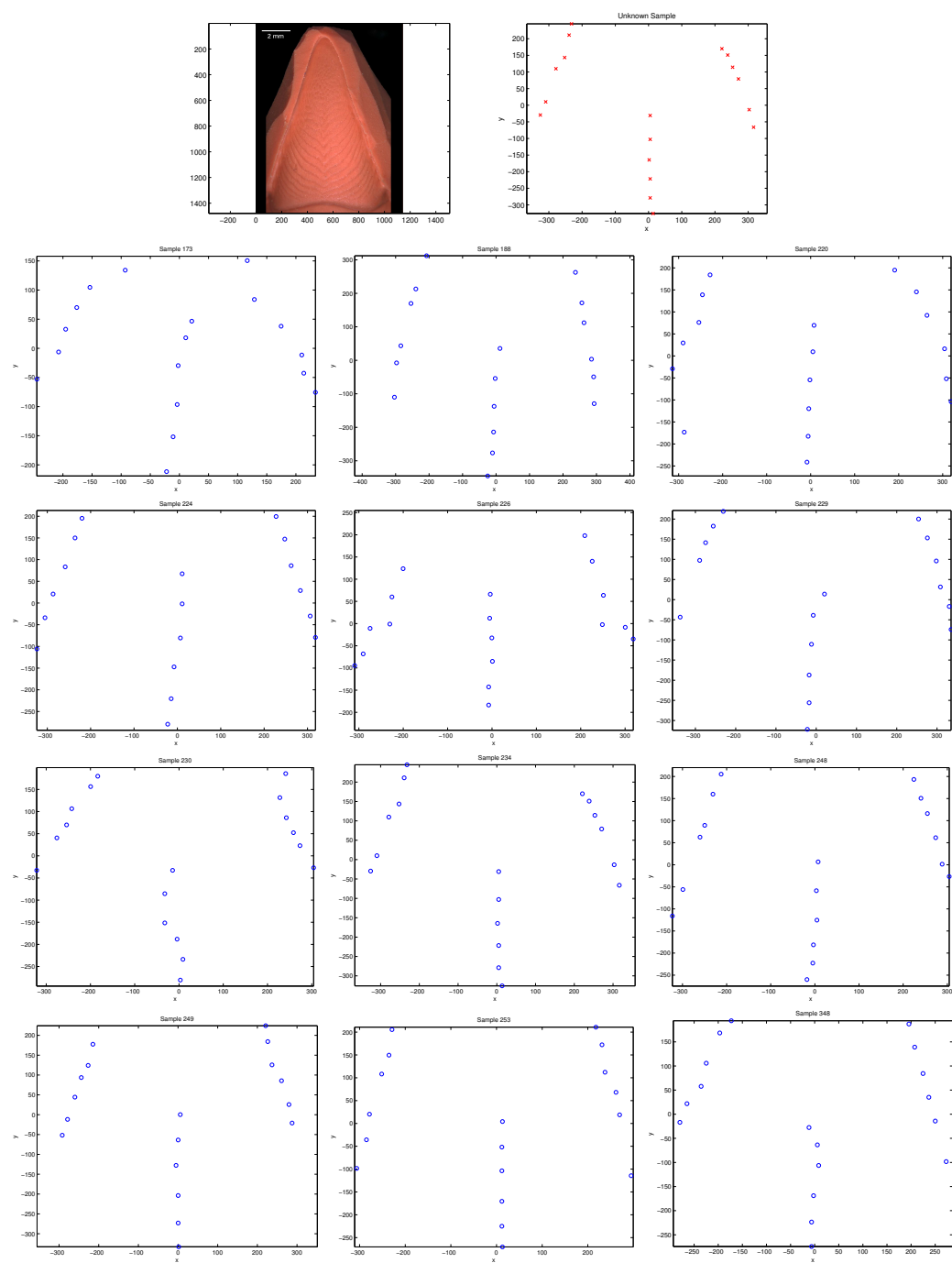


Figure 4: Sample of the palate image and the unknown and comparison data sets, with centroids translated to the origin

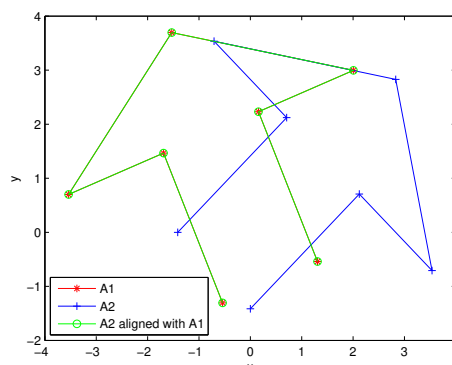


Figure 5: Data sets A_1 and A_2 . The angle calculated using the Q -metric has been applied to A_2 , resulting in excellent alignment with the A_1 matrix

along with some sample database palates are provided in Figure 4. The Q -metric is a measure of the similarity of matrices. Smaller values suggest matrices that are more similar, while larger values suggest matrices that are not very similar. We expect that the sample that returns the smallest value should be the best match.

5 Results and Discussion

5.1 Angle of Optimal Alignment

After calculating the Q -metric value, we substituted into the angle formula 4.6. The angle between the data sets was found to be 50° . The value is exactly the angle that was used to generate the second data set. Applying a rotation matrix to the original data set, A_1 , it is rotated and superimposed onto the plot of the original data sets (Figure 5). In this example, the images were known to be rotated around the same axis. Without *a priori* knowledge of the axis, it would be necessary to center each data set by locating the centroid and translating it to the origin. While the method works effectively in R^2 , application to data sets in R^3 has not been resolved. The additional directional axis will require that the single metric, μ , be able to calculate rotation around up to three different axes.

5.2 Identification within a database

We conduct a single blind test similar to the one performed in the previous study (Homberger (2008)) in which a data set representing an unknown, unregistered palate pattern is compared to a database of known palate patterns. Both data sets are known to be of full rank by nature, and using the Q -metric, it is possible to determine the best match to the data set without performing the registration process (involving a singular value decomposition) used in the previous study. The results from the study are summarized in Figure (7). Using the same data sets, the Q -metric was used to measure similarity between the unknown sample and all of the other members of the data set. The results are summarized in Table (1) and in Figure (6); Table (1) displays the Q metric values for each sample number, while Figure (6) shows the fiducial markers from the unknown data set with the fiducial markers from the best match in the database, based on the Q -metric values. Based on the results in

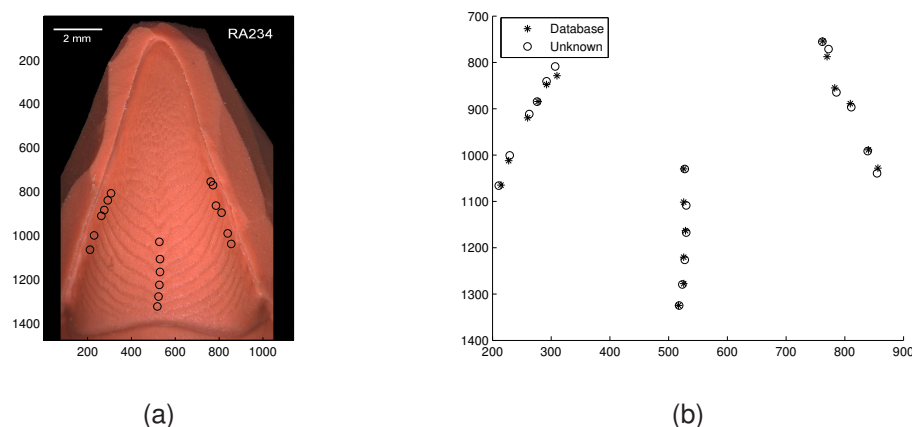


Figure 6: Example of a) a palate and sample points that represent its ridge structures, and b) a registration of an unknown data set (stars) with the database (circles) from palate project from a minimum distance perspective.

Sample	115	135	167	173	188	220	224	226	227	229
Q-metric	0.0703	0.0881	0.0588	0.1531	0.0830	0.1439	0.0837	0.1565	0.0861	0.0713
Sample	230	234	238	240	242	244	245	246	247	248
Q-metric	0.0714	0.0130	0.0687	0.1280	0.1001	0.0566	0.1515	0.0862	0.0604	0.0648
Sample	249	253	348	405	407	420	531			
Q-metric	0.0612	0.0627	0.0624	0.1052	0.0698	0.0555	0.0991			

Table 1: Q-metric values for unknown palate versus database. Sample 234 returns the lowest Q-metric value of 0.0130

Figure (7), the sample in the database that best matches the unknown sample in a lowest Q -metric sense, is number 234. Sample number 234 is verified to be the correct sample. A limitation of this application over the previously mentioned method is that it does not actually perform a registration. If a registration is needed, the method presented in section (4.1) can be used to align the images and then a registration error can be calculated to assess the match. If the data are not well aligned to begin with, or at least known to have the same orientation, then the information gleaned from this could be misleading. In the situation under consideration here however, where the data acquisition process was known to be uniform, then the Q -metric is very appropriate for comparison. While it should not be used in a blindly definitive manner, it can be successfully used to narrow the search window.

Relative to database searches, the Q -metric only conveys which specimen an unknown specimen is *most like*. If the correct match is not known to be in a database, a small, non-zero value cannot necessarily be used to definitively claim a match. It *can*, however, be used to effectively remove members of a database whose values are very large.

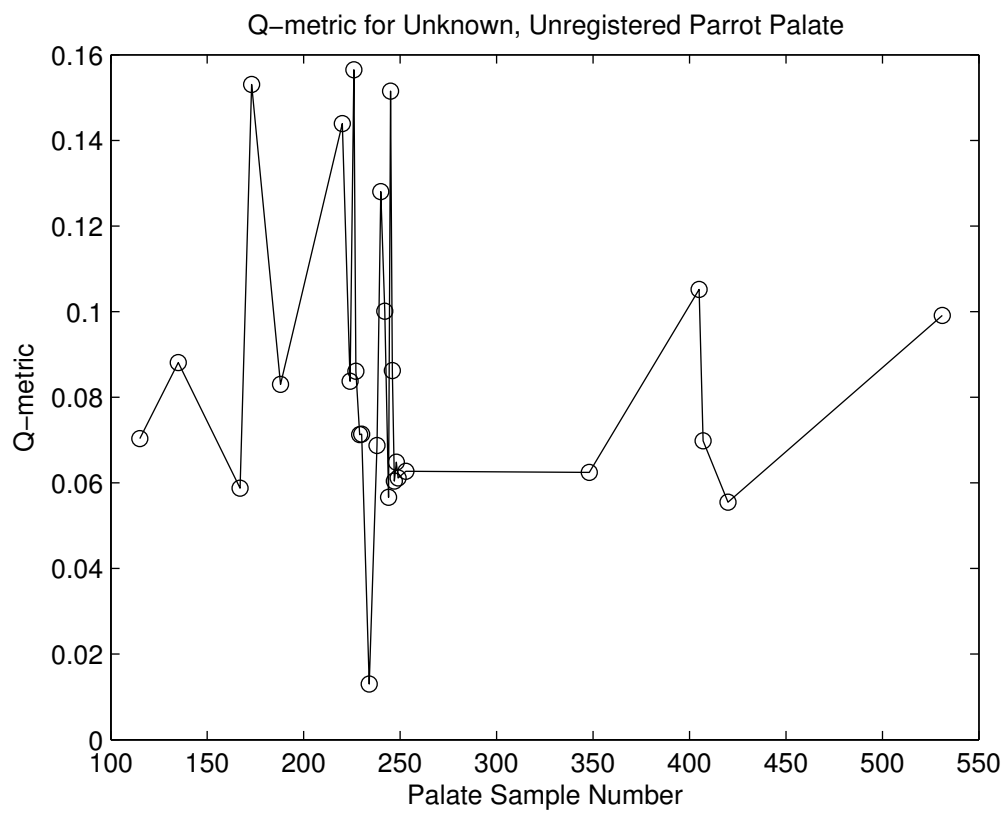


Figure 7: Q-metric value for comparison of unidentified sample with the database

6 CONCLUSIONS

A method for comparing matrix data sets is presented based on QR -factorizations. In general this method will apply to any $m \times n$ of full column rank, and has direct applications in image recognition, comparison, and registration, particularly in instances involving intermodal data acquisition.

- a** In section (2) the measure was defined in terms of the components of the QR -factorizations of the matrices under consideration.
- b** In section (3) the metric properties of the measure were verified. Based on the inherent features of the orthonormal matrix, Q , of a QR -factorization, the introduced metric is bounded and scale invariant, and has predictable rotation variance; its application assumes the use of the center of mass of the objects being compared.
- c** Examples of applications of the metric were provided in section (??). The metric is a measure of the difference between the bases of two given matrices, which is advantageous in situations in which the space spanned by the bases of the matrices is of more interest than the difference between matrices themselves.
- d** In subsection (4.1), the application of the metric to the measure of the angle of rotation between corresponding data sets was investigated. The result was a closed form solution for data sets in R^2 .
- e** In subsection (4.2), another application of the metric was discussed, this time with regards to image registration within a database.

References

- Arun, K. S., Huang, T. S., Blostein, S. D. (1987) Least-squares fitting of two 3 – D point sets. IEEE Trans. On Pattern Anal. and Mach. Intell., 5, 698 – 700
- Belongie, S., Malic, J., Puzicha, J. (2002) Shape matching and object recognition using shape contexts, IEEE Trans. on Pattern Anal. and Mach. Intell., 24, 509 – 522
- Diaz-Garcia, J., Caro-Lopera, F., (2010) Shape theory via QR factorization, arXiv:1003.3439 [math.ST]
- Fitzpatrick, J. M., Hill, D. L. G., Maurer, Jr., C. R., Image Registration, in: Sonka, M., Fitzpatrick, J. M. (Eds.) Handbook of Med. Imag., 2, SPIE Publications, Bellingham, WA pp. 451 – 489
- Fowlkes, E. B., Mallows, C. L., (1983) A method for comparing two hierarchical clusterings. Journ. of Amer. Statist. Assoc., 78, 553 – 569
- Goodall, C. R., Mardia, K. V., (1993) Multivariate aspects of shape theory, Ann. Statist., 21, 848 – 866
- Homberger, D., Hopkins, B., Osborn, M., Castille, A., Bragulla, H., Matthews, K., Butler, L., Tully, T., (2008) The ridge pattern of the cornied oral surface of the upper beak of parrots: Individual genus-level character, ICVM–8

- Jones, D. .F., Mardle, S. J., (2004) A distance-metric methodology for the derivation of weights from a pairwise comparison matrix, Journ. Oper. Research Soc., 55, No. 8, 869 – 875
- Nawrat, A., Rachwal, T., Daniec, K., (2008) Mathematical methods for target detection in the CCD Camera Stream . World Systemics, Cybern., and Inform. Conf. (WMSCI) (*Last accessed on 10 April 2016 at 23:02*).
- Pan, Z., Xiao, Q., Chen, K., Li, Z., (2012) A spectral matching for shape retrieval using pairwise critical points, Foundations of Intell. Systems, AISC, 122, 475 – 482
- Pratt, V., (1987) Direct least-fitting of algebraic surfaces, ACM SIGGRAPH Comp. Graph., 21 No. 4, 145 – 152
- Sebastian, T., Klein, P. N., Kimia, B. B., (2003) On Aligning Curves, PAM1 25, No. 1, 116 – 125
- Selvarajah, S., Kodituwakku, S. R., (2011) Performance evaluation of shape analysis techniques, ARPJ Jour. of Syst. and Software, 1, No. 1, 12 – 18
- Stewart, G. W., (1977) Perturbation bounds for the QR factorization of a matrix, SIAM Journ. Numer. Anal., 14, No. 3, 509 – 518
- Stewart, G. W., (1993) On the perturbation of the LU, Cholesky, and QR factorization of a Matrix, SIAM Journ. on Matrix Anal. and Applic., 14, No. 4, 1141 – 1145
- Stewart, G. W., (2000), The decompositional approach to matrix computation, Comp. in Sci. Eng., Jan/Feb, 50 – 59
- Sun, J. G., (1991) Perturbation bounds for the Cholesky and QR factorizations, BIT, 31, 341 – 352
- Zhang, D., Lu, G., (2004), Review of shape representation and description techniques, Pattern Recogn. 37, 1 – 19