

Original research papers

The behavior of two coupled mechanical oscillators in non-linear fields and the possibility of obtaining controlled halts

ABSTRACT

The proposed mechanical system consists of two magnetic oscillating subsystems, which are mechanically coupled. The first one consists of a spring-magnet-mass subsystem and the other one is a spring-mass subsystem. The non-linear symmetric field created by two other fixed magnets, oriented for attraction, acts only to the first subsystem. The entire system can oscillate horizontally, without friction and without loss of energy. Oscillations occur with conservation of kinetic energy and potential energy stored in the springs. During the movement, depending on the amplitude of oscillations, the main body stops for a period, this can be controlled by the physical constants of the system. The reason for these controlled halts of oscillations is the transfer of mechanical impulse, due to entering of both oscillators with different initially frequencies into the frequencies equalizing status. The frequency of the main oscillator is synchronizing with the frequency of the second oscillator, by deeper or lower entrance in a non-linear magnetic field, which is strongly dependent on distance. Hence, this will be revealed within the present theoretical work. The general case will be solved and, moreover, the method will be verified by a numerical application.

Keywords: Nonlinear second order differential equation system, elliptic functions, 2N parametric vector space, double oscillating mechanical system, elastic forces field, magnetic forces field, controlled halts.

1. INTRODUCTION

The proposed mechanical system consists of two oscillating subsystems, which are mechanically coupled, like in the figure 1. The first subsystem will be the main one, consisting of one spring and a mass with a magnet inside, while the second one is a classical spring-mass subsystem. Both subsystems interact with each other through a connection spring.

The oscillations that occur in various situations are very interesting and will be described in the following. We will treat the ideal case, with no friction and in the assumption that there is no loss of energy during the magnetic and mechanic interactions. One of the most important characteristic is that the system starts with an applied initial impulse and kinetic energy is transformed periodically in potential energy.

The description of the mechanical system:

We will start from the figure below, which describe the proposed mechanical system, where are shown two mechanical subsystems being in nonlinear interaction as it follows:

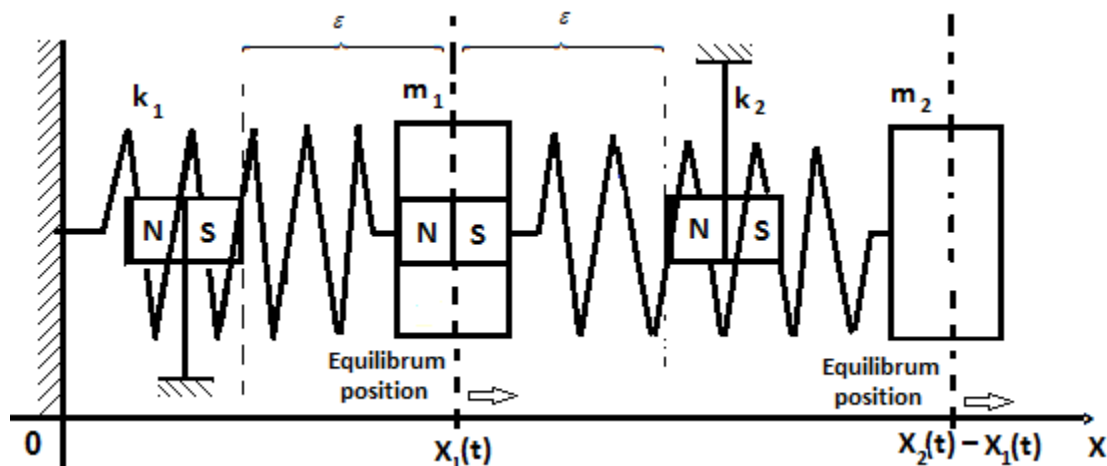
- The first subsystem, called the main subsystem, consists of mass-body m_1 and a resort that will introduce a linear elastic field. In the same time, the subsystem is placed in a magnetic field, nonlinear strong at the ends of the race, thus creating a nonlinear potential field. In the magnetic force description, it was used a simple monopole-monopole expression, for long bar magnets, described in [1].

- The second subsystem, called secondary subsystem, consists of mass-body m_2 and a resort that will produce oscillations under the influence of the potential field of its own resort, but also under the action of the inertial field induced by the first oscillating system.

In the mathematical description of the introduced mechanical system we are taking into account some equations from the physical model already described in the authors' earlier work [2], actually the current study represents a continuation of ideas presented in that article, at a more complex level.

We have to specify that during the oscillations, the magnetic field acts only on the ensemble formed by m_1 mass and its interior magnet. The permanent magnet which is inside the m_1 body, is oriented as it is shown and the entire mechanical system can move freely, without friction, in X axis direction.

Figure 1. Two coupled spring-mass subsystems, interacting in a nonlinear magnetic field



The constants of this mechanical system are:

k_1 – is the elastic constant of the first spring;

k_2 – is the elastic constant of the second spring;

k_3 – is the magnetic constant depending on strength of the permanent magnet;

m_1 – is mass of the magnet-mass body of the principal oscillator;

m_2 – is mass of the mass-body of the second oscillator;

ε - is a magnetic constant depending on the mechanical construction of the system.

From mechanical point of view, the mathematical differential equations what describe the system include the resultant of the static and dynamic forces, as in the following, and some constructive ideas were taken from scientific papers of [3] [4] [5].

The implied forces are:

F_{e1} – the elastic force of the first subsystem, given by the expression:

$$F_{e1} = -k_1 X_1(t) + k_2 [X_2(t) - X_1(t)] \quad (1)$$

F_{e2} – the elastic force of the second subsystem, with the expression:

$$F_{e2} = -k_2 [X_2(t) - X_1(t)] \quad (2)$$

F_{m_n} – the magnetic force acting on the first permanent magnet 1 with nonlinear expression, it was been also described in detail, in the work [2], and has the expressions:

$$F_{m_n}(X_1) = \frac{k_3}{[\varepsilon - X_1(t)]^2} - \frac{k_3}{[\varepsilon + X_1(t)]^2} \quad (3)$$

$F_{r1} = F_{e1} + F_{m_n}$ – the static forces resultant of the first subsystem:

$$F_{r1} = -k_1 X_1(t) + k_2 [X_2(t) - X_1(t)] + \frac{k_3}{[\varepsilon - X_1(t)]^2} - \frac{k_3}{[\varepsilon + X_1(t)]^2} \quad (4)$$

$F_{r2} = F_{e2}$ – the static forces resultant of the second subsystem:

$$F_{r2} = -k_2 [X_2(t) - X_1(t)] \quad (5)$$

F_{i1} – the inertial force acting on the first mass:

$$F_{i1} = m_1 \frac{d^2 X_1(t)}{dt^2} \quad (6)$$

F_{i2} – the inertial force acting on the second mass:

$$F_{i2} = m_2 \frac{d^2 X_2(t)}{dt^2} \quad (7)$$

2. Applied methods:

Given the forces described in the expressions above, we can say that the system of differential equations describing the mechanical system has the following structure:

$$\begin{cases} F_{i1} = F_{e1} + F_{m_n} \\ F_{i2} = F_{e2} \end{cases} \Leftrightarrow \begin{cases} m_1 \frac{d^2 X_1(t)}{dt^2} = -k_1 X_1(t) + k_2 [X_2(t) - X_1(t)] + \frac{k_3}{[\varepsilon - X_1(t)]^2} - \frac{k_3}{[\varepsilon + X_1(t)]^2} \\ m_2 \frac{d^2 X_2(t)}{dt^2} = -k_2 [X_2(t) - X_1(t)] \end{cases}$$

(8)

We have an homogenous, non-linear, of 2nd-order differential system, with coupled equations, difficult to solve through an analytical method. In order to solve these equations, the model proposed by authors, is described in the following:

We will rewrite the system in the form of a matrix system as it follows:

$$\begin{pmatrix} m_1 \frac{d^2 X_1(t)}{dt^2} \\ m_2 \frac{d^2 X_2(t)}{dt^2} \end{pmatrix} = \begin{pmatrix} -(k_1+k_2) & k_3 & k_2 \\ k_2 & 0 & -k_2 \end{pmatrix} \begin{pmatrix} X_1(t) \\ \frac{1}{[\varepsilon - X_1(t)]^2} - \frac{1}{[\varepsilon + X_1(t)]^2} \\ X_2(t) \end{pmatrix} \quad (9)$$

As the system is written in a 2x3 matrix form, with 2 rows and 3 columns and we want to make a 2N-parametric vector transformation, in the vector space $\mathbb{R}^3$, we will perform a mathematical artifice. We

consider a nonlinear mathematical function: $Fm_n(X_1) = \frac{d^2 G_n(X_1)}{dt^2}$, with the following form:

$$\frac{d^2 G_n(X_1)}{dt^2} = \frac{k_3}{[\varepsilon - X_1(t)]^2} - \frac{k_3}{[\varepsilon + X_1(t)]^2} \quad (10)$$

The system becomes:

$$\begin{pmatrix} m_1 \frac{d^2 X_1(t)}{dt^2} \\ \frac{d^2 G_n(X_1)}{dt^2} \\ m_2 \frac{d^2 X_2(t)}{dt^2} \end{pmatrix} = \begin{pmatrix} -(k_1+k_2) & k_3 & k_2 \\ 0 & k_3 & 0 \\ k_2 & 0 & -k_2 \end{pmatrix} \begin{pmatrix} X_1(t) \\ \frac{1}{[\varepsilon - X_1(t)]^2} - \frac{1}{[\varepsilon + X_1(t)]^2} \\ X_2(t) \end{pmatrix} \quad (11)$$

We divide the connexion matrix by k_2 , as it follows:

$$\begin{pmatrix} m_1 \frac{d^2 X_1(t)}{dt^2} \\ \frac{d^2 G_n(X_1)}{dt^2} \\ m_2 \frac{d^2 X_2(t)}{dt^2} \end{pmatrix} = k_2 \begin{pmatrix} -\frac{(k_1+k_2)}{k_2} & \frac{k_3}{k_2} & \frac{k_2}{k_2} \\ 0 & \frac{k_3}{k_2} & 0 \\ \frac{k_2}{k_2} & 0 & -\frac{k_2}{k_2} \end{pmatrix} \begin{pmatrix} X_1(t) \\ \frac{1}{[\varepsilon - X_1(t)]^2} - \frac{1}{[\varepsilon + X_1(t)]^2} \\ X_2(t) \end{pmatrix} \quad (12)$$

By adnotating $q = \frac{k_1}{k_2}$ and $n = \frac{k_3}{k_2}$, we obtain the next system written in E - orthogonal basis:

$$E: \begin{pmatrix} \frac{m_1}{k_2} \frac{d^2 X_1(t)}{dt^2} \\ \frac{1}{k_2} \frac{d^2 G_n(X_1)}{dt^2} \\ \frac{m_2}{k_2} \frac{d^2 X_2(t)}{dt^2} \end{pmatrix} = \begin{pmatrix} -(q+1) & n & 1 \\ 0 & n & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} X_1(t) \\ \frac{1}{[\varepsilon - X_1(t)]^2} - \frac{1}{[\varepsilon + X_1(t)]^2} \\ X_2(t) \end{pmatrix} \quad (13)$$

We are going to transform the system of differential equations in a quasi-diagonal form. This fact will help us to solve separate those three differential equations, in a special vector space, which will be found as in the follows.

This method, which we are going to apply, was already successfully applied in the earlier authors' work [6], where a simple 1N parametric vector transformation in R^2 vector space, was considered.

The problem that implies the usage of vector spaces in solving of systems of liner differential equations, was widely described in specialty scientific paper and also in an original manner in work [7].

We are seeking to transform matrix A, in a matrix B, following a vector transformation, using the associated matrix of a V-vector space together with his inverse, V^{-1} , in respect with condition:

$$\det(A) = \det(B) \quad (14)$$

We are searching to transform the matrix A:

$$A = \begin{pmatrix} -(q+1) & n & 1 \\ 0 & n & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad (15)$$

In a quasi diagonal form one:

$$B = \begin{pmatrix} \frac{\sqrt{q^2+4}-q-2}{2} & 0 & 0 \\ -q & n & 0 \\ 0 & 0 & \frac{-\sqrt{q^2+4}-q-2}{2} \end{pmatrix} \quad (16)$$

That allows us to decouple symbolic those three differential equations, and in compliance with the conditions above, as bellow:

$$\begin{cases} \det(A) = nq \\ \det(B) = \frac{n \left(\sqrt{q^2+4}-q-2 \right) \left(-\sqrt{q^2+4}-q-2 \right)}{4} = \frac{n \left[-q^2-4 - \left(q^2+4q+4 \right) \right]}{4} = nq \end{cases} \quad (17)$$

Looking for a three-dimensional V-basis, in R^3 vector space, the: (a,b,c,d,e,f,g,h,i) - independent parameters, will be found as it follows:

- We are searching to find a generalized vector transformation, regardless of the physical constants of the system, a matrix form, with nine unknown matrix characteristic in R^3 -vector space.
- In order to perform an advance unitary transformation, parametrically point of view, we will consider all the numerical possibilities, through the nine parameters taken into account, and this it was the reason of choosing that those nine parameters.

$$V = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad (18)$$

Calculating symbolically the inverse of this matrix, we find the following form:

$$V^{-1} = \begin{pmatrix} \frac{ei - fh}{aei - afh - dbi + dch + gb - gce} & -\frac{bi - ch}{aei - afh - dbi + dch + gb - gce} & -\frac{bf + ce}{aei - afh - dbi + dch + gb - gce} \\ -\frac{di - fg}{aei - afh - dbi + dch + gb - gce} & \frac{ai - cg}{aei - afh - dbi + dch + gb - gce} & -\frac{af - cd}{aei - afh - dbi + dch + gb - gce} \\ -\frac{-dh + eg}{aei - afh - dbi + dch + gb - gce} & -\frac{ah - bg}{aei - afh - dbi + dch + gb - gce} & \frac{ae - bd}{aei - afh - dbi + dch + gb - gce} \end{pmatrix} \quad (19)$$

where for any R^3 -vector space form, determined by physical constants variations, verifies the equality:

$$V^{-1}V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (20)$$

According to the theory of vector space transformation, used in linear algebra, we obtain:

$$V A V^{-1} = B \quad (21)$$

A matrix requires that data, to transform in a quasi-diagonal matrix B, using a parametric vector space, as it follows:

$$B = V A V^{-1} = \begin{pmatrix} \frac{\sqrt{q^2 + 4 - q - 2}}{2} & 0 & 0 \\ -q & n & 0 \\ 0 & 0 & \frac{-\sqrt{q^2 + 4 - q - 2}}{2} \end{pmatrix} = \begin{pmatrix} eq_{11} & eq_{12} & eq_{13} \\ eq_{21} & eq_{22} & eq_{23} \\ eq_{31} & eq_{32} & eq_{33} \end{pmatrix} \quad (22)$$

Based on the formula above which transforms a matrix according to a vector space and its inverse, we obtain after successive multiplications, next 9 equations with 9 unknowns:

$$eq_{11} = \frac{-aeiq - aei + afhq + afh + baf - cae + nedi - nfdh - ndbi + ndch + ieg - hfg - gb - gce}{aei - afh - dbi + dch + gb - gce} = \frac{\sqrt{q^2 + 4 - q - 2}}{2} \quad (23)$$

$$eq_{12} = \frac{-beiq - bei + bfhq + b^2 f - bce + ne^2 i - nefh - nebi + nech + hei - fh^2 + hce}{aei - afh - dbi + dch + gbf - gce} = 0 \quad (24)$$

$$eq_{13} = -\frac{ceiq - cfhq - cfh - cbf + c^2 e - nfei + nf^2 h + nfbi - nfch - ei^2 + ifh}{aei - afh - dbi + dch + gbf - gce} = 0 \quad (25)$$

$$eq_{21} = \frac{idqa + adi - afgq - a^2 f + acd - nd^2 i + ndfg + ndai - ndcg - gdi + fg^2 - gcd}{aei - afh - dbi + dch + gbf - gce} = -q \quad (26)$$

$$eq_{22} = \frac{dbiq + dbi - bfgq + gbf + baf - cbd - nedi + nfeg + neai - necg + idh - gfg - afh - dch}{aei - afh - dbi + dch + gbf - gce} = n \quad (27)$$

$$eq_{23} = \frac{cidq - cfgq - cfg - caf + c^2 d - nfdi + nf^2 g + nfai - nfcg - di^2 + ifg + iaf}{aei - afh - dbi + dch + gbf - gce} = 0 \quad (28)$$

$$eq_{31} = \frac{-adhq - adh + aegq + a^2 e - abd + nd^2 h - dneg - ndah + ndbg + gdh - eg^2 + gdb}{aei - afh - dbi + dch + gbf - gce} = 0 \quad (29)$$

$$eq_{32} = \frac{-bdhq + begq + beg + bae - b^2 d + nedh - ne^2 g - neah + nebg + dh^2 - heg - hae}{aei - afh - dbi + dch + gbf - gce} = 0 \quad (30)$$

$$eq_{33} = \frac{-cdhq - dch + cegq + gce + cae - cbd + nfdh - nfeg - nfah + nfbg + idh - ieg - aei + dbi}{aei - afh - dbi + dch + gbf - gce} = \frac{-\sqrt{q^2 + 4} - q - 2}{2} \quad (31)$$

We form the next system:

$$S = \{eq_{11}, eq_{12}, eq_{13}, eq_{21}, eq_{22}, eq_{23}, eq_{31}, eq_{32}, eq_{33}\} \quad (32)$$

After solving the nine equations system, with nine unknowns, depending on two parameters (q, n), we obtain solutions for those 9 unknowns as follows:

$$S = \left\{ \begin{array}{ccc} a = -q & b = n+1 & c = \frac{q + \sqrt{q^2 + 4}}{2} \\ d = \frac{2n+2+q+q\sqrt{q^2+4}}{2} & e = \frac{n^2+2n+nq+q}{2} & f = 0 \\ g = 0 & h = 1 & i = -1 \end{array} \right\} \quad (33)$$

It follows that the three-dimensional vector space helped us perform the transformation is V:

$$V = \left(\begin{array}{ccc} -q & n+1 & \frac{q + \sqrt{q^2 + 4}}{2} \\ \frac{2n+2+q+q\sqrt{q^2+4}}{2} & \frac{n^2+2n+nq+q}{2} & 0 \\ 0 & 1 & -1 \end{array} \right) \quad (34)$$

- The associated matrix of V -vector space thus found has a parametric form, depending on a initial given constants, and it can turn the connections matrix A, into a connections matrix B, in any situation, regardless of values of the physical constants (K_1, K_2, K_3), with only condition that the V – matrix to be non singular ones.
- V and his inverse V^{-1} , are parametric type and it is possible to transform matrix A in matrix B, in the general case, regardless of the physical constants of the system.

With matrix B, this 2nd order differential system becomes:

$$V: \left(\begin{array}{c} \frac{m_1}{k_2} \frac{d^2 X_1(t)}{dt^2} \\ \frac{1}{k_2} \frac{d^2 G_n(X_1)}{dt^2} \\ \frac{m_2}{k_2} \frac{d^2 X_2(t)}{dt^2} \end{array} \right) = \left(\begin{array}{ccc} \frac{\sqrt{q^2+4}-q-2}{2} & 0 & 0 \\ -q & n & 0 \\ 0 & 0 & \frac{-\sqrt{q^2+4}-q-2}{2} \end{array} \right) \left(\begin{array}{c} X_1(t) \\ \frac{1}{[\varepsilon - X_1(t)]^2} - \frac{1}{[\varepsilon + X_1(t)]^2} \\ X_2(t) \end{array} \right) \quad (35)$$

Which involves solving a system of a 3 - differential equations in this decoupled form:

$$\left\{ \begin{array}{l} \frac{m_1}{k_2} \frac{d^2 X_1(t)}{dt^2} = \frac{\sqrt{q^2+4}-q-2}{2} X_1(t) \\ \frac{1}{k_2} \frac{d^2 G_n(X_1)}{dt^2} = -qX_1(t) + \frac{n}{[\varepsilon - X_1(t)]^2} - \frac{n}{[\varepsilon + X_1(t)]^2} \\ \frac{m_2}{k_2} \frac{d^2 X_2(t)}{dt^2} = \frac{-\sqrt{q^2+4}-q-2}{2} X_2(t) \end{array} \right. \quad (36)$$

These three differential equations will be separately solved, as it follows:

a) The first 2nd order linear equation, has the form:

$$\frac{m_1}{k_2} \frac{d^2 X_1(t)}{dt^2} - \frac{\sqrt{q^2 + 4} - q - 2}{2} X_1(t) = 0 \quad (37)$$

We denote its own pulsation of the first oscillator:

$$\omega_1^2 = -\frac{k_2}{m_1} \frac{\sqrt{q^2 + 4} - q - 2}{2} \quad (38)$$

and we will obtain the equation:

$$\frac{d^2 X_1(t)}{dt^2} + \omega_1^2 = 0 \quad (39)$$

By solving the characteristic equation,

$$r^2 + \omega_1^2 = 0 \Leftrightarrow r = \pm i\omega_1 \quad (40)$$

the differential equation becomes:

$$X_1(t) = \zeta_1 e^{i\omega_1 t} + \zeta_2 e^{-i\omega_1 t} \quad (41)$$

After transformation, using Euler's formulas, we arrive at:

$$X_1(t) = x_{01} \sin(\omega_1 t + \varphi) \quad (42)$$

b) The second nonlinear differential equation, in this uncoupled form, will have the following resolution,

based on the authors' earlier work [2], where the motion of a single spring-mass-magnet in magnetic field, was considered, and were analytical solution of elliptic sine form functions, was founded.

In the description of the Jacobi elliptical functions, involved of this problem resolution, were taken into account the dedicated scientific paper of [8] and [9].

With the purpose to find the form for the analytic solution of differential equation, we have to proceed as it is shown bellow. Equation solving will be summarized in the following:

To find the form for the differential equation analytic solution, we have to proceed as it is shown bellow. Equation solving will be summarized in the following:

$$\frac{d^2 G_n(X_1)}{dt^2} = -qk_2 X_1(t) + \frac{n}{[\varepsilon - X_1(t)]^2} - \frac{n}{[\varepsilon + X_1(t)]^2} \quad (43)$$

-We calculate first the x_{1p} and p parameters from the roots of the right member of upper differential equation:

$$x_{1p} = \text{Root} \left\{ -qX_1 + \frac{n}{[\varepsilon - X_1]^2} - \frac{n}{[\varepsilon + X_1]^2} = 0 \right\} \quad (44)$$

and

$$p = \text{Root} \left[-(1+p^2 k^2) x_{1p} + 2p^2 k^2 x_{1p}^3 = 0 \right] \quad (45)$$

After this we get the eccentricity argument of elliptic function as follows:

$$k(p, x_0) = \frac{1}{p\sqrt{2x_0^2 - 1}} \quad (46)$$

After it we can calculate the time argument of elliptic function in this mode:

-We find a special prototype function - F_p , starting from our dedicated form of the right member as it follows:

$$F_p(u, v, p, k, x_0) = u \left[-(1 + p^2 k^2) x_0 \right] + v \left(2 p^2 k^2 x_0^3 \right) \quad (47)$$

For finding (u, v) - constants, the next system of integral equations is implied, in order to solve our problem:

$$\begin{cases} F_p = F_r \Leftrightarrow -u(1 + p^2 k^2) x_p + 2vp^2 k^2 x_p^3 = q x_{1p} + \frac{k_3}{[\varepsilon - x_{1p}]^2} - \frac{k_3}{[\varepsilon + x_{1p}]^2} \\ \int_0^{x_p} F_p dX = \int_0^{x_p} F_r dX \end{cases} \quad (48)$$

With specification that the first equation is solved for: X_p, p, k - being constants, and the second one is integrated for variable X , and for constants: p, k .

After the above system is solved to find (u, v) - constants, we have to identify the coefficients of the expression, provided from the right member of (43):

$$-x_0 \alpha^2 X + 2x_0 \alpha^2 k^2 X^3 - x_0 \alpha^2 k^2 X = -u(1 + p^2 k^3) X + 2vp^2 k^2 X^3 \quad (49)$$

Starting from the system:

$$\begin{cases} -x_0 \alpha^2 - x_0 \alpha^2 k^2 = -u(1 + p^2 k^3) \\ 2x_0 \alpha^2 k^2 = 2vp^2 k^2 \end{cases} \quad (50)$$

where after solving it for α , we will be able to find the final form of the function.

$$G_n(X_1(t)) = x_{01} \operatorname{sn} \left[\frac{\zeta_3}{\sqrt{x_{01}}} t, \frac{\zeta_4}{\sqrt{2x_{01}^2 - 1}} \right] \quad (51)$$

In the upper formulation problem, were taken into account some constructive ideas of the authors' work of [10], [11], [12]. The problem resolution consists of finding an analitical general formula to get the interaction between linear and nonlinear subsystems.

c) The third one is a second order linear equation, of the form:

$$\frac{m_2}{k_2} \frac{d^2 X_2(t)}{dt^2} - \frac{-\sqrt{q^2 + 4} - q - 2}{2} X_2(t) = 0 \quad (52)$$

Noting its own pulsation of the first oscillator :

$$\omega_2^2 = -\frac{k_2}{m_2} \frac{-\sqrt{q^2 + 4} - q - 2}{2} \quad (53)$$

we will have:

$$\frac{d^2 X_2(t)}{dt^2} + \omega_2^2 = 0 \quad (54)$$

By solving the characteristic equation:

$$r^2 + \omega_2^2 = 0 \Leftrightarrow r = \pm i\omega_2 \quad (55)$$

The differential equation becomes:

$$X_2(t) = \zeta_5 e^{i\omega_2 t} + \zeta_6 e^{-i\omega_2 t} \quad (56)$$

After transformation, using Euler formulas, we get:

$$X_2(t) = x_{02} \sin(\omega_2 t + \varphi) \quad (57)$$

So that we have the next system solutions following, written in V – basis :

$$V : \begin{cases} X_1(t) = x_{01} \sin(\omega_1 t + \varphi) \\ G_n(X_1(t)) = x_{01} \operatorname{sn} \left[\frac{\zeta_3}{\sqrt{x_{01}}} t, \frac{\zeta_4}{\sqrt{2x_{01}^2 - 1}} \right] \\ X_2(t) = x_{02} \sin(\omega_2 t + \varphi) \end{cases} \quad (58)$$

Finding the final form of those two solutions involves a basic active transformation:

$$E : \begin{pmatrix} X_1(t) \\ G_n(X_1) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} -q & n+1 & \frac{q+\sqrt{q^2+4}}{2} \\ \frac{2n+2+q+\sqrt{q^2+4}}{2} & \frac{n^2+2n+nq+q}{2} & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_{01} \sin(\omega_1 t + \varphi) \\ x_{01} \operatorname{sn} \left[\frac{\zeta_3}{\sqrt{x_{01}}} t, \frac{\zeta_4}{\sqrt{2x_{01}^2 - 1}} \right] \\ x_{02} \sin(\omega_2 t + \varphi) \end{pmatrix} \quad (59)$$

And finally we find the general solution in E space, as it follows. We use the transformation method of differential equations which was described also in work [7]

- We will return to the basic vector space for finding the general solution of this problem in the initial conditions.
- After inverse transformation, the unknown function: $G_n(X_1)$, will have no effect and will not be written.

$$\left\{ \begin{array}{l} X_1(t) = -q x_{01} \sin(\omega_1 t + \varphi) + (n+1)x_{01} \operatorname{sn} \left[\frac{\zeta_3}{\sqrt{x_{01}}} t, \frac{\zeta_4}{\sqrt{2x_{01}^2 - 1}} \right] + \frac{q + \sqrt{q^2 + 4}}{2} x_{02} \sin(\omega_2 t + \varphi) \\ X_2(t) = x_{01} \operatorname{sn} \left[\frac{\zeta_3}{\sqrt{x_{01}}} t, \frac{\zeta_4}{\sqrt{2x_{01}^2 - 1}} \right] - x_{02} \sin(\omega_2 t + \varphi) \end{array} \right. \quad (60)$$

3. A SPECIFIC NUMERICAL APPLICATION:

With the purpose of applying into practice the analytical solution presented above, we take a numerical application, such that the double nonlinear oscillator has the described behavior.

We start to give numerical values of physical constants, to eliminate the parameters considered in the beginning, and to highlight the existence of the functional halts.

Although there are many other possible physical values of the constants, we have selected the next set of values, because the obtained effect is similar to the practical experimental situation, from which we have started, and we developed the theoretical study.

$$\text{Numerical Values} = \left\{ \begin{array}{l} \varepsilon = 5mm \\ m_1 = 1.265g \\ m_2 = 11.81g \\ k_1 = 0.585mN / mm \\ k_2 = 1mN / mm \\ k_3 = 1mN / mm \end{array} \right. \quad (61)$$

We note with:

$$q = \frac{k_1}{k_2} = 0.585 \quad (62)$$

and:

$$n = \frac{k_3}{k_2} = 1 \quad (63)$$

The system of nonlinear second order differential equations becomes:

$$\left\{ \begin{array}{l} 1.265 \frac{d^2 X_1(t)}{dt^2} = -0.585 X_1(t) + [X_2(t) - X_1(t)] + \frac{1}{[5 - X_1(t)]^2} - \frac{1}{[5 + X_1(t)]^2} \\ \frac{d^2 G_n(X_1)}{dt^2} = \frac{1}{[5 - X_1(t)]^2} - \frac{1}{[5 + X_1(t)]^2} \\ 11.81 \frac{d^2 X_2(t)}{dt^2} = -[X_2(t) - X_1(t)] \end{array} \right. \quad (64)$$

Matrix B will take the form:

$$B = \begin{pmatrix} -0.25 & 0 & 0 \\ -0.585 & 1 & 0 \\ 0 & 0 & -2.334 \end{pmatrix} \quad (65)$$

The vector space that decouples the solutions of our system will be:

$$V = \begin{pmatrix} -0.585 & 2 & 1.334 \\ 2.9 & 2.085 & 0 \\ 0 & 1 & -1 \end{pmatrix} \quad (66)$$

So we obtain the system in its decoupled form:

$$V : \begin{cases} 1.265 \frac{d^2 X_1(t)}{dt^2} = -0.585 X_1(t) + [X_2(t) - X_1(t)] + \frac{1}{[5 - X_1(t)]^2} - \frac{1}{[5 + X_1(t)]^2} \\ \frac{d^2 G_n(X_1)}{dt^2} = \frac{1}{[5 - X_1(t)]^2} - \frac{1}{[5 + X_1(t)]^2} \\ 11.8 \frac{d^2 X_2(t)}{dt^2} = -[X_2(t) - X_1(t)] \end{cases} \quad (67)$$

Our system is rewritten in a quasi-diagonal form, in the vector space thus found:

$$\begin{pmatrix} 1.265 \frac{d^2 X_1(t)}{dt^2} \\ \frac{d^2 G_n(X_1)}{dt^2} \\ 11.8 \frac{d^2 X_2(t)}{dt^2} \end{pmatrix} = \begin{pmatrix} -0.25 & 0 & 0 \\ -0.585 & 1 & 0 \\ 0 & 0 & -2.334 \end{pmatrix} \begin{pmatrix} X_1(t) \\ \frac{1}{[5 - X_1(t)]^2} - \frac{1}{[5 + X_1(t)]^2} \\ X_2(t) \end{pmatrix} \quad (68)$$

This implies solving three isolated differential equations as it follows:

$$\begin{cases} a) 1.265 \frac{d^2 X_1(t)}{dt^2} = -0.25 X_1(t) \\ b) \frac{d^2 G_n(X_1)}{dt^2} = -0.585 X_1(t) + \frac{1}{[5 - X_1(t)]^2} - \frac{1}{[5 + X_1(t)]^2} \\ c) 11.8 \frac{d^2 X_2(t)}{dt^2} = -2.334 X_2(t) \end{cases} \quad (69)$$

We solve separately these equations.

a) The first 2nd order linear differential equation will be solved so:

$$1.265 \frac{d^2 X_1(t)}{dt^2} = -0.25 X_1(t) \quad (70)$$

Its attached algebraic equation has the following form:

$$r^2 + 0.197 = 0 \quad (71)$$

With solutions :

$$r = \pm 0.444I \quad (72)$$

The general solution will be of the form:

$$x_1(t) = \zeta_1 e^{0.444 It} + \zeta_2 e^{-0.444 It} \quad (73)$$

Where after Euler fundamental formula development and constants notation, we get a known sinusoidal form:

$$X_1(t) = x_{01} \sin\left(\frac{5\sqrt{506}}{253} t\right) = x_{01} \sin(0.44455 t) \quad (74)$$

b) The non-linear differential equation with elliptical solution is:

$$\frac{d^2 G_n(X_1)}{dt^2} = -0.585 X_1(t) + \frac{1}{[5-X_1(t)]^2} - \frac{1}{[5+X_1(t)]^2} \quad (75)$$

$$F_r = \left[-0.585 X_1(t) + \frac{1}{[5-X_1(t)]^2} - \frac{1}{[5+X_1(t)]^2} \right] \quad (76)$$

We start to find the constant of the system, as it is written bellow:

$$p = \text{Root}\left[-(1+p^2)(4.376)+2p^2(4.376)^3=0\right] = 0.1637 \quad (77)$$

Which is solved for: x_p and k , as following:

$$x_p = \text{Root}\left[-0.585 X_1(t) + \frac{1}{[5-X_1(t)]^2} - \frac{1}{[5+X_1(t)]^2}\right] = (-5.554, -4.376, 0, 4.376, 5.554) \quad (78)$$

and

$$k = k_{\max} = 1 \quad (79)$$

From the expression :

$$2p^2 k^2 x_0^3 - (1 + p^2 k^2) x_0 = 0 \quad (80)$$

we get:

$$k(p, x_0) = \frac{6.107}{\sqrt{2x_0^2 - 1}} \quad (81)$$

Then we take off the time argument of the elliptic function, first finding a prototype function and thus to correlate our function with the new function, by constants u and v, as it follows:

$$F_p(u, v, p, k, x_0) = u \left[- (1 + p^2 k^2) x_0 \right] + v \left(2 p^2 k^2 x_0^3 \right) \quad (82)$$

$$\left\{ \begin{array}{l} F_p = F_r \Leftrightarrow u \left[- (1 + p^2 k^2) x_0 \right] + v \left(2 p^2 k^2 x_0^3 \right) = -0.585 X_1(t) + \frac{1}{[5 - X_1(t)]^2} - \frac{1}{[5 + X_1(t)]^2} \\ \int_0^x F_p dX = \int_0^x F_r dX \end{array} \right. \quad (83)$$

The given function what describes the mechanical system is:

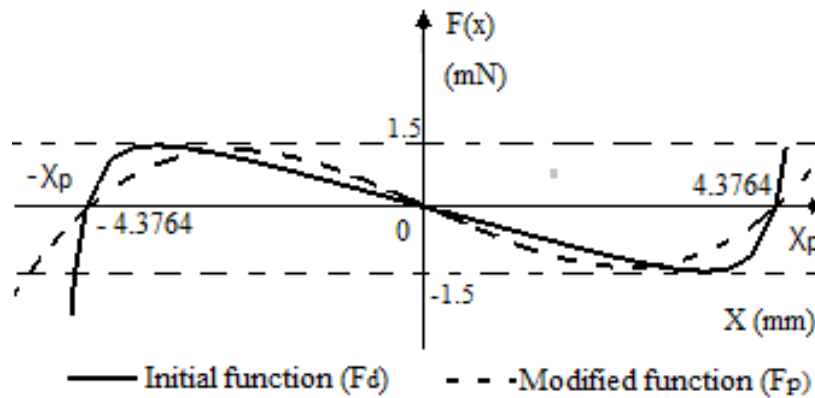
$$F_d = -0.585 X_1(t) + \frac{1}{[5 - X_1(t)]^2} - \frac{1}{[5 + X_1(t)]^2} \quad (84)$$

The found prototype function will be:

$$F_p = -0.89635789X + 0.04679999X^3 \quad (85)$$

The both functions have the same roots, and comparison graph of the two functions is shown bellow.

Figure 2. The graph of (F_p , F_d) functions, with the same roots and the same intergraphic area.



By identifying the polynomial coefficients of the found polynomial with the polynomial resulted from the 2nd order derivation of the proposed formal function, we get:

$$\frac{d^2 X(t)}{dt^2} = -x_0 \alpha^2 X(t) + 2x_0 \alpha^2 k^2 X^3(t) - x_0 \alpha^2 k^2 X(t)$$

The system of elliptic sine derivative type, becomes:

$$\begin{cases} -x_0 \alpha^2 - x_0 \alpha^2 k^2 = -0.896 \\ 2x_0 \alpha^2 k^2 = 0.0467 \end{cases} \quad (87)$$

Thus we find the expression:

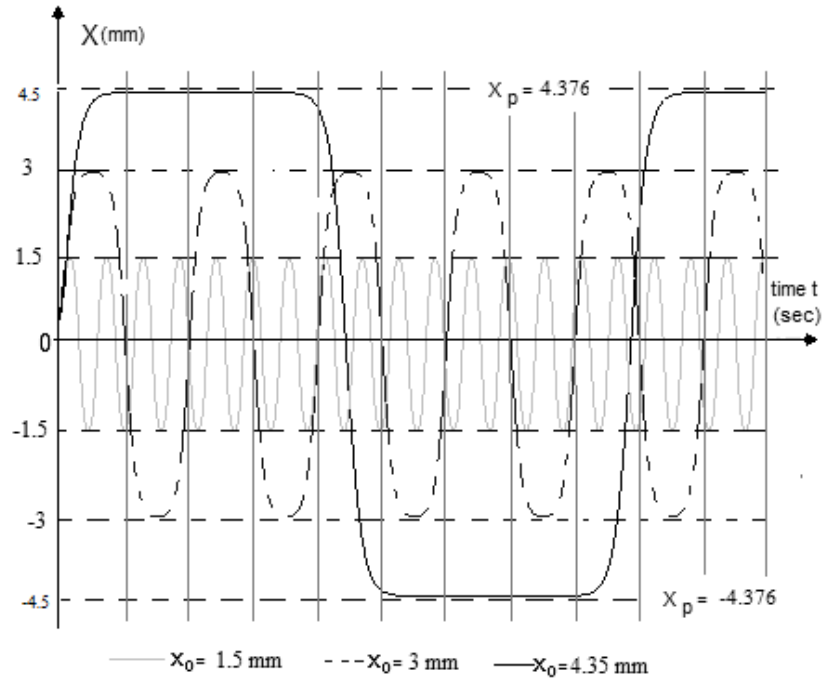
$$\alpha = \frac{0.93432}{\sqrt{x_0}} \quad (88)$$

Finally we obtain the function:

$$G_n(X_1) = x_{01} \operatorname{sn} \left[\frac{0.93432}{\sqrt{x_0}} t, \frac{6.107}{\sqrt{2x_0^2 - 1}} \right] \quad (89)$$

which is accompanied by the multi- graph as a function of t and initial position :

Figure 3. Graph of G_n function, depending on starting position



c) The third 2nd order linear differential equation will be solved as it follows:

$$11.81 \frac{d^2 X_2(t)}{dt^2} = -2.334 X_2(t) \quad (90)$$

The attached algebraic equation is the following:

$$r^2 + 0.19762 = 0 \quad (91)$$

with solutions:

$$r = \pm 0.444I \quad (92)$$

The general solution will be of the form:

$$X_1(t) = \zeta_1 e^{0.444It} + \zeta_2 e^{-0.444It} \quad (93)$$

After Euler's fundamental formula development and after we rewrite the constants expressions, we arrive at a known sinusoidal form:

$$X_2(t) = x_{02} \sin\left(\frac{\sqrt{68853}}{590} t\right) = x_{02} \sin(0.44455 t) \quad (94)$$

CONDITIONS:

In order to produce mechanical oscillations with “controlled halts”, as their existence has been experimentally demonstrated, for the first time, by the authors and presented in figure 9, we have chosed the physical constants of the system, so that those two components in trigonometric sine, have the same frequency of oscillation. The component that oscillates after elliptic sine law, will have a variable frequency, which depends on the strength of the eccentricity of the nonlinear component. It can be adjusted in X_{01} , initial start position, so there is an equal condition of oscillation frequencies.

The system naturally tends to self- synchronize, becoming stronger or weaker in eccentricity to "hold" in frequency.

The oscillatory component that obeys the law in elliptical sine, will have a variable frequency, which depends on deeper or shallow entry in eccentricity.

In this way, the system will always tends to self-regulate in a naturally way, the main oscillator will periodically enter in standby (stops), synchronizing with the frequency of the second oscillator, which requires in fact the repetition frequency of this process.

The novelty is that, although the two coupled oscillators would oscillate with individual frequency, producing a beats phenomenon of oscillations, in our case, in order to get controlled halts (functioning halts) it is indicated to produce a difference between the individual oscillations frequencies (if it would not be the nonlinear magnetic field), by choosing the constants.

Given those specified above , the phenomenon can be simulated for the next value x_{01}

$$\alpha = \frac{0.93432}{\sqrt{x_0}} = 0.446 \Leftrightarrow x_{01} = 4.3734$$

(95)

So the start position for the main oscillator will have the value above.

We found a system with three decoupled solutions, explained as it follows:

$$V : \begin{cases} X_1(t) = x_{02} \sin(0.44 t + \varphi) \\ G_n(X(t)) = x_{01} \operatorname{sn} \left[\frac{0.93432}{\sqrt{x_{01}}} t, \frac{6.107}{\sqrt{2x_{10}^2 - 1}} \right] \\ X_2(t) = x_{03} \sin(0.44 t) \end{cases} \quad (96)$$

To reach the basic vector space, we perform multiplications. We will introduce them in a linear combination, using inverse vector transformation as it follows, in order to obtain the final solution, which we have searched.

We will have to find the final solutions, as it is shown below:

$$E : \begin{pmatrix} X_1(t) \\ G_n(X(t)) \\ X_1(t) \end{pmatrix} = \begin{pmatrix} -0.585 & 2 & 1.334 \\ 2.9 & 2.085 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_{01} \sin(0.444 t + \varphi) \\ x_{01} \operatorname{sn} \left[\frac{0.93432}{\sqrt{x_0}} t, \frac{6.107}{\sqrt{2x_0^2 - 1}} \right] \\ x_{02} \sin(0.444 t + \varphi) \end{pmatrix} \quad (97)$$

- We will return to the basic vector space for finding the general solution of this problem in the initial conditions.
- After inverse transformation, the unknown function: $G_n(X_1)$, will have no effect and will not be written.

$$\begin{cases} X_1(t) = -0.585x_{01} \sin(0.444 t) + 2x_{01} \operatorname{sn} \left[\frac{0.93432}{\sqrt{x_0}} t, \frac{6.107}{\sqrt{2x_0^2 - 1}} \right] + 1.334x_{02} \sin(0.444 t) \\ X_2(t) = x_{01} \operatorname{sn} \left[\frac{0.93432}{\sqrt{x_0}} t, \frac{6.107}{\sqrt{2x_0^2 - 1}} \right] - x_{02} \sin(0.444 t) \end{cases} \quad (98)$$

.....

We associate the graphic representations in the following.

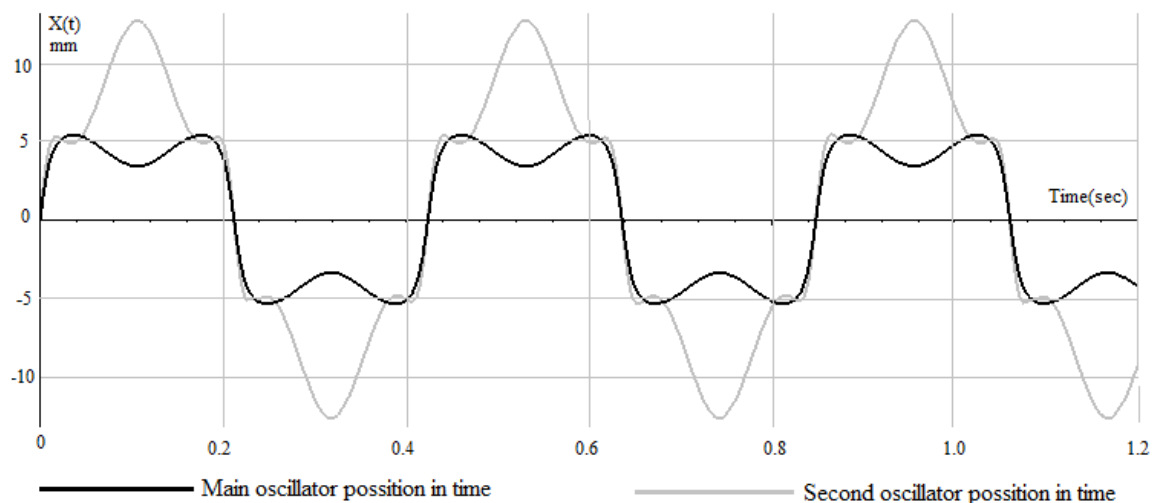
The previous formula has the following graph, which includes the corresponding values of some initial perturbations applied to the two oscillators. It was considered that these two oscillators were trained in motion at time $t = 0$, with zero initial speed and these are the frontier conditions.

a) For

$$\begin{cases} x_{01} = 4.3732 \text{ mm} \\ x_{02} = -1.0 \text{ mm} \end{cases} \quad (99)$$

with the graph below:

Figure 4. Graph for the given physical system and starting conditions.



In the figure above are graphically represented the motion equations (98), as a function of time, for the initial conditions (99). The grey line represents the law of motion for the second oscillator and the black line represents the law of motion of the main oscillator.

The grey line represents the law of motion for the second oscillator and the black line represents the motion law of the main oscillator.

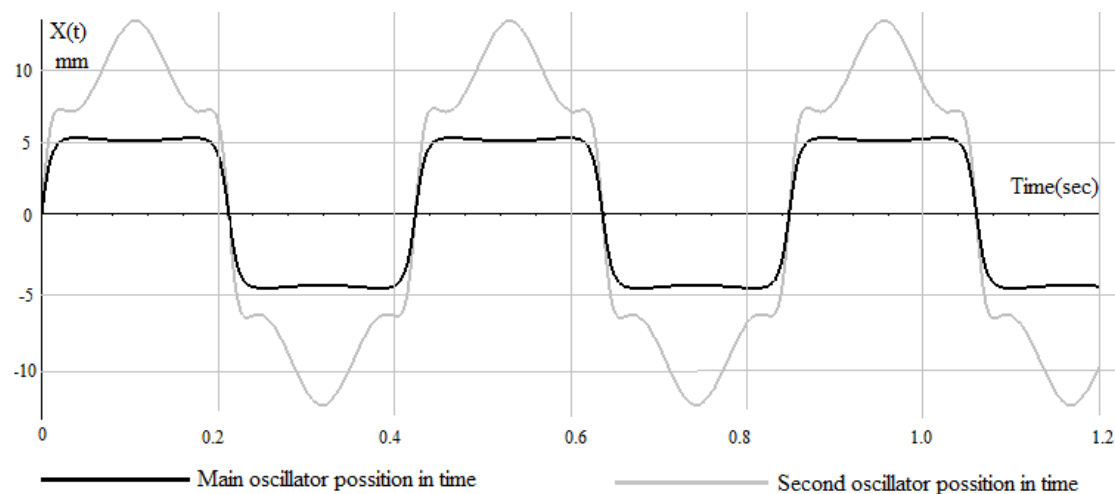
The grey line represents the motion law for the second oscillator and the black line represents the motion law of the main oscillator.

b) For:

$$\begin{cases} x_{01}=4.3732 \text{ mm} \\ x_{02}=-0.1 \text{ mm} \end{cases} \quad (100)$$

If we maintain the initial starting position of the main oscillator, we modify only the starting positions of the secondary oscillator. If the value of the starting positions for the second oscillator is 10 times lower than the previous example, we obtain the changing of the graphic concavity, as it follows in the next graph.

Figure 5. Graph for given physical system and starting conditions.

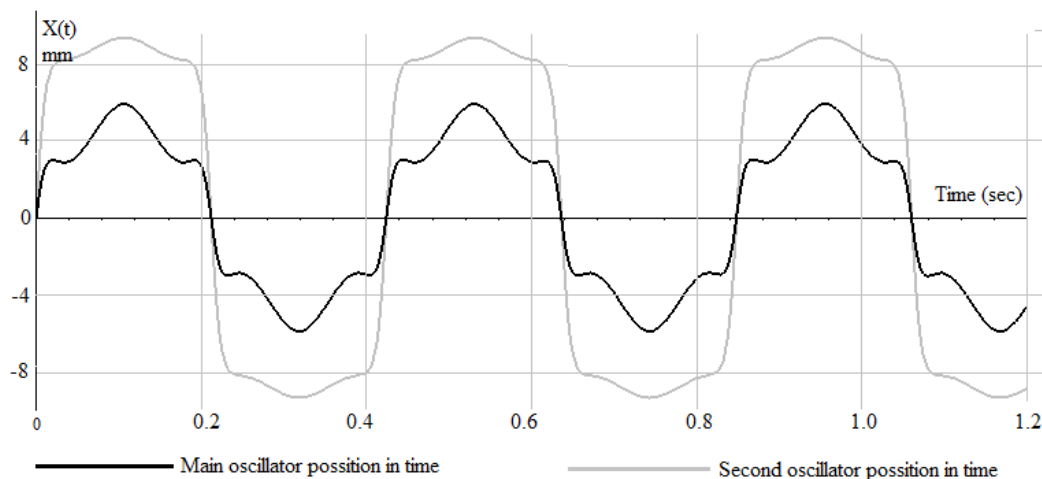


c) For:

$$\begin{cases} x_{01} = 4.373 \text{ mm} \\ x_{02} = 1.5 \text{ mm} \end{cases} \quad (101)$$

If the starting position x_{02} , is increased, we obtain the next graph.

Figure 6. Graph for given physical system and starting condition of item c).

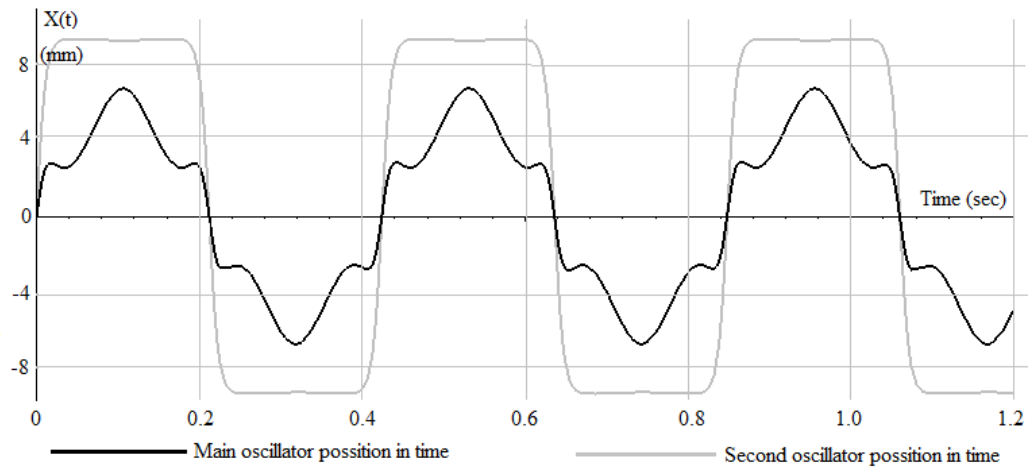


d) For:

$$\begin{cases} x_{01} = 4.373 \text{ mm} \\ x_{02} = 1.95 \text{ mm} \end{cases} \quad (102)$$

If the same value of x_{02} is increasing as it overpasses a certain limit, then we can notice that the subsystems change their roles, as is shown in the next figure.

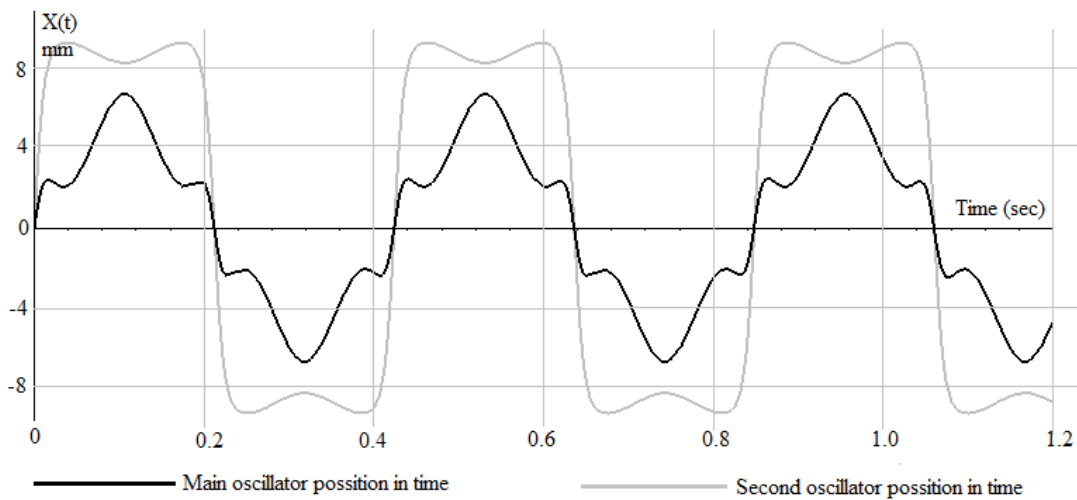
Figure 7. Graph for given physical system and starting condition of item d).



e) For:

$$\begin{cases} x_{01} = 4.373 \text{ mm} \\ x_{02} = 2.3 \text{ mm} \end{cases} \quad (103)$$

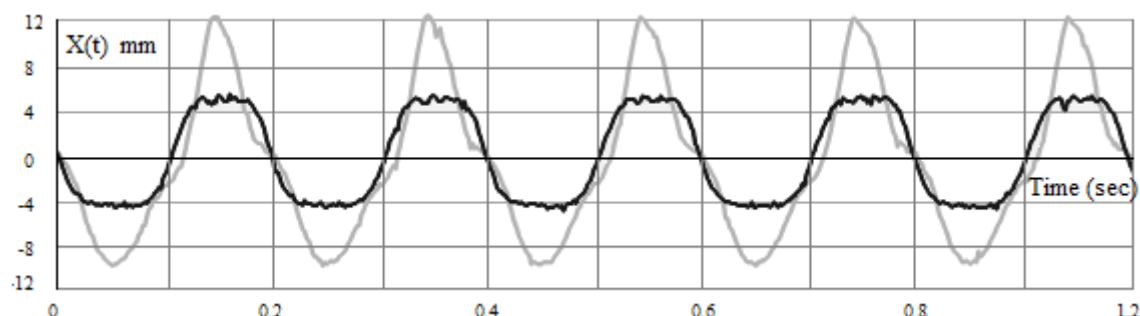
Figure 8. Graph for given physical system and starting condition.



f) The theoretical study presented above is confirmed by the experimental results. In this way we can associate the following graphic representation, illustrating the mechanical system behavior,

which is described in this theoretical work, based on the numerical acquisition of data, and their further processing.

Figure 9. Graph for experimental results



4. CONCLUSIONS:

- We firstly mention that in the mechanical system presented in this work, those two oscillating subsystems do not produce beats phenomenon during the oscillations. Surprisingly, although both subsystems do not have the same individual frequency of oscillation, they synchronize and oscillate in phase, expecting each other, and we find a composition of the oscillations, in subtractive and additive way.
- In terms of application, during the motion laws simulation, we have noticed that this system has an optimum working, in the conditions when m_2 is at least double of m_1 . The oscillations, which are presented in this work, may be also obtained, in an artificial way, in mathematical terms, by composing the elliptic sine and trigonometric sine functions, in certain circumstances
- The halt time of main oscillator depends on the physical constants of the system, especially by the mass difference between the first and the second subsystem.
- Those two subsystems could have different oscillations frequencies and in that situation the beats of oscillations could appear, but the component which oscillates after an elliptic sine law, will have a variable frequency, depending on the strength of the eccentricity in the nonlinear field.
- The both subsystems naturally tends to self- synchronize, the nonlinear field of main oscillator becoming stronger or weaker in elliptic eccentricity, in order to according its frequency with the second.
- In fact the mass difference between these two coupled oscillating subsystems, is offset by a more profound or superficial entrance in a nonlinear field, because the subsystems 1 and 2 naturally tend to come into play and to expect each other. "Controlled halts" are produced without mechanical collision, from distance, by transferring mechanical momentum, through the common spring. We regularly transfer momentum, through spring connection, because during shutdown/startup, the secondary oscillator is accelerated, respectively decelerated.
- The motion equations include three components, two trigonometric equations and one Elliptical equation. It may be said that these allow the description of several other types of movements, not

yet shown from experimental point of view, but they were highlighted by the initially conditions (start) of the mechanical subsystems, and they were shown by the graphics presented.

- We specify that this phenomenon, which is discovered and presented in this study, was used in the construction of a new type of mechanical engine, which is based by controlled halt of movement.

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