

Note on "Mean square solution of second-order random differential equations by using homotopy analysis method"

**Original Research
Article**

Abstract

The Homotopy analysis method is implemented by Golmankhanehi et al. to find the expectation and variance of the approximate solutions of the second-order random differential equations [1]. In this note, we reused the Homotopy analysis method to solve the same problem and we draw a very important corrections and comments on the paper [1]. The results in this paper are coinciding with the results in [3]- [5].

Keywords: Homotopy analysis method; random variable; variance of approximate solution; Expectation of approximate solution.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

1 Introduction

The papers by Golmankhanehi et al. [1], present the Homotopy analysis method for finding the expectation and variance of the approximate solutions of the second-order random differential equations. These authors work on computation of the main statistical functions of the mean square solution, where some basic properties of variance are applied in equation (10).

The present authors are not aware of the following properties, which is used in equation 10 [1],

$$V\left(\sum_{i=0}^n X_i\right) \neq \sum_{i=0}^n V(X_i).$$

Although, It is well known that, in probability theory and statistic [2], the variance of a random variable must be nonnegative since the variance is given by [1].

$$Var(X) = E((X - \mu)^2), \text{ where } E(X) = \mu$$

Golmankhanehi et al. [1] discussed six illustrative examples in section 3. The numerical results are presented graphically in Figures 1-12. We observed that the numerical results of the variance approximation solution by using HAM in [1] are incorrect since the variance curve is negative as shown in Figures 4, 6, and 12.

It is worth noting that from equation (2) in [1], the homotopy analysis method coincides with the homotopy Perturbation method when $h = -1$. As a result the expectation and the variance of the approximate solutions must be consistent with that found by Khalaf [3].

The goal of this work is to correct equation (8) in [1]. A variety of examples using the correct expression will be presented. In order to achieve this we let

$x_N(t) = \sum_{k=0}^N X(k) t^k$ (eq.(7) in [1]), be an approximate solution of equation (2) by using the homotopy analysis method . Then we utilize the independence between Y_0 , Y_1 and A , to yield

$$E[x_N(t)] = \sum_{k=0}^N E[X(k)] t^k,$$

$$V[x_N(t)] = \sum_{j=0}^N \sum_{i=0}^N \text{cov}(X(j), X(i)),$$

where

$$\text{cov}(X(i), X(j)) = E(X(i)X(j)) - E(X(j))E(X(i)).$$

2 Test examples

In this section, we adopt several examples which is considered in [1; 3; 4; 5] to illustrate the using of homotopy analysis method for approximating the expectation and the variance. The results are coincide with result in [3; 4; 5].

2.1 example 1

Consider random initial value problem

$$\frac{d^2 X(t)}{dt^2} + A^2 X(t) = 0, \quad X(0) = Y_0, \text{ and } \frac{dX(t)}{dt} \Big|_{t=0} = Y_1,$$

where,

A^2 is a Beta random variable, $A^2 \sim Be(\alpha = 2, \beta = 1)$, and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 1$, $E[Y_1^2] = 3$ and $E[Y_0 Y_1] = 0$.

$$E[X(t)] = 1 + t - \frac{387420488}{1162261467}t^2 - \frac{387420488}{3486784401}t^3 + \frac{96855113}{4649045868}t^4 + \frac{96855113}{23245229340}t^5 - \dots$$

$$V[X(t)] = 1 - 2t + \frac{1549681958}{1162261467}t^2 + \frac{3099363904}{3486784401}t^3 - \frac{712949527249402925}{2701703435345984178}t^4 - \dots$$

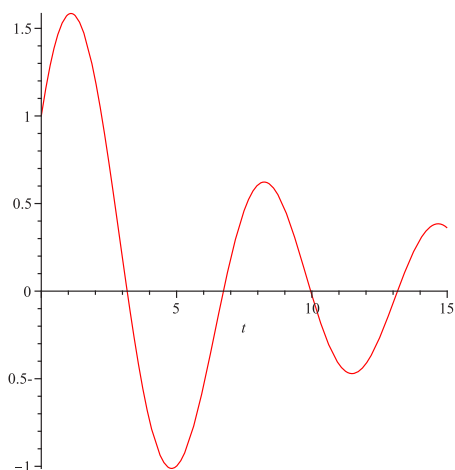


Figure 1: Expectation approximation solution from the HAM with $n = 18$. For $A^2 \sim Be(\alpha = 2, \beta = 1)$.

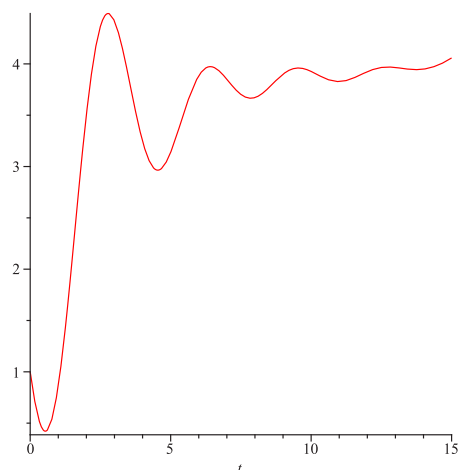


Figure 2: Variance approximation solution from the HAM with $n = 18$. For $A^2 \sim Be(\alpha = 2, \beta = 1)$.

2.2 example 2

Consider random initial value problem

$$\frac{d^2 X(t)}{dt^2} + At X(t) = 0, \quad X(0) = Y_0, \text{ and } \frac{dX(t)}{dt} \Big|_{t=0} = Y_1,$$

where

A is a Beta random variable, $A \sim Be(\alpha = 2, \beta = 3)$, and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 2$, $E[Y_1^2] = 5$.

$$E[X(t)] = 1 + 2t - \frac{1}{15}t^3 - \frac{1}{15}t^4 + \frac{1}{900}t^6 + \frac{1}{1260}t^7 - \frac{1}{113400}t^9 - \frac{1}{198450}t^{10} + \dots$$

$$V[X(t)] = 1 + t^2 - \frac{2}{15}t^3 - \frac{1}{15}t^5 + \frac{2}{225}t^6 + \frac{1}{450}t^7 + \frac{83}{25200}t^8 - \frac{83}{283500}t^9 - \dots$$

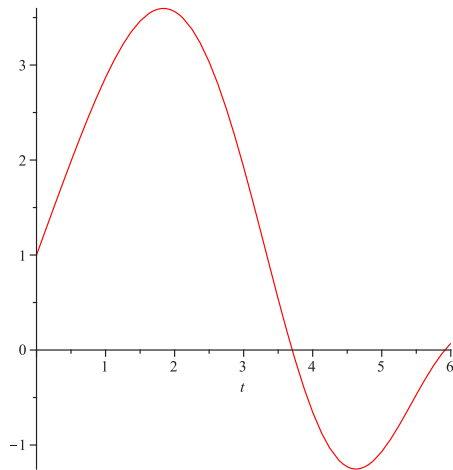


Figure 3: Expectation approximation solution from the HAM with $n = 18$. For $A \sim Be(\alpha = 2, \beta = 3)$.

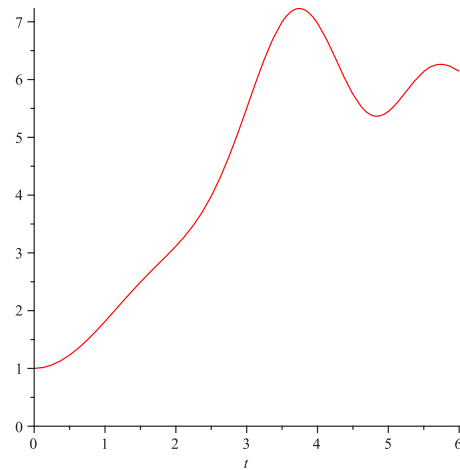


Figure 4: Variance approximation solution from the HAM with $n = 18$. For $A \sim Be(\alpha = 2, \beta = 3)$.

2.3 example 3

Consider random initial value problem

$$\frac{d^2 X(t)}{dt^2} + 2A \frac{dX(t)}{dt} + A^2 X(t) = 0, \quad X(0) = Y_0, \text{ and } \frac{dX(t)}{dt} \big|_{t=0} = Y_1,$$

where

A is a Beta random variable, $A \sim Be(\alpha = 2, \beta = 1)$, and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 1$, $E[Y_1^2] = 1$.

$$E[X(t)] = 1 + t - \frac{11}{12}t^2 + \frac{23}{60}t^3 - \frac{13}{120}t^4 + \frac{59}{2520}t^5 - \frac{83}{20160}t^6 + \frac{37}{60480}t^7 - \frac{143}{1814400}t^8 + \dots$$

$$V[X(t)] = 1 - \frac{1}{2}t^2 + \frac{4}{15}t^3 + \frac{103}{720}t^4 - \frac{649}{2520}t^5 + \frac{8959}{50400}t^6 - \frac{3133}{37800}t^7 + \frac{517}{17280}t^8 - \frac{5923}{665280}t^9 + \dots$$

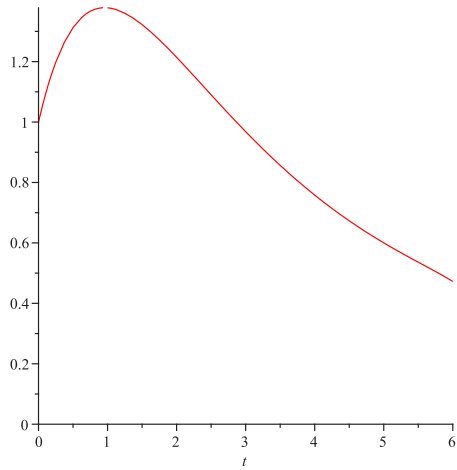


Figure 5: Expectation approximation solution from the HAM with $n = 18$. For $A \sim Be(\alpha = 2, \beta = 1)$.

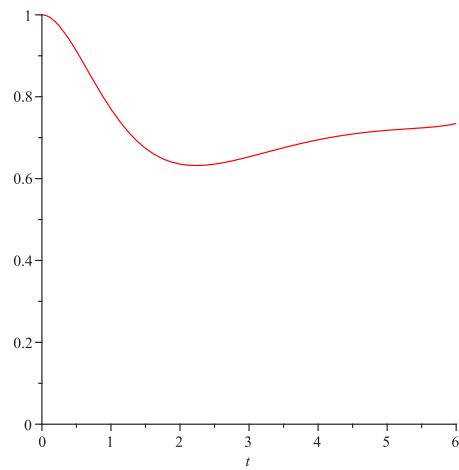


Figure 6: Variance approximation solution from the HAM with $n = 18$. For $A \sim Be(\alpha = 2, \beta = 1)$.

2.4 example 4

Consider random initial value problem

$$\frac{d^2 X(t)}{dt^2} + At X(t) = 0, \quad X(0) = Y_0, \text{ and } \frac{dX(t)}{dt} \Big|_{t=0} = Y_1,$$

where

A is a Uniform random variable, $A \sim U(\alpha = 0, \beta = 1)$, and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 1$, $E[Y_1^2] = 1$.

$$E[X(t)] = 1 + t - \frac{1}{12}t^3 + \frac{1}{120}t^5 - \frac{1}{1344}t^7 + \frac{1}{17280}t^9 - \frac{1}{253440}t^{11} + \frac{1}{4193280}t^{13} - \dots$$

$$V[X(t)] = 1 + \frac{1}{432}t^6 - \frac{1}{1440}t^8 + \frac{131}{1008000}t^{10} - \frac{41}{2177280}t^{12} + \frac{71471}{31294771200}t^{14} - \dots$$

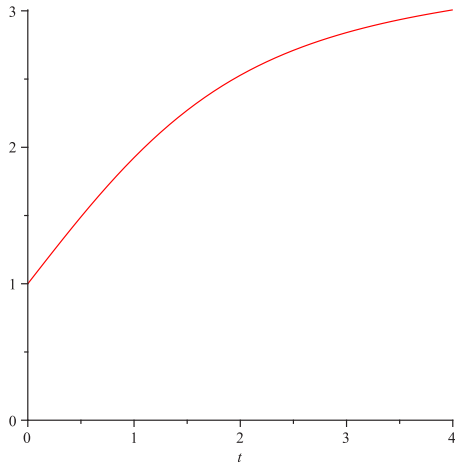


Figure 7: Expectation approximation solution from the HAM with $n = 18$. For $A \sim U(\alpha = 0, \beta = 1)$.

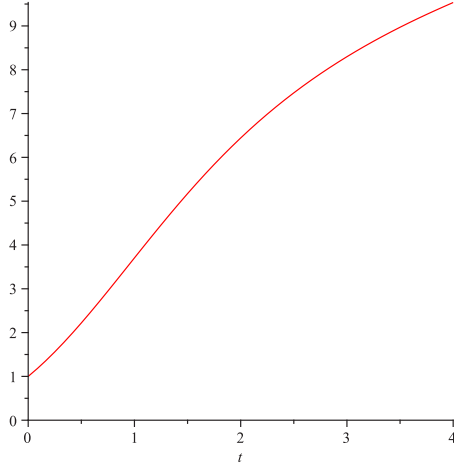


Figure 8: Variance approximation solution from the HAM with $n = 18$. For $A \sim U(\alpha = 0, \beta = 1)$.

2.5 example 5

Consider random initial value problem

$$\frac{d^2 X(t)}{dt^2} + At X(t) = 0, \quad X(0) = Y_0, \text{ and } \frac{dX(t)}{dt} \Big|_{t=0} = Y_1,$$

where

A is a Uniform random variable, $A \sim U(\alpha = 0, \beta = 2)$, and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 4$, $E[Y_1] = 1$, $E[Y_1^2] = 2$.

$$E[X(t)] = 1 + t - \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{18}t^4 + \frac{1}{90}t^5 - \frac{1}{360}t^6 - \frac{1}{2520}t^7 + \frac{1}{12600}t^8 + \frac{1}{113400}t^9 + \dots$$

$$V[X(t)] = 3 - 2t^2 + \frac{13}{12}t^4 + \frac{1}{18}t^5 - \frac{61}{270}t^6 - \frac{2}{135}t^7 + \frac{599}{22680}t^8 + \frac{101}{56700}t^9 - \frac{47}{23625}t^{10} - \dots$$

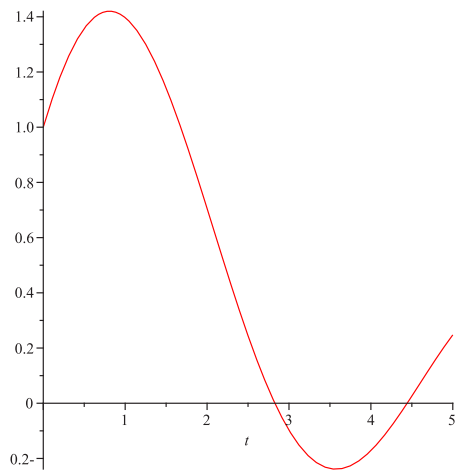


Figure 9: Expectation approximation solution from the HAM with $n = 18$. For $A \sim U(\alpha = 0, \beta = 2)$.

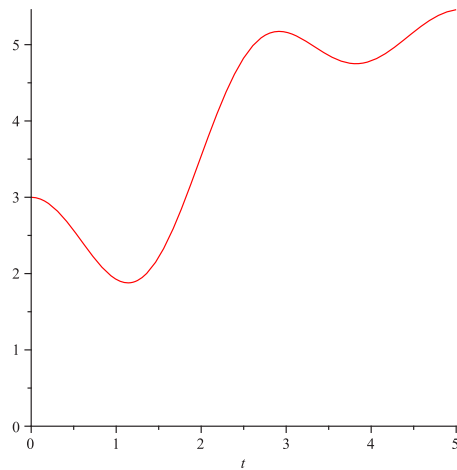


Figure 10: Variance approximation solution from the HAM with $n = 18$. For $A \sim U(\alpha = 0, \beta = 2)$.

2.6 example 6

Consider random initial value problem

$$\frac{d^2 X(t)}{dt^2} + A X(t) = -X(t) + \sin(t), \quad X(0) = Y_0, \text{ and } \frac{dX(t)}{dt} \Big|_{t=0} = Y_1,$$

where

A is a Uniform random variable, $A \sim U(\alpha = 1, \beta = 2)$, and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 1$, $E[Y_1^2] = 6$, $E[Y_0 Y_1] = 0$.

$$E[X(t)] = 1 + t - \frac{5}{4}t^2 - \frac{1}{4}t^3 + \frac{19}{72}t^4 + \frac{17}{720}t^5 - \frac{13}{576}t^6 - \frac{11}{8640}t^7 + \frac{211}{201600}t^8 + \dots$$

$$V[X(t)] = 1 - 2t + \frac{5}{2}t^2 + \frac{10}{3}t^3 - \frac{293}{144}t^4 - \frac{67}{40}t^5 + \frac{185}{288}t^6 + \frac{12221}{30240}t^7 - \frac{198419}{1814400}t^8 - \dots$$

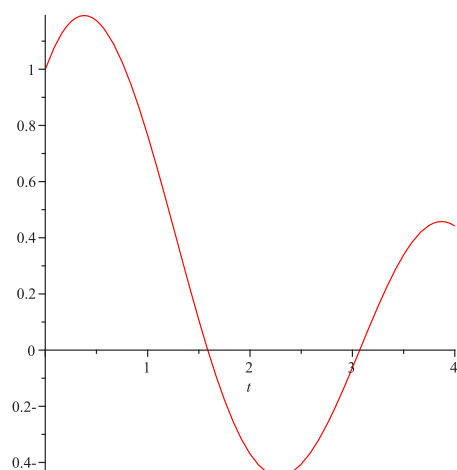


Figure 11: Expectation approximation solution from the HAM with $n = 18$. For $A \sim U(\alpha = 1, \beta = 2)$.

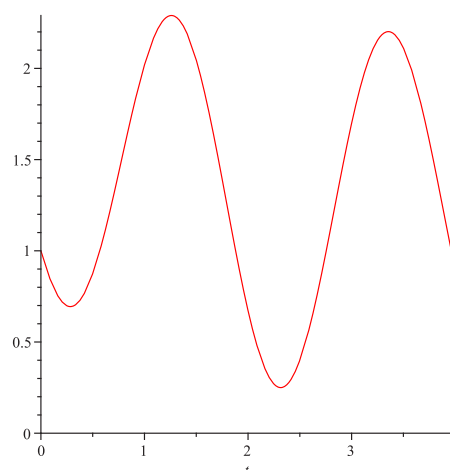


Figure 12: Variance approximation solution from the HAM with $n = 18$. For $A \sim U(\alpha = 1, \beta = 2)$.

References

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