

A New Viewpoint in the study of Quasi b-Components in Bitopological Spaces

Abstract

In This paper we introduce and study Quasi b-components in bitopological spaces using b-open sets due to Andrijevic [2]. This concept is obtained by generalizing the idea of quasi-components due to Reilly and Young [9].

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1 Introduction and Preliminaries

Kelly [5] defined a bitopological space (X, τ_1, τ_2) to be a set X equipped with two topologies τ_1 and τ_2 on X and he initiated a systematic study of bitopological spaces. After the works of Kelly on bitopological space, various authors (e. g. 6, 7, 8, 10) turned their attention to the generalization of various concepts of topology by considering bitopological spaces instead of topological spaces. In 2003, the first author and Abo Khadra [1] extended the concept of b-open sets and b-closed sets to the setting of bitopological spaces. In the present paper we define and study quasi-b-components in bitopological spaces. This is obtained by generalizing the concept of quasi-component in bitopological spaces due to Reilly and Young [9]. For a subset A of X , $\tau_i\text{-Cl}(A)$ (resp. $\tau_i\text{-Int}(A)$) denotes the closure (resp. interior) of A with respect to τ_i for $i=1, 2$. However, $\tau_i\text{-Cl}(A)$ and $\tau_i\text{-Int}(A)$ are briefly denoted by $Cl_i(A)$ and $Int_i(A)$, respectively, if there is no possibility of confusion.

Definition 1.1. (see[1]). A subset A of a bitopological space (X, τ_1, τ_2) is said to be:
 (a) (τ_i, τ_j) - b-open (briefly (i, j) -b-open) or faintly semi-open if $A \subset Int_i(Cl_j(A)) \cup Cl_j(Int_i(A))$;
 (b) (τ_i, τ_j) - b-closed (briefly (i, j) -b-closed) if its complement is (i, j) -b-open, equivalently, $A \supset Cl_j(Int_i(A)) \cap Int_i(Cl_j(A))$ where $i \neq j, i, j = 1, 2$.

Definition 1.2. (see[1]). Let A be a subset of a bitopological space (X, τ_i, τ_j) . Then
 (a) The intersection of all (i, j) -b-closed sets of (X, τ_i, τ_j) containing A is called the (τ_i, τ_j) -b-closure of A and is denoted by (τ_i, τ_j) -bCl(A) (briefly (i, j) -bCl(A)).
 (b) The union of all (i, j) -b-open sets of (X, τ_i, τ_j) contained in A is called the (τ_i, τ_j) -b-interior of A and is denoted by (τ_i, τ_j) -bInt(A) (briefly (i, j) -bInt(A)) where $i \neq j, i, j = 1, 2$.
 The collection of all (i, j) -b-open sets of a bitopological space (X, τ_1, τ_2) will be denoted by (i, j) -BO(X) for $i \neq j$ and $i, j = 1, 2$. A subset A is said to be pairwise b-open if it is $(1, 2)$ -b-open and $(2, 1)$ -b-open. Also A is said to be pairwise b-closed if it is $(1, 2)$ -b-closed and $(2, 1)$ -b-closed.

2 Quasi-b-Components in Bitopological Spaces

In this section we define and study pairwise b-component, pairwise totally b-disconnected and τ_1 -locally b-connected in bitopological space (X, τ_i, τ_j) .

Definition 2.1. A bitopological space (X, τ_1, τ_2) is said to be pairwise b-connected if X cannot be expressed as the union of two non-empty pairwise b-separated sets, where $A, B \subset X$ are called pairwise b-separated if $A \cap \tau_1 - bCl(B) = \phi = B \cap \tau_2 - bCl(A)$.

Remark 2.2. If X can be expressed as the union of two non-empty pairwise b-separated sets A, B . Then we write $X = A/B$ and call this a pairwise b-separation of X .

Definition 2.3. Let (X, τ_1, τ_2) be any topological space. Define a relation R by $(x, y) \in R$ if and only if x and y cannot be separated by a pairwise b-separation A/B of X , where $x, y \in X$. This is equivalent to saying that for each pairwise b-separation A/B of X either $x \in A$ and $y \in A$ or $x \in B$ and $y \in B$. It is easy to see that R is an equivalence relation.

Definition 2.4. Let x be any point of a bitopological space (X, τ_1, τ_2) . The equivalence class of x with respect to R is said to be the quasi-b-component of x and is denoted by $(QB)_x$.

Remark 2.5. A quasi-component of a bitopological space (X, τ_1, τ_2) need not be a quasi semi-component and consequently need not be a quasi b-component as shown as by the following example.

Example 2.6. Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{b\}, \{b, c\}\}$ and $\tau_2 = \{X, \phi, \{c\}, \{b, c\}\}$. Let $A = \{b\}$ and $B = \{a, c\}$. Then $X = A/B$, which is a pairwise b-separation of X . Here $b \in A$ and $c \in B$. Thus b, c do not belong to a quasi b-component of X . But we see that b, c belong to a quasi-component of X , since the points b and c are not (pairwise) separated.

Theorem 2.7. A quasi-b-component of a space (X, τ_1, τ_2) is contained in a quasi-component.

Proof. The proof is obvious in view of the fact that any two pairwise separated sets in (X, τ_1, τ_2) are pairwise b-separated. $\square$

Remark 2.8. Reilly and Young [9] have shown that quasi-components of a bitopological space (X, τ_1, τ_2) are not related to the quasi-components of the spaces (X, τ_1) and (X, τ_2) . Here we take the same example to show that quasi b-components of (X, τ_1, τ_2) are

not related to quasi b-components of (X, τ_1) and (X, τ_2) . The example was: $X = \{a, b\}$, $\tau_1 = \{X, \phi, \{a\}\}$ and $\tau_2 = \{X, \phi, \{b\}\}$. Clearly $\{a\}/\{b\}$ is a pairwise b-separation of X and so $(QB)_a = \{a\}$. But quasi-b- component of a in both (X, τ_1) and (X, τ_2) is X .

Definition 2.9. A subset Y of a bitopological space (X, τ_1, τ_2) is said to be pairwise b-connected if Y cannot be expressed as the union of two non-empty pairwise b-separated sets in X .

Definition 2.10. Let (X, τ_1, τ_2) be any bitopological space. Then a maximal pairwise b-connected set in X is called a b-component of X and if x belongs to this b-component say $(B.C.)$, then it will be denoted by $B.C.(x)$.

Theorem 2.11. In a bitopological space (X, τ_1, τ_2) , every quasi b-component is the union of the b-components of its points.

Proof. Let $(QB)_x$ be a quasi b-component of X containing x . We are to show that $(QB)_x = \cup\{B.C.(q) : q \in (QB)_x\}$, where, $B.C.(q)$ denote b-components in (X, τ_1, τ_2) . Let $y \in B.C.(q)$. Then $(y, q) \in R$. But $q \in (QB)_x$, therefore $(q, x) \in R$ and so $(y, x) \in R$. This implies that $y \in (QB)_x$. Hence, $B.C.(q) \subset (QB)_x$ for each $q \in (QB)_x$. This means that $\cup[B.C.(q) : q \in (QB)_x] \subset (QB)_x$. The inclusion $(QB)_x \subset \cup[B.C.(q) : q \in (QB)_x]$ is clear. $\square$

Theorem 2.12. The following statements are true for any bitopological space (X, τ_1, τ_2) .
 (i) Every b-component is contained in a quasi b-component.
 (ii) Every quasi b-component is a union of b-components.
 (iii) A quasi-b-component is a b-component if and only if it is pairwise b-connected.

Proof. Follows directly from Theorem 2.11. $\square$

Definition 2.13. A bitopological space (X, τ_1, τ_2) is said to be pairwise totally b-disconnected if each pair of points of X can be separated by a pairwise b-separation of X .

Theorem 2.14. A bitopological space (X, τ_1, τ_2) is pairwise totally b-disconnected if and only if quasi-b-components of X are singletons.

Proof. Obvious. $\square$

Definition 2.15. (EL-Atik [4]) A function $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is said to be b-irresolute if the inverse image of every b-open set is b-open.

Theorem 2.16. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau_1^*, \tau_2^*)$ be a pairwise b-irresolute pairwise open surjection. Then the image of a quasi b-component of X lies in a quasi b-component of Y .

Proof. The proof is obvious in view of Theorem 2.11. $\square$

Definition 2.17. Let (X, τ_1, τ_2) be a bitopological space. Then X is said to be τ_1 -locally b-connected with respect to τ_2 if for each $x \in X$ and every τ_1 -b-open set U containing x , there exists a pairwise b-connected τ_1 -open set G such that $x \in G \subset U$. A bitopological space (X, τ_1, τ_2) is said to be pairwise locally b-connected if it is τ_1 -locally b-connected with respect to τ_2 and τ_2 -locally b-connected with respect to τ_1 .

Theorem 2.18. *If a bitopological space (X, τ_1, τ_2) is pairwise locally b-connected, then the b-components of X are bi-b-closed.*

Theorem 2.19. *In a pairwise locally b-connected space (X, τ_1, τ_2) , each quasi b-component is a b-component.*

Proof. Let (X, τ_1, τ_2) be a pairwise locally b-connected space and let $x \in X$. Then with the usual notations of quasi b-components of X , we have $(QB)_x = \cup\{B.C.(g) : g \in (QB)_x\}$. Clearly, $B.C.(x) \subset (QB)_x$. We claim that $(QB)_x = B.C.(x)$. Let $y \in (QB)_x - B.C.(x)$. Since $B.C.(x)$ is τ_1 -open and τ_1 -b-closed (Theorem 2.18), therefore, $X = B.C.(x)/[X - B.C.(x)]$ is a pairwise b-separation of X with $x \in B.C.(x)$ and $X - B.C.(x)$. Thus $y \notin (QB)_x$, which is a contradiction. Hence $(QB)_x = B.C.(x)$. $\square$

Theorem 2.20. *If x is any point of a bitopological space (X, τ_1, τ_2) , $\{W_a : a \in A\}$ is the family of all τ_1 -b-open τ_2 -b-closed sets containing x , and $\{V_b : b \in B\}$ is the family of all τ_1 -b-closed τ_2 -b-open sets containing x , then $(QB)_x = (\bigcap_a W_a) \cap (\bigcap_b V_b)$.*

Proof. Follows directly from the above definitions. $\square$

Corollary 2.21. *Any quasi-b-component $(QB)_x$ of bitopological space (X, τ_1, τ_2) satisfies the relation: $(QB)_x = \tau_1 - bCl[(QB)_x] \cap \tau_2 - bCl[(QB)_x]$.*

Proof. With the notations of Theorem 2.20, $\bigcap_b V_b$ is a $\tau_1 - b$ -closed set containing $(QB)_x$, so that $\tau_1 - bCl[(QB)_x] \subset \bigcap_b V_b$. Similarly $\tau_2 - bCl[(QB)_x] \subset \bigcap_a W_a$. Hence $\tau_1 - bCl[(QB)_x] \cap \tau_2 - bCl[(QB)_x] \subset (\bigcap_a W_a) \cap (\bigcap_b V_b) = (QB)_x$. Obviously, $(QB)_x \subset \tau_1 - bCl[(QB)_x] \cap \tau_2 - bCl[(QB)_x]$ and so the relation is satisfied. $\square$

3 Conclusion

The study of Bitopological spaces generalized most of near open sets, beside the concepts of b-open sets, b-closed sets and the notion of quasi-components in bitopological spaces. So we introduce the concept of quasi b-components in bitopological spaces and basic theorems of quasi b-components in bitopological spaces.

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