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Original Research Article

The non-singular periodic solutions for Kundu equation and derivative nonlinear Schrödinger equation

Abstract In this paper, exact periodic solutions of Kundu equation are obtained through transformation, direct integration method and trial function method when the parameters satisfy certain conditions. By the way, exact periodic solutions for the derivative nonlinear Schrödinger equation are also obtained. Two periodic solutions' images were displayed. These results greatly enriched the solutions' structural diversity for these equations.
Keywords Kundu equation; Derivative nonlinear Schrödinger equation; Transformation and direct integration method; Trial function method; Periodic solution.

1 Introduction

Nonlinear Schrödinger type equations have been widely applied to the field of physics, such as plasma physics, nonlinear fluid mechanics, nonlinear optics and quantum physics. In recent decades, this type equations have attracted many researchers to study and get fruitful results[1-10]. Anjan Kundu derived the following higher-order nonlinear equation (Kundu equation) from nonlinear Schrödinger type equations[11]

$$iu_t + u_{xx} + \beta|u|^2u + \gamma|u|^4u + i\alpha(|u|^2u)_x + is(|u|^2)_xu = 0, \tag{1}$$

Kundu equation can be denoted the following equivalent form

$$iu_t + u_{xx} + \beta|u|^2u + \gamma|u|^4u + i(2\alpha + s)|u|^2u_x + i(\alpha + s)u^2\bar{u}_x = 0. \tag{2}$$

When $s = 0$, Eq.(1) becomes the derivative nonlinear Schrödinger equation as follows

$$iu_t + u_{xx} + \beta|u|^2u + \gamma|u|^4u + i\alpha(|u|^2u)_x = 0. \tag{3}$$

When $\alpha = 0$, $\beta = 2$, $\gamma = 4\delta^2$ and $s = -4\delta$, Eq.(1) becomes Kundu - Eckhaus equation

$$iu_t + u_{xx} + 2|u|^2u + 4\delta^2|u|^4u - 4i\delta(|u|^2)_xu = 0,\ \delta \in R, \tag{4}$$

In addition, Chen-Lee-Lin equation $iu_t + u_{xx} + i\delta^2|u|^2u_x = 0$ and Gerdjikov-Ivanov equation $iu_t + u_{xx} + \beta|u|^2u + 2\delta^2|u|^4u + 2i\delta u^2\bar{u}_x = 0$ are also special cases of Eq.(2).

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Many scholars have obtained exact solitary wave solutions, travelling wave solutions and singular periodic solutions for Eq.(1)[12-15]. Some scholars have also investigated Kundu – Eckhaus equation which rogue-wave solutions were obtained[16,17]. However, to our knowledge, the non-singular periodic solutions of Eq.(1) have not been reported. Difficulty to look for non-singular periodic solutions lies in the presence of the fifth-order nonlinear term in this type equations. After efforts to explore, we find that Eq.(1) possesses non-singular periodic solutions under a special condition, which is $4\,s^2 - 16\gamma + 4\,\alpha\,s - 3\,\alpha^2 = 0$. The aim of the present paper will be to investigate Eq.(1) through transformation, direct integration method and trial function method to obtain exact periodic solutions including trigonometric and elliptic function periodic solutions.

Rest of the paper is organized as follows: In Sect.2, we will simplify the structure of Eq.(1) by transformation, and solve simplified equation. Sect.3 will be our conclusions.

2 The non-singular periodic solutions

Now, we consider Kundu equation. By using a transformation

$$u(x,t) = Q(x,t)e^{(iW(x,t)+ivt)}, \tag{5}$$

Eq.(1) can be transformed into the following equation

$$\begin{aligned}&((3\,\alpha + 2\,s)Q^2(x,t)Q_x(x,t) + Q(x,t)W_{xx}(x,t) + Q_t(x,t) + 2Q_x(x,t)W_x(x,t))i + \gamma Q^5(x,t)\\&+(\beta - \alpha\,W_x(x,t))Q^3(x,t) - (v + W_x^2(x,t) + W_t(x,t))Q(x,t) + Q_{xx}(x,t) = 0.\end{aligned} \tag{6}$$

where $Q(x,t)$ and $W(x,t)$ are real functions to be determined, and v is real constant to be determined.

Case 1: If set

$$W_x(x,t) = A + B\,Q^2(x,t),\ W_t(x,t) = E + F\,Q^2(x,t), \tag{7}$$

then Eq.(6) becomes the following form

$$\begin{aligned}&((3\alpha + 2s + 4B)Q^2(x,t)Q_x(x,t) + 2AQ_x(x,t) + Q_t(x,t))i + Q_{xx}(x,t) - (v + A^2 + E)Q(x,t)\\&-(A\,\alpha + 2\,A\,B + F - \beta)Q^3(x,t) + (\gamma - \alpha\,B - B^2)Q^5(x,t) = 0,\end{aligned} \tag{8}$$

where A, B, E and F are real constants to be determined. When the parameters satisfy the following conditions

$$3\alpha + 2s + 4B = 0,\ A\,\alpha + 2\,A\,B + F - \beta = 0,\ \gamma - \alpha\,B - B^2 = 0, \tag{9}$$

or

$$A = 2\tfrac{F-\beta}{\alpha+2s},\ B = -\tfrac{1}{2}s - \tfrac{3}{4}\alpha, \gamma = \tfrac{1}{4}s\alpha - \tfrac{3}{16}\alpha^2 + \tfrac{1}{4}s^2, \tag{10}$$

then, Eq.(8) is simplified to the following form

$$(2AQ_x(x,t) + Q_t(x,t))i + Q_{xx}(x,t) - (v + A^2 + E)Q(x,t) = 0. \tag{11}$$

Solving equations $2AQ_x(x,t) + Q_t(x,t) = 0$, $Q_{xx}(x,t) - (v + A^2 + E)Q(x,t) = 0$, we have

$$Q(x,t) = C_1\,sin(\sqrt{-v - A^2 - E}(x - 2At)) + C_2\,cos(\sqrt{-v - A^2 - E}(x - 2At)), \tag{12}$$

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where C_1, C_2, E and v are arbitrary constants, they should satisfy the condition $-v - A^2 - E > 0$.

Substituting Eq.(12) into Eq.(7) and integralling it , we obtain

$$W(x,t) = -\frac{1}{4\sqrt{-v-A^2-E}}(B(C_1^2 - C_2^2)\, sin(2\sqrt{-v-A^2-E}(x-2At))$$

$$+2BC_1C_2\, cos(2\sqrt{-v-A^2-E}(x-2At)) + \sqrt{-v-A^2-E}(4B(C_1^2+C_2^2)(-\tfrac{1}{2}x + At) \tag{13}$$

$$-4Et - 4Ax - 4C_3) + 2BC_1C_2).$$

By using $W_{xt}(x,t) = W_{tx}(x,t)$, we obtain $F = -2AB$. Therefore, when A, B, α, β, s and γ satisfy the relationships

$$A = \frac{\beta}{\alpha},\ B = -\frac{1}{4}(2s+3\alpha), \gamma = \frac{1}{4}s\alpha - \frac{3}{16}\alpha^2 + \frac{1}{4}s^2, \tag{14}$$

periodic solutions of Eq.(1) are expressed as

$$u(x,t) = (C_1\, sin(\sqrt{-\frac{v\alpha^2+\beta^2+E\alpha^2}{\alpha^2}}(\frac{\alpha x-2\beta t}{\alpha})) + C_2\, cos(\sqrt{-\frac{v\alpha^2+\beta^2+E\alpha^2}{\alpha^2}}(\frac{\alpha x-2\beta t}{\alpha})))e^{(iW(x,t)+ivt)}, \tag{15}$$

where C_1, C_2, E and v are arbitrary constants, and

$$W(x,t) = \frac{(2s+3\alpha)}{16\sqrt{-\frac{v\alpha^2+\beta^2+E\alpha^2}{\alpha^2}}}((C_1^2 - C_2^2)\, sin(\sqrt{-\frac{v\alpha^2+\beta^2+E\alpha^2}{\alpha^2}}(\frac{\alpha x-2\beta t}{\alpha}))$$

$$+2C_1C_2\, cos(\sqrt{-\frac{v\alpha^2+\beta^2+E\alpha^2}{\alpha^2}}(\frac{\alpha x-2\beta t}{\alpha}))) + \frac{C_1C_2(2s+3\alpha)}{8\sqrt{-\frac{v\alpha^2+\beta^2+E\alpha^2}{\alpha^2}}} \tag{16}$$

$$-\frac{1}{8\alpha}(C_1^2+C_2^2)(2s+3\alpha)(\alpha x - 2\beta t) + Et + \frac{\beta}{\alpha}x + C_3,$$

α, β, v and E satisfy $v\alpha^2+\beta^2+E\alpha^2 < 0$. Eg.(15) is a non-singular periodic solution of Kundu equation (see Fig.1).

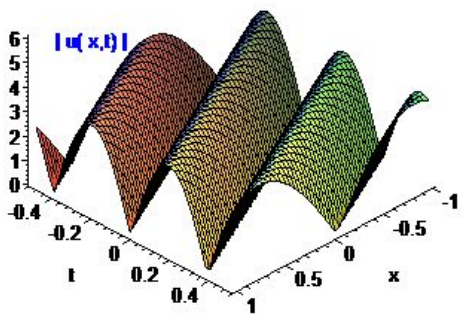

Fig. 1: The trigonometric functions periodic solution Eq.(15) with $\alpha = 1$, $\beta = 2$, $v = 2$, $E = -10$, $C_1 = 2$, $C_2 = 3$.

Case 2: If set $X = P(x - k\,t)$ and $W_X(X) = A + B\,Q(X)^2$, then Eq.(6) can be converted to the

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following from

$$\begin{aligned} &i((3\alpha+2s+4PB)Q(X)^2+(2PA-k))Q_X(X)+P^2\,Q_{XX}(X)+(-v+P^2\,A^2)Q(X) \\ &+(\beta-\alpha\,P\,A)Q(X)^3+(-\alpha\,P\,B+\gamma-P^2\,B^2)Q(X)^5=0. \end{aligned} \tag{17}$$

When the parameters satisfy the following conditions

$$k=2PA,\ B=-\tfrac{1}{4P}(2s+3\alpha),\gamma=\tfrac{1}{4}s\alpha-\tfrac{3}{16}\alpha^2+\tfrac{1}{4}s^2, \tag{18}$$

Eq.(17) is simplified to the following form

$$P^2\,Q_{XX}(X)+(-v+P^2\,A^2)Q(X)+(\beta-\alpha\,P\,A)Q(X)^3=0. \tag{19}$$

We use trial function method to look for elliptic function periodic solutions for Eq.(19). Suppose solution of Eq.(19) as follows

$$Q(X)=a_0+\,a_1\,JacobiSN(X,m), \tag{20}$$

where a_0, a_1, b_1 and $m\,(0<m<1)$ are constants to be determined. Substituting Eq.(20) into Eq.(19), we easily obtain the following results.

When $a_0=0,\ A=\frac{2\,P^2m^2+\beta\,a_1^2}{\alpha P\,a_1^2},\ v=-\frac{P^2m^2a^2a_1^4-4P^4m^4-4\beta P^2m^2a_1^2-\beta^2a_1^4+\alpha^2P^2a_1^4}{\alpha^2a_1^4}$, Eq.(19) has solutions

$$Q(X)=a_1\,JacobiSN(X,m), \tag{21}$$

where a_1, P and $m\,(0<m<1)$ are arbitrary constants. At this time, the elliptic function periodic solutions of Eq.(1) are expressed as

$$u(x,t)=a_1\,JacobiSN(X,m)\,e^{(i\int(\frac{2\,P^2m^2+\beta\,b_1^2}{\alpha P\,b_1^2}-\frac{1}{4P}(2s+3\alpha)(a_1\,JacobiSN(X,m))^2)dX+ivt)}, \tag{22}$$

where $X=P(x-2PAt)$ and $v=-\frac{P^2m^2a^2a_1^4-4P^4m^4-4\beta P^2m^2a_1^2-\beta^2a_1^4+\alpha^2P^2a_1^4}{\alpha^2a_1^4}$. This is a non-singular elliptic function periodic solution of Kundu equation (see Fig.2).

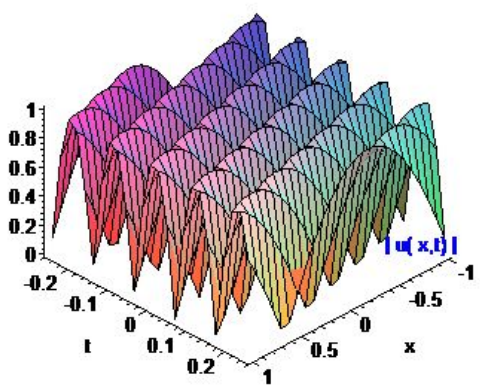

Fig. 2: The elliptic function periodic solution Eq.(22) with $\alpha=\frac{1}{2}$, $\beta=2$, $m=\frac{1}{2}$, $a_1=1$, $P=2$.

In the above solutions, setting $s=0$, we can obtain solutions of the derivative nonlinear Schrödinger equation.

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3 Conclusion

When parameters satisfy condition $4\,s^2-16\gamma+4\,\alpha\,s-3\,\alpha^2=0$ in Kundu equation, its non-singular periodic solutions including trigonometric and elliptic function periodic solutions, are obtained. Prior to this, we have not seen reported. Based on the solutions of Kundu equation, we easily obtain solutions to the derivative nonlinear Schrödinger equation. These results contribute to a better understanding of the structure of the solutions for the nonlinear Schrödinger type equations.

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