

Review Article

Analytical approximate solutions of fractional convection-diffusion equation by means of local fractional derivative operators

Abstract

In this article, an analytical approximate solution of nonlinear fractional convection-diffusion was obtained with the help of local fractional decomposition method (LFDM). A new application of local fractional decomposition method (LFDM) was extended to derive analytical solutions in the form of a series for this equation. It is indicated that the solutions obtained by the LFDM are reliable, simple and an effective method for strongly nonlinear partial equations.

Keywords: Local fractional decomposition method; fractional convection-diffusion equation; Riemann-Liouville derivative

1. Introduction

The local fractional calculus (LFC) was applied to model and process the non-differentiable phenomena in several fractal physical phenomena [1–15]. Here are some local fractional models, such as the local fractional Fokker-Planck equation (LFFPE) [1], the local fractional stress-strain relations (LFSSR) [2], the local fractional heat conduction equation (LFHCE) [12], wave equations on the Cantor sets (WECSs) [14], local fractional Laplace equation (LFLE) [15], Newtonian mechanics (NM) on fractals subset of real-line [16]. Das et al. have obtained an approximate analytical solution of the fractional diffusion equation with absorbent term and external force [17], Fractional convection-diffusion equation with nonlinear source term by Momani and Yildirim [18], space-time fractional advection-dispersion equation by Yildirim and Kocak [19], fractional Zakharov-Kuznetsov equations by Yildirim and Gulkanat [20], integro-differential equation by El-Shahed [21], non-Newtonian flow by Siddiqui et al. [22], fractional PDEs in fluid mechanics by Yildirim [23], reaction-diffusion Brusselator system with fractional time derivative [24].

The homotopy perturbation method (HPM) [25–26], the Adomian decomposition method [27–29], homotopy analysis method (HAM) [30] and the variational iteration method (VIM), proposed by Ji-Huan He [31–33], was successfully applied to autonomous ordinary and partial differential equations and other fields. Ji-Huan He [34] was the first to apply the variational iteration method to fractional differential equations. Recently, a new modified Riemann-Liouville left derivative is suggested by Jumarie [5,35–38]. More recently, Momani [39] used the Adomian decomposition method, HPM [18] and FVIM [40] for solving the nonlinear fractional convection-diffusion equation.

In this paper, we extend the application of the LFDM in order to derive analytical approximate solutions to nonlinear fractional convection-diffusion problem

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - c \frac{\partial u(x,t)}{\partial x} + \Phi(u(x,t)) + f(x,t), \quad (1)$$

$$0 < x \leq 1, 0 < \alpha \leq 1, t > 0,$$

$$u(x,0) = h(x), 0 < x \leq 1, \quad (2)$$

where $\Phi(u)$ is some reasonable nonlinear function of u which is chosen as a potential energy, c is a constant, α is a parameter describing the order of the time-fractional derivative. The fractional derivative is considered in the modified Riemann-Liouville derivative. $u(x, t)$ is assumed to be a causal function of time, i.e., vanishing for $t < 0$. The convection–diffusion equations are widely used in science and engineering as mathematical model for computational simulations, such as in oil reservoir simulations, transport of mass and energy, and global weather production, in which an initially discontinuous profile is propagated by diffusion and convection, the latter with a speed of c .

The aim of this paper is to extend the application of the local fractional PDEs (LFPDEs) within local fractional derivative (LFD) to solve fractional nonlinear convection-diffusion equations with modified Riemann-Liouville derivative.

This paper is organized as follows:

In section 2, we are given brief definitions related to the local fractional calculus theory. In section 3, we define the solution procedure of the local fractional decomposition method to show inefficiency of this method, we present the application of the LFD for the fractional nonlinear convection-diffusion equations with modified Riemann-Liouville derivative and numerical results in Section 4. The conclusions are then given in the final section 5.

2. Basic definitions

Here, some basic definitions and properties of the local fractional continuity(LFC), local fractional derivative(LFD) and local fractional integral(LFI) of non-differential functions which can be found in [1-10, 35-39] are given.

Definition 1 If there exists the relation [9,41-42]

$$|f(u) - f(u_0)| < \varepsilon^\alpha \quad (3)$$

with $|u - u_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in \mathbb{R}$. Now $f(u)$ is called local fractional continuous at $u = u_0$, denote by $\lim_{u \rightarrow u_0} f(u) = f(u_0)$. Then $f(u)$ is called local fractional continuous on the interval (a, b) , denoted by

$$f(u) \in C_\alpha(a, b) \quad (4)$$

Definition 2 A function $f(x)$ is called a non-differentiable function of exponent α , $0 < \alpha \leq 1$, which satisfy Hölder function of exponent α , then for $u, v \in X$ such that [9,41-42]

$$|f(u) - f(v)| < C|u - v|^\alpha. \quad (5)$$

Definition 3 A function $f(x)$ is called to be continuous of order α , $0 < \alpha \leq 1$, or shortly α continuous, when we have the following relation [9,41-42] $|f(u) - f(u_0)| < \varepsilon^\alpha$,

$$f(u) - f(u_0) = o((u - u_0)^\alpha). \quad (6)$$

Compared with (6), (3) is standard definition of local fractional continuity(LFC). Here (5) is unified local fractional continuity(ULFC).

Definition 4. Setting $f(u) \in C_\alpha(a, b)$, LFD of $f(x)$ of order α at $u = u_0$, is defined by [9,41-42]:

$$f^{(\alpha)}(u_0) = \left. \frac{d^\alpha f(u)}{du^\alpha} \right|_{u=u_0} = \lim_{u \rightarrow u_0} \frac{\Delta^\alpha(f(u) - f(u_0))}{(u - u_0)^\alpha}, 0 < \alpha \leq 1, \quad (7)$$

Where $\Delta^\alpha(f(u) - f(u_0)) \cong \Gamma(1 + \alpha)\Delta(f(u) - f(u_0))$. For any $u \in (a, b)$, there exists

$f^{(\alpha)}(u) = D_u^\alpha f(u)$, denoted by $f(u) \in D_u^\alpha(a, b)$. LFD of higher order can be expressed as follows:

$$f^{(k\alpha)}(u) = \underbrace{D_u^\alpha \dots D_u^\alpha}_{k \text{ times}} f(u),$$

and local fractional partial derivative(LFPD) of higher order is shown in the following form:

$$\frac{\partial^{(k\alpha)} f(u)}{\partial u^{k\alpha}} = \underbrace{\frac{\partial^\alpha}{\partial u^\alpha} \dots \frac{\partial^\alpha}{\partial u^\alpha}}_{k \text{ times}} f(u).$$

Definition 5. Setting $f(u) \in C_\alpha(a, b)$, local fractional integral(LFI) of $f(u)$ of order α in the interval $[a, b]$ is defined by [41-43]:

$$\begin{aligned} {}_a I_b^\alpha f(u) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha, \quad 0 < \alpha \leq 1, \end{aligned} \quad (8)$$

Where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_3, \dots\}$ and $[\Delta t_j, \Delta t_{j+1}]$, $j = 0, 1, \dots, N-1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a, b]$. For any $u \in (a, b)$, there exists ${}_a I_b^\alpha f(u)$, denoted by $f(u) \in I_u^{(\alpha)}(a, b)$. If $f(u) = D_u^\alpha f(a, b)$, or $I_u^{(\alpha)}(a, b)$, we have $f(u) \in C_\alpha(a, b)$.

For any $f(u) \in C_\alpha(a, b)$, $0 < \alpha \leq 1$, we have local fractional multiple integrals (LFMIs)

$${}_{u_0} I_u^{(k\alpha)} f(u) = \overbrace{{}_{u_0} I_u^{(\alpha)} \dots {}_{u_0} I_u^{(\alpha)}}^{k \text{ times}} f(u),$$

For $0 < \alpha \leq 1$, $f^{(k\alpha)}(u) \in C_\alpha^k(a, b)$, then we have

$$\left({}_{u_0} I_u^{(k\alpha)} f(u) \right)^{(k\alpha)} = f(u),$$

where ${}_{u_0} I_u^{(k\alpha)} f(u) = \underbrace{{}_{u_0} I_u^{(\alpha)} \dots {}_{u_0} I_u^{(\alpha)}}_{k \text{ times}} f(u)$ and $f^{(k\alpha)}(u) = \underbrace{D_u^\alpha \dots D_u^\alpha}_{k \text{ times}} f(u)$.

Definition 6. The Mittag-Leffler function(MLF) in fractional space is defined by(see[9,41-43])

$$E_\alpha(u^\alpha) = \sum_{n=0}^{\infty} \frac{u^{(n\alpha)}}{\Gamma(1+n\alpha)}, \quad 0 < \alpha \leq 1. \quad (9)$$

Some useful formulas of LFD were summarized [9,43] as follows:

$$\frac{d^\alpha u^{n\alpha}}{du^\alpha} = \frac{\Gamma(1+n\alpha)u^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}, \quad (10)$$

$$\frac{d^\alpha E_\alpha(u^\alpha)}{du^\alpha} = E_\alpha(u^\alpha), \quad (11)$$

$$\frac{d^\alpha E_\alpha(nu^\alpha)}{du^\alpha} = nE_\alpha(nu^\alpha), \quad (12)$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b E_\alpha(u^\alpha)(du)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha), \quad (13)$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b u^{n\alpha}(du)^\alpha = \frac{\Gamma(1+n\alpha)(b^{(n+1)\alpha} - a^{(n+1)\alpha})}{\Gamma(1+(n+1)\alpha)}. \quad (14)$$

3. Local Fractional Decomposition Method

To describe the solution procedure of local fractional decomposition method, we consider the following fractional differential equation [18,39-40]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - c \frac{\partial u(x,t)}{\partial x} + \Phi(u(x,t)) + f(x,t), \quad (15)$$

$$0 < x \leq 1, 0 < \alpha \leq 1, t > 0,$$

According to the LFD, we can build a local fractional differential operator form for Eq. (15) as follows

$$L_t^{(\alpha)} u(x,t) = u_{xx}(x,t) - u_x(x,t) + \Phi(u(x,t)) + f(x,t), \quad (16)$$

where $0 < \alpha \leq 1$, $u(x,t)$ is a local fractional continuous function. Applying the inverse operator $L_t^{(-\alpha)}$ to both sides of (16) yields

$$\begin{cases} u_{n+1}(x,t) = L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_n(x,t) - \frac{\partial}{\partial x} u_n(x,t) + \Phi(u_n(x,t)) + f(x,t) \right], n \geq 0 \\ u_0(x,t) = u(x,0). \end{cases} \quad (17)$$

The successive approximations $u_{n+1}(x,t), n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon by using any selective function u_0 . The initial values are usually used for choosing the zeroth approximation u_0 . Consequently, the exact solution may be procured by using

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \quad (18)$$

Thus, we can find that the following condition

$$|f(x) - f(x_0)| < \varepsilon^\alpha,$$

Where fractional dimension $f(x)$ is equal to α for any $x \in (a,b)$.

4. Applications

In this section, we present the solution of nonlinear fractional partial differential equations as the applicability of local fractional decomposition method(LFD).

Examples 4.1. Consider the nonlinear fractional convection-diffusion equation where $0 < \alpha \leq 1, c = 1, 0 < x \leq 1, t > 0$,

and $\Phi(u) = u \frac{\partial^2 u}{\partial x^2} - u^2 + u$, we get

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + u(x,t) \frac{\partial^2 u(x,t)}{\partial x \partial t} - u(x,t)^2 + u(x,t) \quad (19)$$

with initial conditions

$$u(x, 0) = e^x, [18,39-40]. \quad (20)$$

Making use of Eq.(20), the recurrence relation reads as follows:

$$u_0(x, t) = u(x, 0) \text{ and}$$

$$u_{n+1}(x, t) = L_t^{(-\alpha)} \left[\begin{aligned} &\frac{\partial^2}{\partial x^2} u_n(x, t) - \frac{\partial}{\partial x} u_n(x, t) \\ &+ u_n(x, t) \frac{\partial^2}{\partial x^2} u_n(x, t) - u_n(x, t)^2 + u_n(x, t) \end{aligned} \right] n \geq 0. \quad (21)$$

Applying the recursive relation (21) and the conditions in (20), we get the followings results:

$$\begin{aligned} u_0(x, t) &= e^x, \\ u_1(x, t) &= L_t^{(-\alpha)} \left[\begin{aligned} &\frac{\partial^2}{\partial x^2} u_0(x, t) - \frac{\partial}{\partial x} u_0(x, t) \\ &+ u_0(x, t) \frac{\partial^2}{\partial x^2} u_0(x, t) - u_0(x, t)^2 + u_0(x, t) \end{aligned} \right] \\ &= \frac{e^x t^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \quad (22)$$

$$\begin{aligned} u_2(x, t) &= L_t^{(-\alpha)} \left[\begin{aligned} &\frac{\partial^2}{\partial x^2} u_1(x, t) - \frac{\partial}{\partial x} u_1(x, t) \\ &+ u_1(x, t) \frac{\partial^2}{\partial x^2} u_1(x, t) - u_1(x, t)^2 + u_1(x, t) \end{aligned} \right] \\ &= \frac{e^x t^{2\alpha}}{\Gamma(1+2\alpha)}, \end{aligned} \quad (23)$$

$$\begin{aligned} u_3(x, t) &= L_t^{(-\alpha)} \left[\begin{aligned} &\frac{\partial^2}{\partial x^2} u_2(x, t) - \frac{\partial}{\partial x} u_2(x, t) \\ &+ u_2(x, t) \frac{\partial^2}{\partial x^2} u_2(x, t) - u_2(x, t)^2 + u_2(x, t) \end{aligned} \right] \\ &= \frac{e^x t^{3\alpha}}{\Gamma(1+3\alpha)}, \end{aligned} \quad (24)$$

$\vdots$,

$$\begin{aligned} u_n(x, t) &= L_t^{(-\alpha)} \left[\begin{aligned} &\frac{\partial^2}{\partial x^2} u_{n-1}(x, t) - \frac{\partial}{\partial x} u_{n-1}(x, t) \\ &+ u_{n-1}(x, t) \frac{\partial^2}{\partial x^2} u_{n-1}(x, t) - u_{n-1}(x, t)^2 + u_{n-1}(x, t) \end{aligned} \right] = \frac{e^x t^{n\alpha}}{\Gamma(1+n\alpha)} \end{aligned} \quad (25)$$

(26)

Then, the approximate solutions in a series form are

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = e^x \left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \dots \right) = e^x \sum_{n=0}^{\infty} \frac{t^{(n\alpha)}}{\Gamma(1+n\alpha)} \quad (27)$$

which has the exact solution

$$u(x, t) = e^x E_\alpha(t^\alpha), \quad (28)$$

For the special case, $\alpha = 1$ is

$$u(x, t) = e^x \sum_{k=0}^{\infty} \frac{(t^k)}{\Gamma(k+1)} = e^{x+t} \quad (29)$$

which is an exact solution to the nonlinear convection-diffusion equation. Fig. 1 shows the approximate solutions for the nonlinear convection-diffusion equation (19) obtained using local fractional decomposition method(LFDM).

Figure 2 shows approximate solution of Eq. (19) for $\alpha = 0.9, 0.8, 0.7, 0.6$. Figures 3. and 4 are prepared to show the influence of α on the function $u(x, t)$.

From the numerical results in Figs 3 and 4, it is clear that a decrease in the fractional order α by selecting the fixed $x = 0.5$ corresponds to an increase in the function. From Figs. 3 and 4 five sequential values of $\alpha = 1, 0.9, 0.8, 0.7, 0.6$ are seen.

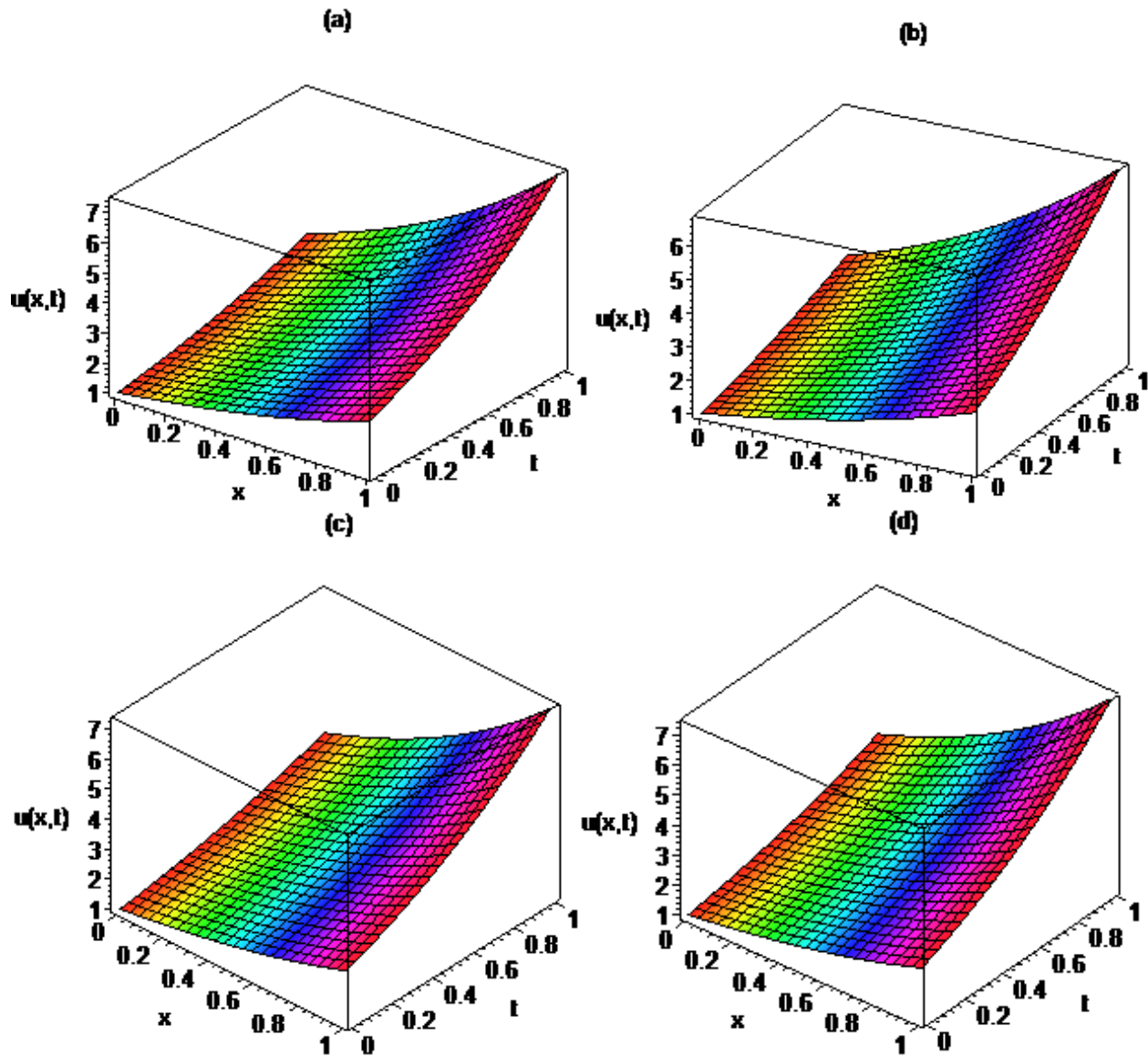
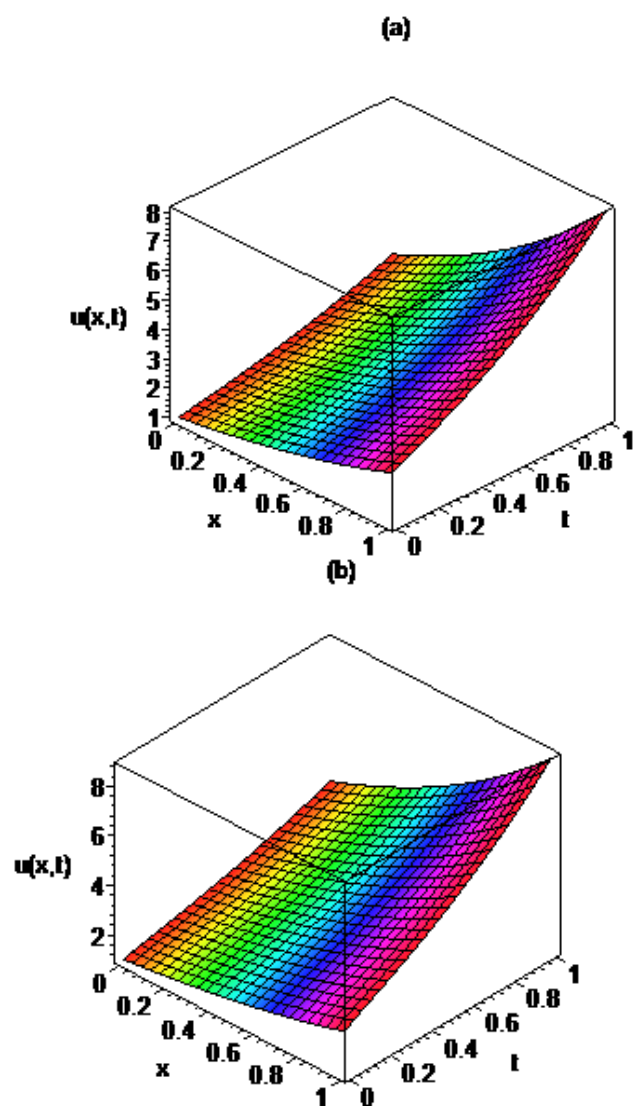


Fig. 1. The surface indicates the solution $u(x,t)$ for Eq.(19) when $\alpha = 1$. (a) Exact solution (b) 2-iterate LFD approximate solution, (c) 3-iterate LFD approximate solution and (d) 4-iterate LFD approximate solution



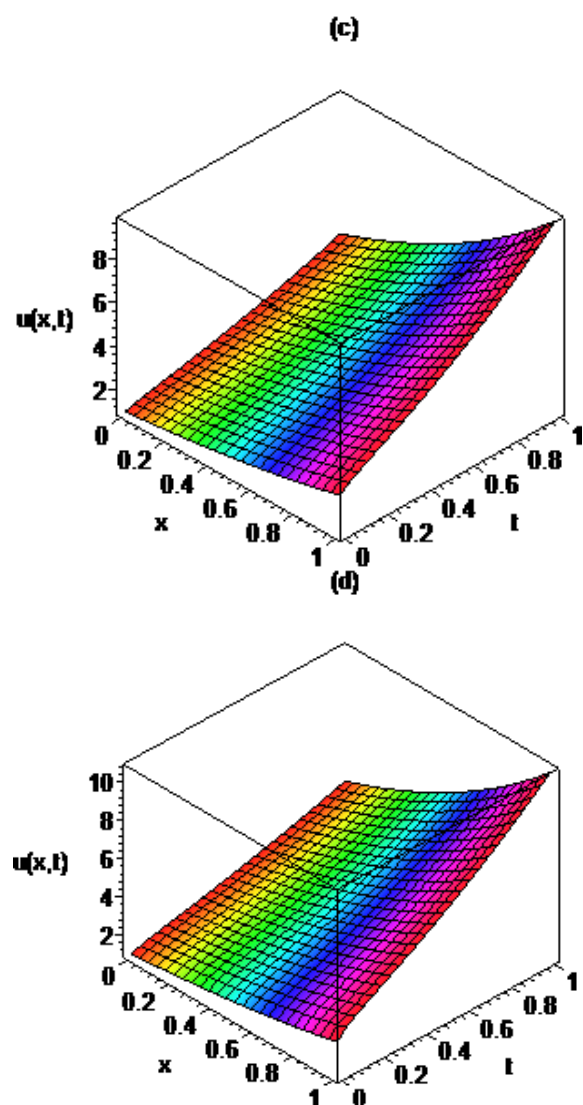


Fig. 2. The surface indicates the solution $u(x,t)$ for Eq.(19) (a) 4-iterate LFD approximate solution for $\alpha = 0.9$ (b) 4-iterate LFD approximate solution for $\alpha = 0.8$ (c) 4-iterate LFD approximate solution for $\alpha = 0.7$ and (d) 4-iterate LFD approximate solution for $\alpha = 0.6$

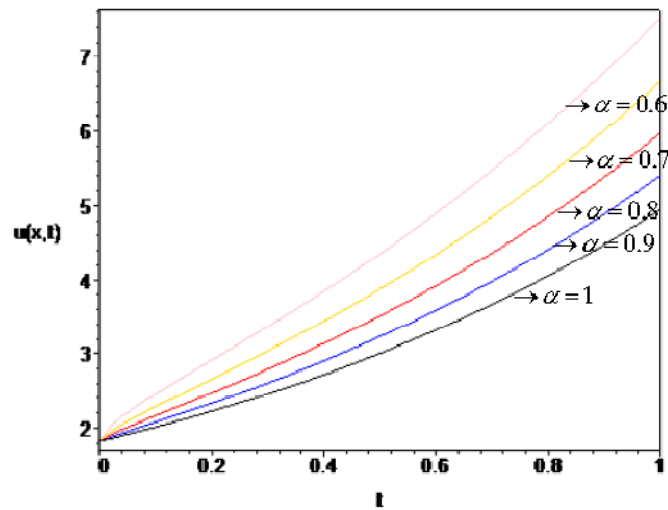


Fig. 3. 5-iterate LFD approximate solution for $x = 0.6$

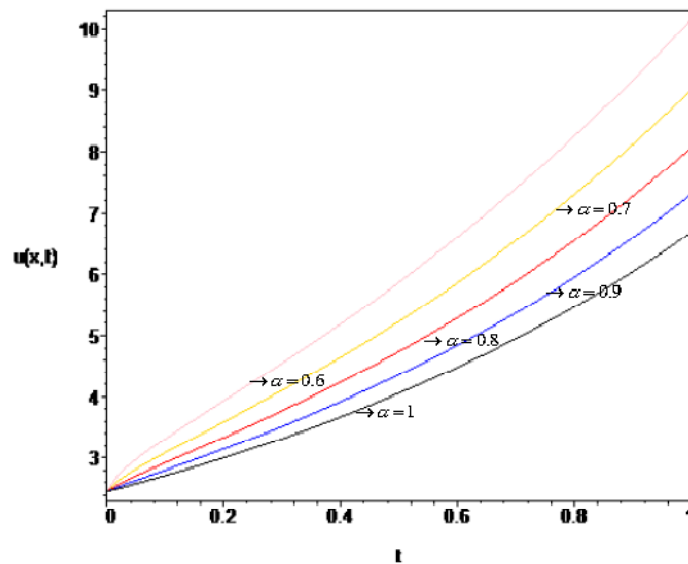


Fig. 4. 5-iterate LFD approximate solution for $x = 0.9$

Eq. (19) is solved in [39] using the Adomian decomposition method, FVIM [40] and HPM [18]. Local fractional decomposition method (LFD) shown that the present algorithm performs with considerable efficiency, simplicity and reliability. The results obtained from LFD are fully compatible with modified variational iteration method (FVIM), ADM and HPM.

Examples 4.2. We next consider the non-homogeneous nonlinear fractional convection-diffusion equation where $0 < \alpha \leq 1$, $c = 1$, $0 < x \leq 1$, $t > 0$,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - c \frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} \frac{\partial u(x,t)}{\partial x} + u(x,t) \frac{\partial^2 u(x,t)}{\partial x \partial t} - 2x \quad (30)$$

with initial conditions[18,39-40]

$$u(x, 0) = x^2. \quad (31)$$

According to LFAMD, the recurrence relation reads as follows:

$$u_0(x, t) = u(x, 0) \text{ and}$$

$$u_{n+1}(x, t) = L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_n(x, t) - \frac{\partial}{\partial x} u_n(x, t) + \frac{\partial}{\partial t} u_n(x, t) \frac{\partial}{\partial x} u_n(x, t) + u_n(x, t) \frac{\partial^2}{\partial x^2} u_n(x, t) - 2x \right] n \geq 0. \quad (32)$$

Applying the recursive relation (32) and the conditions in (31), we get the followings results:

$$u_0(x, t) = x^2, \quad (33)$$

$$u_1(x, t) = L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_0(x, t) - \frac{\partial}{\partial x} u_0(x, t) + \frac{\partial}{\partial t} u_0(x, t) \frac{\partial}{\partial x} u_0(x, t) + u_0(x, t) \frac{\partial^2}{\partial x^2} u_0(x, t) - 2x \right] \\ = \frac{(2-4x)t^\alpha}{\Gamma(1+\alpha)}, \quad (34)$$

$$u_2(x, t) = L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_1(x, t) - \frac{\partial}{\partial x} u_1(x, t) + \frac{\partial}{\partial t} u_1(x, t) \frac{\partial}{\partial x} u_1(x, t) + u_1(x, t) \frac{\partial^2}{\partial x^2} u_1(x, t) - 2x \right] \\ = \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{(32x-16)\alpha\Gamma(2\alpha)t^{3\alpha-1}}{\Gamma(1+\alpha)^2\Gamma(3\alpha)} - \frac{2xt^\alpha}{\Gamma(1+\alpha)}, \quad (35)$$

$$u_3(x, t) = L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_2(x, t) - \frac{\partial}{\partial x} u_2(x, t) + \frac{\partial}{\partial t} u_2(x, t) \frac{\partial}{\partial x} u_2(x, t) + u_2(x, t) \frac{\partial^2}{\partial x^2} u_2(x, t) - 2x \right] \\ = -\frac{32\alpha\Gamma(2\alpha)t^{4\alpha-1}}{\Gamma(\alpha+1)\Gamma(4\alpha)} + \frac{256\alpha^2\Gamma(2\alpha)\Gamma(5\alpha-1)t^{6\alpha-2}}{\Gamma(\alpha+1)^2\Gamma(3\alpha)\Gamma(1+2\alpha)\Gamma(6\alpha-1)} - \frac{16\alpha\Gamma(3\alpha)t^{4\alpha-1}}{\Gamma(1+2\alpha)\Gamma(1+\alpha)\Gamma(4\alpha)} + \\ \frac{32(32x-16)\alpha^2(3\alpha-1)\Gamma(2\alpha)^2\Gamma(6\alpha-2)t^{7\alpha-3}}{\Gamma(\alpha+1)^4\Gamma(3\alpha)^2\Gamma(7\alpha-2)} - \frac{2(32x-16)\alpha(3\alpha-1)\Gamma(2\alpha)\Gamma(4\alpha-1)t^{5\alpha-2}}{\Gamma(\alpha+1)^3\Gamma(3\alpha)\Gamma(5\alpha-1)} - \\ \frac{64x\alpha^2\Gamma(2\alpha)\Gamma(4\alpha-1)t^{5\alpha-2}}{\Gamma(\alpha+1)^3\Gamma(3\alpha)\Gamma(5\alpha-1)} + \frac{4x\alpha\Gamma(2\alpha)t^{3\alpha-1}}{\Gamma(\alpha+1)\Gamma(3\alpha)} - \frac{2xt^\alpha}{\Gamma(\alpha+1)}, \quad (36)$$

⋮,

Then, the approximate solutions in a series form are

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = x^2 + \frac{(2-8x)t^\alpha}{\Gamma(1+\alpha)} + \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{(32x-16)\alpha\Gamma(2\alpha)t^{3\alpha-1}}{\Gamma(1+\alpha)^2\Gamma(3\alpha)} - \\ - \frac{32\alpha\Gamma(2\alpha)t^{4\alpha-1}}{\Gamma(\alpha+1)\Gamma(4\alpha)} + \frac{256\alpha^2\Gamma(2\alpha)\Gamma(5\alpha-1)t^{6\alpha-2}}{\Gamma(\alpha+1)^2\Gamma(3\alpha)\Gamma(1+2\alpha)\Gamma(6\alpha-1)} - \frac{16\alpha\Gamma(3\alpha)t^{4\alpha-1}}{\Gamma(1+2\alpha)\Gamma(1+\alpha)\Gamma(4\alpha)} +$$

$$\frac{32(32x-16)\alpha^2(3\alpha-1)\Gamma(2\alpha)^2\Gamma(6\alpha-2)t^{7\alpha-3}}{\Gamma(\alpha+1)^4\Gamma(3\alpha)^2\Gamma(7\alpha-2)} - \frac{2(32x-16)\alpha(3\alpha-1)\Gamma(2\alpha)\Gamma(4\alpha-1)t^{5\alpha-2}}{\Gamma(\alpha+1)^3\Gamma(3\alpha)\Gamma(5\alpha-1)} - \frac{64x\alpha^2\Gamma(2\alpha)\Gamma(4\alpha-1)t^{5\alpha-2}}{\Gamma(\alpha+1)^3\Gamma(3\alpha)\Gamma(5\alpha-1)} + \frac{4x\alpha\Gamma(2\alpha)t^{3\alpha-1}}{\Gamma(\alpha+1)\Gamma(3\alpha)} \dots = \quad (37)$$

For the special case $\alpha = 1$ is [44]

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) = x^2 + 2t - 4tx + 2t^2 + 4tx - 12x^2t + 16xt^2 - 8t^3 + \dots, \quad (38)$$

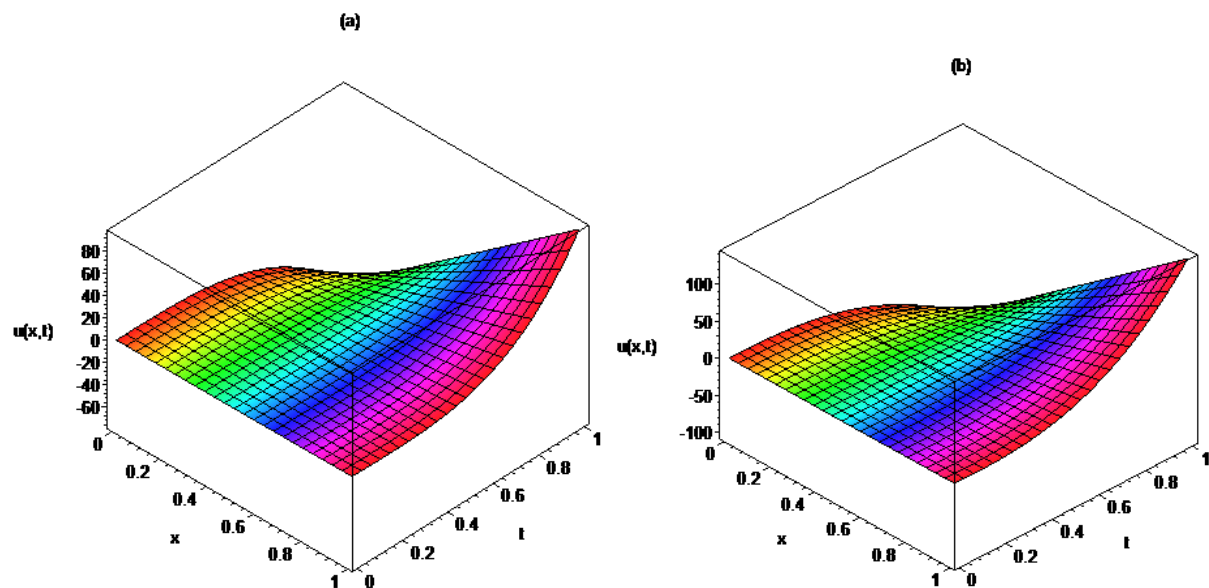
Canceling the noise terms and keeping the non-noise terms yields the exact solution of Eq.(38) given by

$$u(x, t) = x^2 + 2t \quad (39)$$

which is easily confirmed. This is formally proved right in [45].

Finally, the solution surfaces of the non-homogenous nonlinear fractional convection-diffusion equation are depicted in Fig. 5 for different values of α . From the graphical results in Figs. 5 and 6, it can be seen the influence of α on the function $u(x, t)$. It is clearly seen that a $u(x, t)$ increase with the increases in t for $\alpha = 1, 0.9, 0.8, 0.7, 0.6 \dots$

Eq. (30) is solved in [39] using the Adomian decomposition method ,FVIM[40], HPM[18] and the results in Fig. 5 compare well with those obtained from the Adomian decomposition method and HPM.



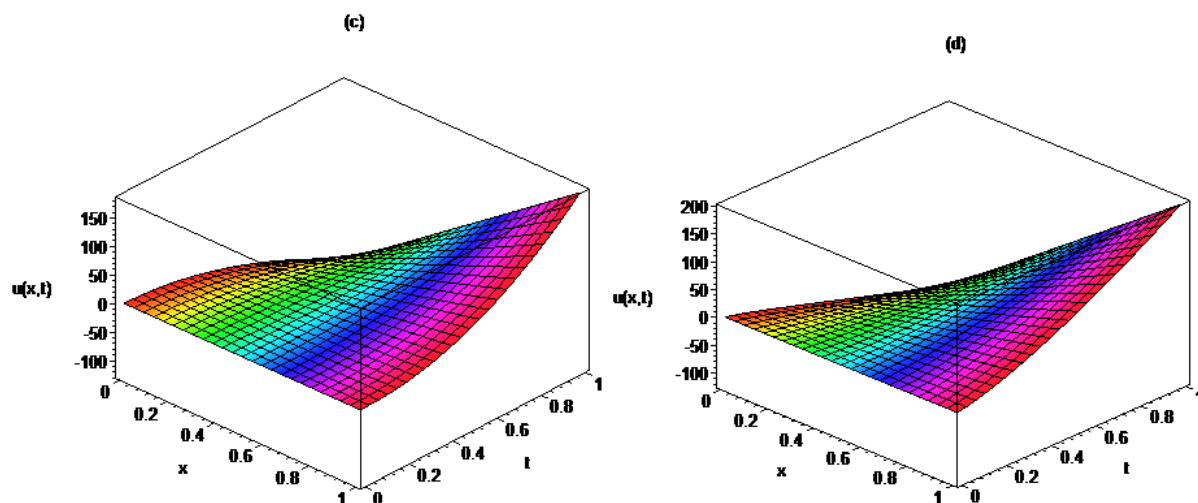


Fig. 5. The surface indicates the solution $u(x,t)$ for Eq.(30) (a) 3-iterate LFD approximate solution for $\alpha = 0.9$ (b) 3-iterate LFD approximate solution for $\alpha = 0.8$ (c) 3-iterate LFD approximate solution for $\alpha = 0.7$ and (d) 3-iterate LFD approximate solution for $\alpha = 0.6$

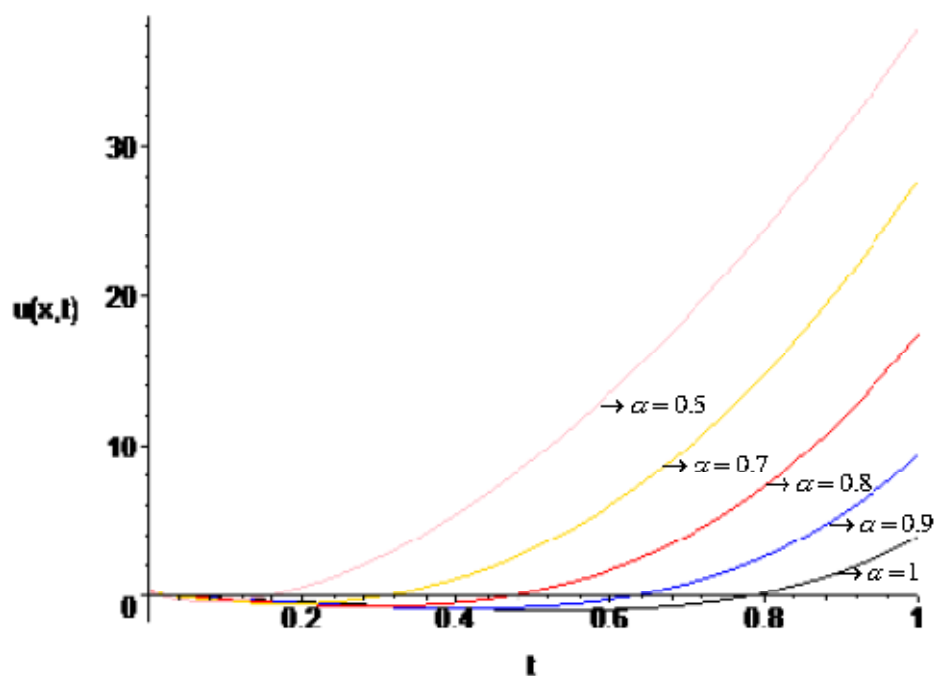


Fig. 6. 3-iterate LFD approximate for $x = 0.5$

5. Conclusions

Local fractional decomposition method(LFDM) may be used in solving nonlinear problems and ordinary, partial, fractional, integral equations. In this paper, we have taken Local fractional decomposition method having integral w. r. t. $(d\tau)^\alpha$ used for the first time by Jumarie. The obtained results indicate that this method is powerful and meaningful for solving the nonlinear fractional differential equations. Two examples indicate that the results of local fractional decomposition method having integral w. r. t. $(d\tau)^\alpha$ are in excellent agreement with those obtained by classical VIM,MVIM, GDTM, HPM and Adomian decomposition method which is available in the literature.

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