

Original Research Article

On Rings where Certain Subsets are Multiplicatively Generated by Idempotents

Abstract

In a Boolean ring, every element is trivially multiplicatively generated by idempotent elements. In this paper, we study the structure of rings in which certain subsets are multiplicatively generated by idempotents or multiplicatively generated by idempotents and nilpotents.

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Introduction

In a Boolean ring every element is trivially a product of idempotents. This naturally motivates the study of the structure of a ring, which as a semi-group, is generated by its idempotents. That is, rings which are multiplicatively generated by idempotents. This also motivated the study of rings which are multiplicatively generated by idempotents and nilpotents in [1]. Theorems A and B below were proved by Putcha and Yaqub (see[1]), and Theorem C below was proved by Abu-Khuzam in [1].

Theorem A: Let R be a ring which is multiplicatively generated by idempotents. If R has an identity element, then R is Boolean.

Theorem B: Let R be a ring which is multiplicatively generated by idempotents. If R is finite, then R is Boolean.

Theorem C: Let R be a ring which is multiplicatively generated by idempotents and nilpotents such that the set N of nilpotent elements is commutative, then N forms an ideal of R and R/N is Boolean.

Main Results

Throughout, R is a ring, N is the set of nilpotents, C is the center, J is the Jacobson radical of R , $C(R)$ is the commutator ideal of R , and Z is the ring of integers. As usual $[x,y]$ will denote the commutator $xy-yx$.

49 Theorem 1. Let R be a finite ring such that
 50 (1) The set $R \setminus N$ is multiplicatively generated by idempotents.
 51 Then N is an ideal of R and R/N is Boolean.

52
 53 The following Lemmas are needed for the proof of Theorem 1 .
 54

55 Lemma 1. Any homomorphic image of a ring satisfying (1) must satisfy (1).
 56

57 Proof. Let $\phi: R \rightarrow R'$ be a surjective ring homomorphism. Let $x' = \phi(x)$ be any
 58 element of R' . Using (1),

59 $x \in N$ or $x = e_1 e_2 \dots e_n$, where e_i is idempotent in R . Thus,

60 $x^n = 0$ or $x = e_1 e_2 \dots e_n$. So,

61 $x'^n = \phi(x^n) = 0$ or $x' = \phi(x) = \phi(e_1)\phi(e_2)\dots\phi(e_n)$.

62 Clearly, $\phi(e_i)^2 = \phi(e_i^2) = \phi(e_i)$. So R' satisfies (1).
 63

64 The following Lemma is well known:
 65

66 Lemma 2. Let R be a ring. If the commutator ideal $C(R)$ of R is nil, then the set N of
 67 nilpotents is an ideal of R .
 68

69 Lemma 3. Let R be a ring with identity such that R satisfies (1). Then R is Boolean.
 70

71 Proof:

72 Let $a \in N$. Then $a+1$ is invertible and $a+1 \notin N$. Hence, by (1),

73 $1 + a = e_1 e_2 \dots e_k$, e_i is idempotent.

74 Multiplying both sides by e_1 from the left, we get

75 $e_1(1 + a) = e_1 e_2 \dots e_k$, since e_1 is idempotent
 76 $= 1 + a$.

77 But $1 + a$ has an inverse in R , hence $e_1 = 1$.

78 Similarly, we show that $e_2 = \dots = e_k = 1$.

79 Hence $1 + a = 1$ and $a = 0$. So $N=0$, and hence R is multiplicatively generated by
 80 idempotents. So, to show that R is a Boolean ring, it is enough to prove that the
 81 idempotents are central.

82 Let e be an idempotent element in R and let $a = ex - exe$. Clearly,

83 $a^2 = (ex - exe)(ex - exe) = 0$.

84 So $a = 0$. Therefore, $ex = exe$ for every $x \in R$.

85 Following the same above argument, we consider $a = xe - exe$ and we show that $xe =$
 86 exe for every x in R . So, $ex = xe$ for all x in R .

87 Thus e is central and R is Boolean.
 88

89 Proof of Theorem 1:
 90

91 Let R be a finite ring satisfying (1).
 92

93 Case 1. If $J=0$, then R is a finite semisimple Artinian ring with identity satisfying (1).
 94 So R is Boolean, by Lemma 3, and the theorem follows.
 95

96 Case 2. If $J \neq 0$, then R/J is a semisimple Artinian ring with identity satisfying (1), by
 97 Lemma 1. So by Lemma 3, R/J is Boolean and hence commutative. Thus, $xy-yx \in J$
 98 for all x, y in R . But J is nil, since R is finite (hence Artinian). Hence,
 99 $xy-yx \in N$ for all x, y in R .

100 Since $xy-yx \in J$, therefore $C(R) \subseteq J$, and hence

$$101 \quad C(R) \subseteq J \subseteq N.$$

102 Thus, The commutator ideal $C(R)$ is nil and N is an ideal of R , by Lemma 2.

103 Using (1) we have

104 $x \in R$ implies that $x \in N$ or $x = e_1 e_2 \dots e_n$, e_i is idempotent.

105 Moreover, $x \notin N$ implies that

$$106 \quad \begin{aligned} x+N &= e_1 e_2 \dots e_n + N \\ &= (e_1+N)(e_2+N)\dots(e_n+N) \end{aligned}$$

108 and clearly, $(e_i+N)^2 = e_i+N$.

109 So every element $x+N$ in R/N is multiplicatively generated by idempotents, and since
 110 R/N is finite, R/N is Boolean, by Theorem B. This completes the proof of Theorem 1.

111
 112 **Theorem 2.** Let R be a finite ring such that
 113 (2) The set $R \setminus J$ is multiplicatively generated by idempotents.

114 Then N is an ideal of R and R/N is Boolean.
 115

116 The following Lemmas are needed for the proof of Theorem 2.
 117

118 Lemma 4. Any homomorphic image of a ring satisfying (2) must satisfy (2).
 119

120 *Proof.* Let $\phi: R \rightarrow R'$ be a surjective ring homomorphism. Let $x' \in R' = \phi(R)$,
 121 then $x' = \phi(x)$ for some $x \in R$. Now using (2),

122 $x \in J$ or $x = e_1 e_2 \dots e_n$, where e_i is idempotent in R .

123 Let J' denote the Jacobson radical of R' . It is readily verified that $\phi(J) \subseteq J'$, and
 124 thus, for all $x' = \phi(x) \in R' = \phi(R)$,

$$125 \quad x' = \phi(x) \in \phi(J) \subseteq J' \text{ or } \phi(x) = \phi(e_1 e_2 \dots e_n).$$

126 Thus for all $x' = \phi(x) \in R' = \phi(R)$

127 $x' \in J'$ or x' is a product of idempotents.

128 So, R' satisfies (2).
 129

130 Lemma 5. Let R be a ring with identity such that R satisfies (2).
 131 Then R is Boolean.

132
 133 *Proof.* Let $a \in J$. Then $a+1 \notin J$, and hence
 134 $a+1 = e_1 e_2 \dots e_n$, e_i is idempotent.

135 So, $e_1(a+1) = e_1 e_2 \dots e_n = a+1$, and thus $e_1=1$, since $a+1$ is invertible.
 136 Continue this way to show that $e_2=e_3=\dots=e_n=1$, which implies that $a=0$.

137 So, $J=0$, and thus (2) implies that R is multiplicatively generated by idempotents.
 138 Therefore, R is Boolean, by Theorem A.

139
 140 Proof of Theorem 2
 141

142 Let R be a finite ring satisfying (2).
 143

144 Case 1. If $J=0$, then R is a semisimple Artinian ring and hence has an identity. Thus,
145 by Lemma 5, R is Boolean and the Theorem follows.

146
147 Case 2. Suppose $J \neq 0$. Then R/J is a semisimple Artinian ring and satisfies (2) by
148 Lemma 4. Hence R/J is a ring with identity satisfying (2), and thus R/J is Boolean by
149 Lemma 5.

150 So, R/J is commutative, and thus

$$151 \quad xy - yx \in J, \text{ for all } x, y \text{ in } R.$$

152 Hence $C(R) \subseteq J$. But J is nilpotent, since R is Artinian. So

$$153 \quad C(R) \subseteq J \subseteq N,$$

154 and hence the commutator ideal $C(R)$ is nil, which implies that N is an ideal of R , by
155 Lemma 2.

156
157 To complete the proof, since $J \subseteq N$, $x \notin N$ implies that $x \notin J$ and by (2)

$$158 \quad x = e_1 e_2 \dots e_n, \quad e_i \text{ is idempotent.}$$

159 Hence $x \notin N$ implies that

$$160 \quad x + N = e_1 e_2 \dots e_n + N$$

$$161 \quad = (e_1 + N)(e_2 + N) \dots (e_n + N)$$

162 where, $(e_i + N)^2 = e_i + N$.

163 Thus, R/N is multiplicatively generated by idempotents, and since R/N is finite, R/N
164 is Boolean, by Theorem B.

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167

168 **Theorem 3.** Let R be a ring with identity 1 such that

169 (3) The set $R \setminus C$ is multiplicatively generated by idempotents.

170 Then R is commutative.

171

172 **Proof.** Let e be any idempotent of R and x any element of R . Then,

$$173 \quad a = ex - exe \in N.$$

174 Suppose that $a \notin C$, then $1 + a \notin C$, and hence using (3),

175 $1 + a = e_1 e_2 \dots e_n$, where e_i is idempotent. So

176 $e_1(1 + a) = e_1 e_2 \dots e_n = 1 + a$. This implies that $e_1 = 1$ since $1 + a$ is a unit. Similarly, we

177 get $e_2 = e_3 = \dots = e_n = 1$. So $a = 0$ and $ex = exe$. Similarly we can show that

$$178 \quad xe = exe, \text{ and hence } e \in C.$$

179 On the other hand, if $a \in C$, then it is easy to show that also the idempotents are
180 central.

181 So, in all cases, the idempotents are central which implies using (3) that R is
182 commutative.

183

184 **Theorem 4:** Let R be a ring such that

185 (4) The set $R \setminus J$ is multiplicatively generated by idempotents and nilpotents.

186 If R satisfies the polynomial identity $x^m = x^{m+1} f(x)$, where f is a polynomial with
187 integer coefficients, and if the set N of nilpotent elements is commutative, then N is
188 an ideal of R and R/N is Boolean.

189

190 **Proof.**

191

192 A simple induction shows that $x^m = x^{2^m} f(x)^m$. This implies that $e = x^m f(x)^m$ is
 193 idempotent for each x in R .
 194 Let $x \in J$. Since J is an ideal then $e = x^m f(x)^m$ is an idempotent element in J . But J
 195 contains no nonzero idempotent elements and hence $e = 0$.
 196 So, $x^m f(x)^m = 0$, and hence $x^m = x^{2^m} f(x)^m = 0$.
 197 So every element in J is nilpotent. This implies, using (4), that every element in R is
 198 multiplicatively generated by idempotents and nilpotents.
 199 Now, using Theorem C, N is an ideal and R/N is Boolean.

200
 201 The following example shows the essentiality that R is finite in Theorems 1 and 2,
 202 and the essentiality that R has identity in Lemmas 3 and 5.

203
 204 Example Let R_0 be a ring with identity, and let R be the ring of all
 205 $\infty \times \infty$ matrices over R_0 in which at most a finite number of entries are non zero. For
 206 every $X \in R$, there exists a finite $n \times n$ matrix A over R_0 such that

$$207 \quad X = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where } 0\text{'s are zero matrices.}$$

208 Let L , M and N be matrices in R defined by

$$209 \quad L = \begin{pmatrix} I_n & I_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0_n & 0_n & 0 \\ A & I_n & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and } N = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$$

211 It is readily verified that L , M , and N are idempotents.
 212 Clearly for every X in R , $X = LMN$. Thus, every X in R can be expressed as a product
 213 of idempotents. However, N is not an ideal, since

$$214 \quad S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R, \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in N, \quad \text{but } ST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is not in}$$

217 N .

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