

Original Research Article

An Application of the Maximum Theorem and the Distance Method in Set-Valued Optimization Related to Efficient Continuous Selections

Abstract

In the present paper, we consider relationships between the basic solution concepts of set-valued optimization, the properties of continuous multifunctions with compact and convex values and the existence of efficient continuous selections using the Maximum Theorem and the Distance method. We also discuss two solution concepts in set-valued optimization problem and prove existence of two types of efficient continuous selections which correspond to the two set-valued optimization solutions, respectively.

Keywords: set-valued optimization, continuous multifunction, efficient continuous selection, Maximum Theorem, compact, convex.

2010 Mathematics Subject Classification: 90C29

1. Introduction

Set-valued optimization appears naturally in many branches of science, such as game theory, economics, finance, physics, control theory, biomathematics and so on. The aim of this paper is to show some relationships between the basic solution concepts of set-valued optimization, the properties of continuous multifunctions with compact and convex values and the existence of efficient continuous selections in finite dimensional Euclidean spaces.

In present paper, we will consider an application of the Maximum Theorem and the Distance method in set-valued optimization problem. So, let $\|\cdot\|$ be the Euclidean norm on R^n , $d(x,y) = \|x-y\|$ for $x,y \in R^n$ be the Euclidean distance and τ be the topology induced by d . It is known that the Euclidean norm is strictly convex.

We will focus our attention on set-valued optimization. The general form of this optimization problem is to find a variable $x \in X \subset R^m$, $m \geq 1$, so as to maximize $F(x)$ subject to $x \in X$, where the feasible domain X is nonempty and $F: X \Rightarrow R^n$, $n \geq 2$, is a given multifunction [1].

It is known that the image of each point by a multifunction is nonempty and if the multifunction F is function with vector values, then the above problem reduces to a multi-criteria, multi-objective or vector optimization problem. Additionally, if $n=1$, then F is a real function and we are in the framework of classical optimization.

47

48 **Definition 1.** Let $M \subset R^n$ be a nonempty set. A point $x \in M$ is called an efficient point of M
 49 if and only if $\{x\} = M \cap (\{x\} + R_+^n)$. The set of the all efficient points of M is denoted by
 50 $Eff(M)$ and it is called an efficient set.

51

52 **Remark 1.** It is well-known that if M is nonempty compact and $x \in M$, then $Eff(M)$ is
 53 nonempty, $Eff(M) \subset \partial M$ and there exists a point $y \in Eff(M)$ such that $y \in (\{x\} + R_+^n)$, see
 54 also [2] for more details.

55

56 Recall that $z(z_1, z_2, \dots, z_n) \in R^n$ is an ideal point for compact set M if and only if
 57 $z_i = \max\{x_i \mid x(x_1, x_2, \dots, x_n) \in M\}$ for every $i \in \{1, 2, \dots, n\}$. It is easy to show that there are two
 58 cases: $z \in \partial M$ or $z \notin M$.

59

60 The multifunction $F : X \Rightarrow R^n$ is usually identified with its graph
 61 $Graph(F) = \{(x, y) \in X \times F(X) \mid y \in F(x)\}$.

62

63 For our set-valued optimization problem we study the following solution concepts.

64

65 **Definition 2.** (a) A pair $(x, y) \in Graph(F)$ is called an efficient solution if and only if $x \in X$
 66 and $y \in Eff(F(x))$. The set of the all efficient solutions of $Graph(F)$ is denoted by $Eff(X, F)$
 67 and it is called an efficient-graph set.

68 (b) A pair $(x, y) \in Graph(F)$ is called a Pareto-optimal solution if and only if $x \in X$, $y \in F(x)$
 69 and $y \in Eff(F(x))$. The set of the all Pareto-optimal solutions of $Graph(F)$ is denoted by
 70 $PO(X, F)$ and it is called a Pareto-optimal set.

71

72 From above definition it follows that $PO(X, F) \subset Eff(X, F)$.

73

74 **Remark 2.** In our set-valued optimization problem it can be shown that if the feasible domain
 75 X is compact and F is upper semi-continuous with compact and convex values, then $F(X)$
 76 is compact [3 – Proposition 9.7]. As a result there exists a point $y \in F(X)$ such that
 77 $y \in Eff(F(X))$, i.e. there exists a point $x \in X$ such that $y \in F(x)$ and $(x, y) \in PO(X, F)$, see
 78 also [1] and [2]. Usually, a Pareto-optimal solution is not necessarily uniquely determined, but
 79 there are several Pareto-optimal solutions [4].

80

81 **Remark 3.** Consider a multifunction $F : X \Rightarrow R^n$.

82 (1) By a continuous selection f for F , we mean a continuous function $f : X \rightarrow F(X)$ such
 83 that $f(x) \in F(x)$ for all $x \in X$ [5].

84 (2) Let F have compact values. By an efficient selection f for F , we mean a function
 85 $f : X \rightarrow F(X)$ such that $f(x) \in Eff(F(x))$ for all $x \in X$.

86

87 In this paper, we study our set-valued optimization problem where the multifunction F is
 88 continuous on X and it has compact and convex values. In this case we get that $Eff(X, F)$
 89 is nonempty and if X is compact, then $PO(X, F)$ is also nonempty. We prove the existence
 90 of two different efficient continuous selections which correspond to the two set-valued

optimization solutions, respectively. Here we use the Maximum Theorem and the Distance method in set-valued optimization.

2. Main result

In this section, we start our analysis with the first basic theorem.

Theorem 1.

Let $X \subset R^m$ and $F: X \Rightarrow R^n$ be a continuous multifunction with compact and convex values. The following statements are true:

(a) There exists an efficient continuous selection $f: X \rightarrow F(X)$ for F such that $(x, f(x)) \in \text{Eff}(X, F)$ for all $x \in X$.

(b) If $X \subset R^m$ is compact, then there exist an efficient continuous selection $f: X \rightarrow F(X)$ for F and a point $x \in X$ such that $(x, f(x)) \in \text{PO}(X, F)$.

Remark 4. We often use the Maximum Theorem (or Berge Maximum Theorem) as a mathematical tool in optimization and nonlinear analysis. In order to give the proof of Theorem 1 we will use this theorem in its classical variant: "Let $S \subset R^n$ and $\Theta \subset R^m$, $g: S \times \Theta \rightarrow R$ be a continuous function, and $D: \Theta \Rightarrow S$ be a compact-valued and continuous multifunction. Then, the function $g^*: \Theta \rightarrow R$ defined by $g^*(\theta) = \max\{g(x, \theta) \mid x \in D(\theta)\}$ is continuous on Θ , and the multifunction $D^*: \Theta \Rightarrow S$ defined by $D^*(\theta) = \{x \in D(\theta) \mid g(x, \theta) = g^*(\theta)\}$ is compact-valued and upper semi-continuous on Θ " [6] [7].

The above remark allows us to prove the theorem.

Proof of Theorem 1.

(a) Let $S = F(X)$ and $x \in X$. For each $i \in \{1, 2, \dots, n\}$ we define a function $m_i: X \rightarrow R$ by $m_i(x) = \max\{y_i \mid y(y_1, y_2, \dots, y_n) \in F(x)\}$. It is easy to see that these functions $\{m_i\}_{i=1}^n$ are continuous on X , see Remark 4. As a result we get a continuous function $m: X \rightarrow R^n$ such that $m(x) = (m_1(x), m_2(x), \dots, m_n(x))$ for all $x \in X$. Observe that for each $x \in X$, $m(x)$ is an ideal point for $F(x)$.

Now, let us define a function $g: S \times X \rightarrow R$ by $g(y, x) = -\sqrt{\sum_{i=1}^n (y_i - m_i(x))^2}$ for all $y(y_1, y_2, \dots, y_n) \in S$ and for all $x \in X$. Clearly, g is continuous on $S \times X$ and $F(x) \subset (\{m(x) - R_+^n\})$ for all $x \in X$.

Let $x \in X$. We also see that either $m(x) \notin F(x)$ or $m(x) \in \partial F(x)$.

First, let $m(x) \notin F(x)$. From $F(x)$ is compact and convex it immediately follows that there exists a unique $z \in F(x)$ such that $g(z, x) = \max\{g(y, x) \mid y \in F(x)\}$ and $g(z, x) < 0$.

Second, let $m(x) \in \partial F(x)$. This means that $d(m(x), F(x)) = 0$ and $d(m(x), y) > 0$ for all $y \in F(x) \setminus \{m(x)\}$. In this case, let $z = m(x)$. We get that $z \in F(x)$ is unique and $g(z, x) = \max\{g(y, x) \mid y \in F(x)\}$.

133 Finally, from the construction of $z \in F(x)$ it follows that $\{z\} = F(x) \cap (F(x) + R_+^n)$, i.e.
134 $z \in \text{Eff}(F(x))$, see also [2].

135 From Remark 4 (or the Maximum Theorem) it follows the multifunction $F^*: X \Rightarrow Y$ defined
136 by $F^*(x) = \{y \in F(x) \mid g(y, x) = g^*(x)\}$ is compact-valued and upper semi-continuous on X .

137 But, we also get that $z \in F^*(x)$ and $|F^*(x)| = 1$. As a result we deduce that $f = F^*$ is a
138 continuous function on X and $z = f(x)$, i.e. $f(x) \in \text{Eff}(F(x))$.

139 Finally, we obtain f is an efficient continuous selection for F and $(x, f(x)) \in \text{Eff}(X, F)$ for
140 all $x \in X$.

141

142 (b) In Remark 2 we have shown that $S = F(X)$ is compact and $PO(X, F)$ is nonempty.

143 Let $v(v_1, v_2, \dots, v_n) \in R^n$ be an ideal point for S and define a function $g: S \times X \rightarrow R$ by

144 $g(y, x) = -\sqrt{\sum_{i=1}^n (y_i - v_i)^2}$ for all $y(y_1, y_2, \dots, y_n) \in S$ and for all $x \in X$. Clearly, g is

145 continuous on $S \times X$ and $F(x) \subset \{v - R_+^n\}$ for all $x \in X$.

146 As in the proof of (a) we construct an unique point $z \in F(x)$ such that $z \in \text{Eff}(F(x))$ and an
147 efficient continuous selection f for F such that $(x, f(x)) \in \text{Eff}(X, F)$ for all $x \in X$.

148 It is easy to show that $f(X)$ is compact; therefore, there exist a point $w \in f(X)$ and a point
149 $x \in X$ such that $d(v, w) = d(v, F(X))$ and $w = f(x)$. Thus we obtain $(x, f(x)) \in PO(X, F)$.

150

151 The theorem is proven.

152

153 Note that the efficient selections in (a) and (b) are different, but we use one idea for their
154 constructions.

155

156 **Remark 5.** We used the Maximum Theorem, the Distance method (Approximation of the
157 Ideal Point or Compromise Method) and the Euclidean metric in the proof of Theorem 1. This
158 allows us to transform our set-valued optimization problem into mono-objective
159 optimization problem [2] [8].

160

161 It is interesting to note that in Theorem 1(a) we do not prove that $(x, f(x)) \in PO(X, F)$. Recall
162 that $PO(X, F)$ is nonempty when X is compact. If X is no compact, then $PO(X, F) = \emptyset$
163 is possible.

164

165 **Remark 6.** Theorem 1 shows that the following statement is true: "If $(x, y) \in \text{Graph}(F)$ and
166 $(x, y) \in PO(X, F)$, then $y \in \text{Eff}(F(x))$ ". But, the converse is not true in general.

167

168 **Definition 3.** A multifunction $F: X \Rightarrow R^n$ is called strongly injective if and only if
169 $F(x) \cap F(y) = \emptyset$ for any two distinct points $x, y \in X$.

170

171 **Remark 7.** Let $X \subset R^m$ and $F: X \Rightarrow R^n$ be a strongly injective multifunction. It is easy to
172 show that there exists a function $v: F(X) \rightarrow X$ such that $v(F(z)) = z$ for all $z \in X$. This
173 function v is called an inverse function for F . In view into Definition 3 we understand that
174 the condition $F(x) \cap F(y) = \emptyset$ for any two distinct points $x, y \in X$ is very important. By
175 analogy, for an injective function f we have that $f(x) \neq f(y)$.

176

177 It is easy to show that Theorem 1 implies the following corollary.

178

179 **Corollary 1.**

180 *Let $X \subset R^m$ be compact and $F: X \Rightarrow R^n$ be a strongly injective continuous multifunction*
 181 *with compact and convex values. Define f as in Theorems 1(a) or 1(b). Then X and $f(X)$*
 182 *are homeomorphic.*

183

184 **Proof.**

185 According to Theorems 1(a) or 1(b) there exists an efficient continuous selection f for F .
 186 From $F(x) \cap F(y) = \emptyset$ for all $x, y \in X$ such that $x \neq y$ we obtain that f is bijective. The set

187 X is compact and f is a continuous bijective function on X imply f^{-1} is continuous on
 188 $f(X)$, i.e. f is homeomorphism.

189 The corollary is proven.

190

191

192 **3. Conclusion**

193

194 In this paper, we have shown an application of the Maximum Theorem and the Distance
 195 method in set-valued optimization for the construction of the two continuous selections and
 196 the role of these selections to consider the two solution sets – efficient-graph set and Pareto-
 197 optimal set. As a future work, we intend to study the Maximum Theorem as a generator for
 198 continuous selections and solutions in multi-criteria and set-valued optimization problems.

199

200

201

202 **References**

203

204 [1] Khan A., C. Tammer, C. Zalinescu, Set-valued Optimization: Introduction with
 205 Applications, Springer, 2015.

206

207 [2] Ehrgott M., Multi-criteria Optimization. Springer, 2005.

208

209 [3] Sundaran R., A First Course in Optimization Theory. Cambridge University Press, 1996.

210

211 [4] Luc D., Theory of Vector Optimization. Springer, 1989.

212

213 [5] Repovš D., P. Semenov, Continuous Selections of Multivalued Mapping, Springer, 1998.

214

215 [6] Berge C., Topological Spaces, Including Treatment of Multi-Valued Function, Vector
 216 Spaces, and Convexity, Oliver and Boyd, Edinburgh, 1963.

217

218 [7] Slavov Z., C. Evans, An Application of the Maximum Theorem in Multi-Criteria
 219 Optimization, Properties of Pareto-Retract Mapping, and the Structure of Pareto Sets,
 220 Applied Mathematics (3) 2012; 1415-1422.

221

222 [8] Collette Y., P. Siarry, Multiobjective Optimization and Case Studies, Springer, 2004.

223

224

225

226