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Original Research Article

Eta-Einstien S-Manifolds and spectral geometry*†

Abstract

In this paper results of Patodi are extended to Eta-Einstien S-Manifolds and some new facts are derived related to spectral geometry of Eta-Einstien S-Manifolds.

1 Introduction

Generalizing the notion of η-Einstien contact manifolds, M. Kobayashi and S. Tsuchiya defined the notion of η-Einstien S-manifolods [10], for more detail on eta-Einstien S-manifolds see [2].
Geometry of Riemannian manifolds has been studied by many authors through the spectrum of operators of Laplace type [8], [4].
Let Δ_p be the Laplace Biltrami operator acting on the space of smooth p-forms, p=0, 1, 2 over a compact m-dimentional Riemannian manifold M. Patodi [8] set up the following result for the space forms.

Theorem 1.1. *Let (M_i, g_i) be the comapct Riemanian manifolds without boundary. Assume that $Spec(\Delta_p, M_1) = Spec(\Delta_p, M_2)$, for $p = 0, 1, 2$. Then*
(1) The manifold M_1 has constant scalar curvature c if and only if the manifold M_2 has constant scalar curvature c.
(2) The manifold M_1 is Einstein if and only if the manifold M_2 is Einstein.
(3) The manifold M_1 has constant sectional curvature c if and only if the manifold M_2. has constant sectional curvature c.

Later on Donnely [3] and Gilkey and Sacks [9] extended the Theorem 1.1 *to complex manifolds. Later on J.Park extended this result to almost contact manifolds. In this paper, after introduction, the second section is related to S-manifolds. Third section is related to some new results of heat trace asymptotics on η-Einstien S-manifolds.*

2 S-manifolds

As a generalization of both almost complex (in even dimension) and almost contact (in odd dimension) structures, Yano introduced in [11] the notion of f-structure on a smooth manifold of dimension $2n + s$, i.e. a tensor field of type (1,1) and rank $2n$ satisfying $f^3 + f = 0$. The existence of such a structure is equivalent to a reduction of the structural group of the tangent bundle to $U(n) \times O(s)$. Let N be a $(2n + s)$-dimensional manifold

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with an f-structure of rank $2n$. If there exist s global vector fields $\xi_1, \xi_2, \ldots, \xi_s$ on N such that:

$$f\xi_\alpha = 0, \qquad \eta_\alpha \circ f = 0, \qquad f^2 = -I + \sum \xi_\alpha \otimes \eta_\alpha, \tag{1}$$

where η_α are the dual 1-forms of ξ_α, we say that the f-structure has complemented frames. For such a manifold there exists a Riemnnian metric g such that

$$g(X,Y) = g(fX, fY) + \sum \eta_\alpha(X)\eta_\alpha(Y)$$

for any vector fields X and Y on N.

An f-structure f is normal, if it has complemented frames and

$$[f,f] + 2\sum \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f,f]$ is Nijenhuis torsion of f.

Let F be the fundamental 2-form defined by $F(X,Y) = g(X, fY)$, X, $Y \in T(N)$. A normal f-structure for which the fundamental form F is closed, $\eta_1 \wedge, \ldots, \wedge \eta_s \wedge (d\eta_\alpha)^n \neq 0$ for any α, and $d\eta_1 = \cdots = d\eta_s = F$ is called to be an S-structure. A smooth manifold endowed with an S-structure will be called an S-manifold. These manifolds were introduced by Blair in [1].

We have to remark that that if we take $s = 1$, S-manifolds are natural generalizations of Sasakian manifolds. In the case $s \geq 2$ some interesting examples are given in [1, 6].

If N is an S-manifold, then the following formulas are true (see [1]):

$$\overline{\nabla}_X \xi_\alpha = -fX, \qquad X \in T(N), \quad \alpha = 1, \ldots, s, \tag{2}$$

$$(\overline{\nabla}_X f)Y = \sum \{g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X\}, \quad X, Y \in T(N), \tag{3}$$

where $\overline{\nabla}$ is the Riemannian connection of g. Let L be the distribution determined by the projection tensor $-f^2$ and let M be the complementry distribution which is determined by $f^2 + I$ and spanned by $\xi_1, \ldots, \xi_s$. It is clear that if $X \in L$ then $\eta_\alpha(X) = 0$ for any α, and if $X \in M$, then $fX = 0$. A plane section π on N is called an invariant f-section if it is determined by a vector $X \in L(x)$, $x \in N$, such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature of π is called the f-sectional curvature. If N is an S-manifold of constant f-sectional curvature k, then its curvature tensor has the form

$$\begin{aligned} R(X,Y,Z,W) = \sum_{\alpha,\beta} \{ & g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W) + \\ & + g(fY, fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY, fW)\eta_\alpha(X)\eta_\beta(Z)\} + \\ & + \frac{1}{4}(k+3s)\{g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)\} + \\ & + \frac{1}{4}(k-s)\{F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)\}, \end{aligned} \tag{4}$$

X, Y, Z, W $\in T(N)$. Such a manifold $N(k)$ will be called an S-space form. The euclidean space E^{2n+s} and the hiperbolic space H^{2n+s} are examples of S-space forms.

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Definition 2.1. *[2], [10] S-manifold $(M_m, \eta_\alpha, g, f, \xi_\alpha)$ is said to be η-Einstien if the Ricci tensor ρ of M is of the form*

$$\rho = ag + b\sum_{\alpha=1}^{s} \eta_\alpha \otimes \eta_\alpha,$$

where a, b are constants on M.

3 Main results

Let M be a (2m+s)-dimensional S-space form then from (4), the ricci tensor ρ is given by

$$\rho = \frac{4s + (k+3s)(2m-1) + 3(k-1)}{4} g + \frac{(2m+s-2)(4-k-3s) - 3(k-s)}{4} \eta_\alpha \otimes \eta_\alpha. \quad (5)$$

Hence M is an eta-Einstien manifold.
Define the tensor fields $S_{\alpha,\beta}$ and T_c of M respectively by

$$S_{a,b}(X,Y) = \rho(X,Y) - (ag(X,Y) + b\sum_{\alpha=1}^{s} \eta_\alpha(X) \otimes \eta_\alpha(X))$$

$$\begin{aligned} T_c(X,Y,Z,W) &= R(X,Y,Z,W) - \{\sum_{\alpha,\beta}\{g(fX,fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX,fZ). \\ &. \quad \eta_\alpha(Y)\eta_\beta(W) + g(fY,fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY,fW)\eta_\alpha(X)\eta_\beta(Z)\} \\ &+ \quad \frac{1}{4}(k+3s)\{g(fX,fW)g(fY,fZ) - g(fX,fZ)g(fY,fW)\} \\ &+ \quad \frac{1}{4}(k-s)\{F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - \\ &- \quad 2F(X,Y)F(Z,W)\}\}, \end{aligned}$$

for vector fields X, Y, Z, W on M, where a, b are smooth functions on M and k is a constant f-sectional curvature.
Let $\{e_i\}$ be an orthonormal basis of T_pM at any point $p \in M$. Next in this paper, we shall use the following notations:

$$R_{ijkl} = g(R(e_i,e_j)e_k, e_l), \quad \rho_{i,j} = \rho(e_i,e_j), \quad f_{ij} = g(fe_i,e_j),$$
$$\nabla_i f_{jk} = g((\nabla_{e_i} f)e_j, e_k), \nabla_i \eta_{\alpha j} = g((\nabla_{e_i}\xi_\alpha, e_j)$$

and so on, where indices run over the range $1, 2, \ldots, 2m+s$.
The equations (2), (3) can be written as:

$$\nabla_i f_{jk} = \sum_\alpha g_{ij}\eta_{\alpha k} - \eta_{\alpha j} g_{ik} \quad (6)$$
$$\nabla_i \eta_{\alpha j} = -f_{ij}$$

By the definition of the tensor field $S_{\alpha,\beta}$, (6) and some computations, we have

$$|S_{a,b}|^2 = |\rho|^2 + (2m+s)a^2 + sb^2 - 2a\tau + 2abs - 4bms \quad (7)$$

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Here we see that M is η-Einstien if and only if $S_{a,b} = 0$ and

$$a = \frac{\tau}{2m} - s, \quad b = s(2m+s) - \frac{s\tau}{2m}.$$

Now first we shall prove the following lemma for further computations of $\|T_k\|^2$ of the tensor field T_k on S-manifold M. In case of Sasakian manifolds, it was proved by J. Park [7].

Lemma 3.1. *On S-manifold, we have*

$$R_{ijkl}\{f_{ki}f_{jl} - f_{kj}f_{il} + 2f_{ji}f_{kl}\} = 6\tau - 12ms + 12m(2m+s-2)$$

Proof. It is clear from the properties of curvature tensor that

$$R_{ijkl}f_{ki}f_{jl} = \frac{1}{2}(R_{ijkl} - R_{kjil})f_{ki}f_{jl} = \frac{1}{2}(R_{ijkl} + R_{jkil})f_{ki}f_{jl} = -\frac{1}{2}R_{kijl}f_{ki}f_{jl} \tag{8}$$

Similarly

$$-R_{ijkl}f_{kj}f_{il} = -\frac{1}{2}R_{jkil}f_{jk}f_{il} \tag{9}$$

$$R_{ijkl}f_{ji}f_{kl} = -R_{ijkl}f_{ij}f_{kl} \tag{10}$$

from (8), (9) and (10) we have

$$R_{ijkl}\{f_{ki}f_{jl} - f_{kj}f_{il} + 2f_{ji}f_{kl}\} = -3R_{ijkl}f_{ij}f_{kl} \tag{11}$$

Also

$$\nabla_l\nabla_i f_{jk} = \sum_\alpha g_{ij}\nabla_l\eta_{\alpha k} - g_{ik}\nabla_l\eta_{\alpha j} = \sum_\alpha -g_{ij}f_{lk} + g_{ik}f_{lj} \tag{12}$$

$$\nabla_l\nabla_i f_{jk} - \nabla_i\nabla_l f_{jk} = \sum_\alpha -g_{ij}f_{lk} + g_{ik}f_{lj} + g_{lj}f_{ik} + g_{lk}f_{ij} \tag{13}$$

Apply Ricci identity to (13) and taking sum by setting $i = k$ in the rsulting equality, we get

$$-R_{lij\beta}f_{\beta i} - \rho_{l\beta}f_{j\beta} = (2m+s-2)f_{lj} \tag{14}$$

$$-R_{lij\beta}f_{\beta i}f_{lj} - \rho_{l\beta}f_{j\beta}f_{lj} = (2m+s-2)f_{lj}f_{lj} = (2m+s-2)(2m)$$

Hence

$$\begin{aligned} -R_{il\beta}jf_{\beta i}f_{jl} &= -\rho_{l\beta}(g_{l\beta} - \eta_l\eta_\beta) + (2m+s-2)(2m) \\ &= -\tau + 2ms + (2m+s-2)(2m) \end{aligned} \tag{15}$$

From (8) and (15)

$$\frac{1}{2}R_{jkil}f_{jk}f_{il} = -\tau + 2ms + (2m+s-2)(2m) \tag{16}$$

From (11) and (16)

$$R_{ijkl}\{f_{ki}f_{jl} - f_{kj}f_{il} + 2f_{ji}f_{kl}\} = 6\tau - 12ms + 12m(2m+s-2) \quad \square$$

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Let M be a compact Riemannian manifold of real dimension m without boundary and let D ba a operator of Laplace type on the space of smooth sections to a smooth vector bundle over M. Let e^{-tD} be the fundamental solution of the heat equation. This operator is of trace class and as $t \downarrow 0$ there is a complete asymptotic expansion with locally computable coefficients in the form:

$$Tr_{L^2} e^{-tD} \sim \sum_{n \geq 0} t^{(n-m)/2} a_n(D).$$

Consider the following result [5].

Theorem 3.1. *Let D be an operator of Laplace type on the space of sections $C^\infty(V)$ to a vector bundle V over a compact manifold M. Let I be the identity endomorphism of V. We have*

$$\begin{aligned} a_0(D) &= (4\pi)^{-m/2} \int_M Tr\{I\}, \qquad (17) \\ a_2(D) &= (4\pi)^{-m/2} \frac{1}{6} \int_M Tr(6E + \tau I), \\ a_4(D) &= (4\pi)^{-m/2} \frac{1}{360} \int_M Tr\{60E + 60\tau E + 180E^2 + 30\Omega^2 + (12\tau + 5\tau^2 - 2|\rho|^2 + \\ &+ 2|R|^2)I\} \end{aligned}$$

Now we will prove Theorem 1.1 for s-manifolds.

Theorem 3.2. *Let $M_i = (M_i, \eta_i, g_i, f_i, \xi_i)$ be m_i-dimensional compact S-manifolds without boundary with $m_i \geq 5$. Assume that $Spec(\triangle_p, M_1) = Spec(\triangle_p, M_2)$ for $p = 0, 1, 2$, then:*
(1) $m_1 = m_2$ and $Vol(M_1) = Vol(M_2)$.
(2) M_1 has constant scalar curvature k if and only if the manifold M_2 has constant scalar curvature k.
(3) M_1 is η-Einstien if and only if M_2 is η-Einstien.
(4) M_1 is S-space form with constant f-sectional curvature k if and only if M_2 is S-space form with constant f-sectional curvature k.

The values p=0, 1, 2 are not particularly special.

Proof. Let $M(\eta, g, f, \xi)$ be a $(2m+s)$-dimensional compact S-manifold without boundary. From (3.1) Theorem for $D = \triangle_p$(p=0, 1, 2), we have

$$\begin{aligned} Tr_{L^2}(e^{-t}\triangle_0) &= (4\pi t)^{-m/2}\{Vol(M) + O(t)\}, \\ a_2(\triangle_0, M) &= \frac{1}{6}(4\pi)^{-m/2} \int_M \tau, \\ a_2(\triangle_1, M) &= \frac{1}{6}(4\pi)^{-m/2} \int_M (2m+s)\tau. \end{aligned}$$

The work of Patodi shows that there exist universal constants so:

$$a_4(\triangle_p, M) = (4\pi)^{-m/2} \int_M \{c^1_{m,p}\tau^2 + c^2_{m,p}\rho^2 + c^1_{m,p}R^2 + c^1_{m,p}\tau\}, \quad p = 0, 1, 2.$$

Let $M_i = (M_i, \eta_i, g_i, f_i, \xi_i)$ be $m_i + s$-dimensional compact S-manifolds without boundary (i = 1, 2). Assume that $Spec(\triangle_p, M_1) = Spec(\triangle_p, M_2)$ for p=0, 1, 2. Let R_i, ρ_i and

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τ_i denote the curvature tensor, the Ricci tensor and the scalar curvature of $M_i (i = 1, 2)$ respectively. Then by Theorem 3.1(1) we have
$m_1 = m_2$ and $Vol(M_1) = Vol(M_2)$.
Also

$$\begin{aligned} \tau_1 &= (4\pi)^{m/2} Vol(M_1)^{-1}\{ma_2(\triangle_0, M_1) - a_2(\triangle_1, M_1)\} \\ &= (4\pi)^{m/2} Vol(M_2)^{-1}\{ma_2(\triangle_0, M_2) - a_2(\triangle_1, M_2)\} \\ &= \tau_2 \end{aligned}$$

Further suppose that M_1 is an η-Einstien S-manifold with coefficient functions α_1 and β_1. Here α_1 and β_1 are constant and hence the scalar curvature τ_1 of M_1 is also constant given as $\tau_1 = (2m+s)\alpha_1+\beta_1$. Thus from second assertion of theorem the scalar curvature τ_2 of M_2 is also constant and $\tau_1 = \tau_2$. Since $Vol(M_1) = Vol(M_2)$, the integrals of τ^2 are equal. Since $\tau_{;ii} = 0$, from a_4, we have

$$\int_{M_1} (c^2_{m,p}\rho_1^2 + c^3_{m,p}R_1^2) = \int_{M_2} (c^2_{m,p}\rho_2^2 + c^3_{m,p}R_2^2), \tag{18}$$

for $p = 1, 2$ these two equations are independent [8]. Consequently

$$\int_{M_1} \rho_1^2 = \int_{M_2} \rho_2^2, \quad \int_{M_1} R_1^2 = \int_{M_2} R_2^2. \tag{19}$$

Thus from (7), we have

$$0 = \int_{M_1} |S^1_{a_1,b_1}|^2 = \int_{M_1} |\rho_1|^2 + (2m+s)a_1^2 + sb_1^2 - 2a\tau_1 + 2a_1b_1s - 4b_1ms \tag{20}$$

$$= \int_{M_2} |\rho_2|^2 + (2m+s)a_1^2 + sb_1^2 - 2a_1\tau_1 + 2a_1b_1s - 4b_1ms \tag{21}$$

Here we may note that

$$a_1 = \frac{\tau_1}{2m} - s = \frac{\tau_2}{2m} - s \tag{22}$$

$$b_2 = s(2m+s) - \frac{s\tau_1}{2m} = s(2m+s) - \frac{s\tau_2}{2m} \tag{23}$$

Here we may take

$$S^2_{a,b} = \rho_2 - (a_2g_2 + b_2\sum_{\alpha=1}^{s} \eta_{\alpha 2} \otimes \eta_{\alpha 2}),$$

where

$$a_2 = \frac{\tau_2}{2m} - s \quad b_2 = s(2m+s) - \frac{s\tau_2}{2m},$$

then we have

$$\int_{M_2} |S^2_{a_2,b_2}|^2 = \int_{M_2} |\rho_2|^2 + (2m+s)a_2^2 + sb_2^2 - 2a_2\tau_2 + 2a_2b_2s - 4b_2ms. \tag{24}$$

This implies

$$a_1 = a_2 \quad b_1 = b_2. \tag{25}$$

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Therefore all above equation implies

$$0 = \int_{M_2} |S^2_{a_2,b_2}|^2,$$

then M_2 is an η-Einstien manifold with the same coefficients in the defining equation. This completes the Theorem 3.2 (3). Lastly, suppose that M_1 is s-space form with constant f-sectional curvature k. Then from (5), M_1 is η-Einstien with constant coeficients $a_1 = \frac{4s+(k+3s)(2m-1)+3(k-1)}{4}$ and $b_1 \frac{(2m+s-2)(4-k-3s)-3(k-s)}{4}$ Thus the scalar cutvature is

$$\tau_1 = \frac{m}{2}\{(2m+2)k + 6m + 8s - 6\}$$

Therefore from assertion (3) and hypothesis that $Spec(\triangle_p, M_1) = Spec(\triangle_p, M_2)(p = 0, 1, 2)$, we see that M_2 is η-Einstien manifold with constant coefficients a_2 and b_2 such that $a_2 = a_1$, $b_2 = b_1$ and hence $\tau_2 = \tau_1$. Now we define the tensor field for the S-manifold M_2 as:

$$(T^2_c)_{ijkl} = R_{ijkl} - K_{ijkl}, \tag{26}$$

where

$$\begin{aligned} K_{ijkl} = \sum_{\alpha,\beta}\{&g(fe_i, e_l)\eta_\alpha(e_j)\eta_\beta(e_k) - g(fe_i, e_k)\eta_\alpha(e_j)\eta_\beta(e_l) + \\ &+g(fe_j, fe_k)\eta_\alpha(e_i)\eta_\beta(e_l) - g(fe_j, fe_l)\eta_\alpha(e_i)\eta_\beta(e_k)\} + \\ &+\frac{1}{4}(k+3s)\{g(fe_i, fe_l)g(fe_j, fe_k) - g(fe_i, fe_k)g(fe_j, fe_l)\} + \\ &+\frac{1}{4}(k-s)\{F(e_i, e_l)F(e_j, e_k) - F(e_i, e_k)F(e_j, e_l) - 2F(e_i, e_j)F(e_k, e_l)\}, \end{aligned} \tag{27}$$

After some computations, we have

$$|K|^2 = m\{(2m+2)k^2 + 6m + 8s - 6\} \tag{28}$$

Now using lemma

$$R_{ijkl}K_{ijkl} = 2k\tau - m(k-s)(6m - 6 + 8s) \tag{29}$$

from (26), (28) and (29), we obtain

$$|T^2_c|^2 = |R_2|^2 - 4k\tau + m\{(2m+2)k^2 + 6m + 8s - 6\} + 2m(k-s)(6m - 6 + 8s). \tag{30}$$

Also, since M_1 is m-dimensional S-space form with constant f-sectional curvature k, we have

$$|R_1|^2 = m\{(2m+2)k^2 + 6m + 8s - 6\}, \tag{31}$$

and

$$0 = |T^1_c|^2 = |R_1|^2 - 4k\tau_1 + m\{(2m+2)k^2 + 6m + 8s - 6\} + 2m(k-s)(6m - 6 + 8s). \tag{32}$$

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Thus from (19), (30), (32) and $\tau_1 = \tau_2$ we obtain

$$\begin{aligned} 0 &= \int_{M_1} |T_c^1|^2 \\ &= \int_{M_1} |R_1|^2 - 4k\tau_1 + m\{(2m+2)k^2 + 6m + 8s - 6\} + 2m(k-s)(6m - 6 + 8s) \\ &= \int_{M_2} |R_2|^2 - 4k\tau_2 + m\{(2m+2)k^2 + 6m + 8s - 6\} + 2m(k-s)(6m - 6 + 8s) \\ &= \int_{M_2} |T_c^2|^2, \end{aligned}$$

and hence $T_c^2 = 0$ on M_2. Therefore we see that M_2 is also an (2m+s)-dimensional S-space form with constant f-sectional curvature k. This completes the proof of Theorem. □

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