

Original Research Article

FIXED POINT THEOREMS IN PARTIALLY ORDERED RECTANGULAR METRIC SPACES

In the manuscript, a concept of a mixed monotone mapping is acquainted and a coupled fixed point theorems is substantiated for such nonlinear shrinkage mappings in partially ordered exact rectangular metric spaces. We enlarge and universalize the conclusions of these theory.

Keywords: Rectangular metric spaces, Coupled fixed point, Hausdorff spaces, Partially ordered set, Mixed monotone mapping.
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1. Introduction

Stefan Banach is a Polish mathematician, who described a benchmark fixed point theorem familiar with the Banach Shrinkage Principle (BSP). The BSP has been universalized in many different respects. Many researcher enlarged to the situation of nonlinear shrinkage mappings. Presence of a fixed point in partially ordered metric spaces and imagined in [1, 7, 9, 11, 13, 15, 16, 17, 18, 19, 20, 21, 22] so far, here implementations to matrix equations, ordinary differential equations and integral equations are asserted.

A coupled fixed point is acquainted by Bhaskar and Lakshmikantham [7] and some coupled fixed point theorems is demonstrated for mixed monotone mappings in ordered metric spaces. Subsequently, another conclusions on coupled fixed point theory subsist in the theory, for further particulars, the commentator is mentioned to [5, 6, 14, 15, 16, 19, 20, 21].

Branciari [8] defined a rectangular metric space (RMS) by replacing the sum at the righthand side of the triangle inequality by a three-term expression. He also substantiated an alike of the BSP. The intriguing nature of these spaces has attracted attention, and fixed points theorems for various shrinkage on rectangular metric spaces have been established [3, 4, 10, 12].

In the article, the presence of a coupled fixed point is attested for a mixed monotone mapping $T : X \times X \rightarrow X$ under a universalized shrinkage, build the exclusiveness under a supplementary presumption on partially ordered exact rectangular metric space.

We state primary definitions and notations to be used throughout this paper. Rectangular metric spaces are designated as below.

40 **Definition 1.1** Assume X is a not null set and $d : X \times X \rightarrow [0, \infty]$ satisfy
 41 the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of which is
 42 dissimilar from x and y .

43 (RM1) $d(x, y) = 0 \Leftrightarrow x = y$,

44 (RM2) $d(x, y) = d(y, x)$,

45 (RM3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$. [12]

46 At that time the map d is named a rectangular metric and the pair (X, d)
 47 is named a rectangular metric spaces (RMS).

48 **Definition 1.2** Suppose $(X, \circ)$ is a partially ordered set, $T : X \times X \rightarrow X$.
 49 T has got the mixed monotone property if $T(x, y)$ is monotone nondecreasing
 50 with respect to x , is monotone non-increasing with respect to y , namely, for any
 51 $x, y \in X$,

52
$$x_1, x_2, y \in X, x_1 \circ x_2 \Rightarrow T(x_1, y) \circ T(x_2, y)$$

53 and

54
$$y_1, y_2, x \in X, y_1 \circ y_2 \Rightarrow T(x, y_1) \pm T(x, y_2).$$

55 [21]

56 **Definition 1.3** A member $(x, y) \in X \times X$ is named a coupled fixed point
 57 of the mapping T if

58
$$T(x, y) = x \text{ and } T(y, x) = y.$$

59 [21]

60 For any $(x, y), (u, v) \in X \times X$, the product space $X \times X$ is equipped with the
 61 metric ρ described by

62
$$\rho((x, y), (u, v)) = \frac{d(x, u) + d(y, v)}{2}.$$

63 So ρ is a rectangular metric.

64 **Definition 1.4** (i) A sequence x_n in a RMS (X, d) is RMS convergent to
 65 a limit $x \Leftrightarrow d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.

66 (ii) A sequence x_n in a RMS (X, d) is RMS Cauchy $\Leftrightarrow$ for every $\varepsilon > 0$
 67 there subsist a positive integer $N(\varepsilon)$ with $d(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.

68 (iii) A RMS (X, d) is named exact if every RMS Cauchy sequence in X
 69 is RMS convergent [12].

70 The fixed point theorem is described by, Lakzian and Samet [16] as
 71 below.

72 **Theorem 1.5** Suppose (X, d) is a Hausdorff and exact RMS and assume
 73 $T : X \rightarrow X$ is a self-map ensuring

74
$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

75 for all $x, y \in X$, and $\psi, \phi \in \Psi$, where ψ is non-decreasing. At the time T has
 76 got an individual fixed point in X [16].

77 Recently, İ. M. Erhan et al. [12] secured the fabulous conclusion and
 78 given a universalization of Theorem 1.5 for a larger class of (ψ, ϕ) weakly
 79 shrinkage mappings.

80 **Theorem 1.6** Suppose (X, d) is a Hausdorff and exact RMS and assume
 81 $T : X \rightarrow X$ is a self-map convincing

$$82 \quad \psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)) + Lm(x, y)$$

83 for all $x, y \in X$ and $\psi, \phi \in \Psi$, where $L > 0$, the function ψ is nondecreasing and
 84 at the time T has got an individual fixed point in X .

$$85 \quad M(x, y) = \max\{d(x, y), d(x, Tx)d(y, Ty)\},$$

$$86 \quad m(x, y) = \min\{d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx)\}.$$

87 At the time T has got an individual fixed point in X [12].

88 In next section, we give a universalization of Theorem 1.6 for a mixed
 89 monotone mappings. We verify a coupled fixed point theorem in partially ordered
 90 exact rectangular metric spaces.

91 2. Main Consequence

92 Now the main conclusion is certified.

93 **Theorem 2.1** Supposing $(X, \circ)$ is partially ordered set and (X, d)
 94 Hausdorff and exact RMS. Suppose there subsist a function $\phi, \theta : [0, \infty) \rightarrow [0, \infty)$
 95 with $\phi(t) < t$, $\theta(t) < t$ and for each $t > 0$ $\lim_{r \rightarrow t^+} \phi(r) < t$, $\lim_{r \rightarrow t^+} \theta(r) < t$ and also The
 96 mixed monotone property on X is verified by $T : X \times X \rightarrow X$ be a continuous
 97 mapping and

$$98 \quad d(T(x, y), T(u, v)) \leq \phi(\rho((x, y), (u, v))) - \theta(\rho((x, y), (u, v))) \quad (1)$$

99 for all $x, y, u, v \in X$ for which $x \circ u$, $y \pm v$ There subsist $(x_0, y_0) \in X \times X$ with
 100 $x_0 \prec T(x_0, y_0)$, $y_0 \pm T(y_0, x_0)$. So T has got an individual coupled fixed point.

101 **Proof.** Suppose $x_0, y_0 \in X$ is with $x_0 \circ T(x_0, y_0)$, $y_0 \pm T(y_0, x_0)$. Let
 102 $x_1 = T(x_0, y_0)$, $y_1 = T(y_0, x_0)$. Then $x_0 \circ x_1$, $y_0 \pm y_1$. Again, let $x_2 = T(x_1, y_1)$,
 103 $y_2 = T(y_1, x_1)$. The mixed monotone property is verified by T , obtaining $x_1 \circ x_2$,
 104 $y_1 \pm y_2$. To continue to do so through, two sequences $\{x_n\}$, $\{y_n\}$ are built in X
 105 with $x_{n+1} = T(x_n, y_n)$, $y_{n+1} = T(y_n, x_n)$,

$$106 \quad x_0 \circ x_1 \circ x_2 \circ \dots \circ x_n \circ x_{n+1} \circ \dots \quad (2)$$

107 and,

$$108 \quad y_0 \pm y_1 \pm y_2 \pm \dots \pm y_n \pm y_{n+1} \pm \dots \quad (3)$$

109 Denote

$$110 \quad \delta_n = \rho((x_n, y_n), (x_{n+1}, y_{n+1})) = \frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2}. \quad (4)$$

111 We expose that $\delta_n < \delta_{n-1}$. Now, enforcing if the inequality (1) is
 112 implemented with $(x, y) = (x_n, y_n)$, $(u, v) = (x_{n-1}, y_{n-1})$, for all $n \geq 0$. Utilizing
 113 properties of φ , we get

$$114 \quad d(T(x_n, y_n), T(x_{n-1}, y_{n-1})) = d(x_{n+1}, x_n) \\
 115 \quad \leq \varphi(\rho((x_n, y_n), (x_{n-1}, y_{n-1}))) - \theta(\rho((x_n, y_n), (x_{n+1}, y_{n+1}))) \\
 116 \quad \leq \varphi(\rho((x_n, y_n), (x_{n-1}, y_{n-1}))) < \rho((x_n, y_n), (x_{n-1}, y_{n-1})). \quad (5)$$

117 Similarly, we can obtain

$$118 \quad d(y_{n+1}, y_n) < \rho((x_n, y_n), (x_{n-1}, y_{n-1})). \quad (6)$$

119 Thus we acquire

$$120 \quad \frac{d(x_{n+1}, x_n) + d(y_{n+1}, y_n)}{2} < \rho((x_n, y_n), (x_{n-1}, y_{n-1})).$$

121 That is $\delta_n < \delta_{n-1}$. Thus $\{\delta_n\}$ is monotone decreasing bounded to the bottom.
 122 Hereby, there subsist a $\delta \geq 0$ with

$$123 \quad \lim_{n \rightarrow \infty} \delta_n = \delta.$$

124 Demonstrating $\delta = 0$. Supposing the contrary $\delta > 0$. At the time from (1)
 125 obtaining

$$126 \quad d(T(x_n, y_n), T(x_{n-1}, y_{n-1})) = d(x_{n+1}, x_n) \leq \varphi(\rho((x_n, y_n), (x_{n-1}, y_{n-1}))) \\
 127 \quad - \theta(\rho((x_n, y_n), (x_{n+1}, y_{n+1}))) \leq \varphi(\rho((x_n, y_n), (x_{n-1}, y_{n-1}))). \quad (7)$$

128 Similarly, we can obtain

$$129 \quad d(y_{n+1}, y_n) < \varphi(\rho((x_n, y_n), (x_{n-1}, y_{n-1}))). \quad (8)$$

130 Thus we get

$$131 \quad \frac{d(x_{n+1}, x_n) + d(y_{n+1}, y_n)}{2} < \rho((x_n, y_n), (x_{n-1}, y_{n-1})). \quad (9)$$

132 While $n \rightarrow \infty$ in (9), getting

$$133 \quad \delta = \lim_{n \rightarrow \infty} \delta_n < \lim_{n \rightarrow \infty} \delta_{n-1} = \delta.$$

134 So the incompatibility is obtained. Hence $\delta = 0$. Namely

$$135 \quad \lim_{n \rightarrow \infty} \rho((x_n, y_n), (x_{n+1}, y_{n+1})) = \lim_{n \rightarrow \infty} \frac{d(x_{n+1}, x_n) + d(y_{n+1}, y_n)}{2} = 0.$$

136 Thus

$$137 \quad \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0 \text{ and } \lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0. \quad (10)$$

138 Evidencing $\{x_n\}$, $\{y_n\}$ are RMS Cauchy sequences. To accept the
 139 opposite that at least one of $\{x_n\}$ or $\{y_n\}$ is not a RMS Cauchy sequence. At the
 140 time there subsist an $\varepsilon > 0$ when obtaining two subsequences $\{(x_{n(i)})\}$ and $\{(x_{m(i)})\}$
 141 of $\{x_n\}$ with $n(i)$ is the smallest index where

$$142 \quad n(i) > m(i) > i, \quad d(x_{m(i)}, x_{n(i)}) + d(y_{m(i)}, y_{n(i)}) \geq \varepsilon. \quad (11)$$

143 It is videlicet

$$144 \quad d(x_{m(i)}, x_{n(i)-1}) + d(y_{m(i)}, y_{n(i)-1}) < \varepsilon. \quad (12)$$

145 By (RM3), we obtain

$$146 \quad d(x_{m(i)}, x_{n(i)}) \leq d(x_{m(i)}, x_{m(i)-1}) + d(x_{m(i)-1}, x_{n(i)-1}) + d(x_{n(i)-1}, x_{n(i)}). \quad (13)$$

147 Similarly from (RM3), we can obtain

$$148 \quad d(y_{m(i)}, y_{n(i)}) \leq d(y_{m(i)}, y_{m(i)-1}) + d(y_{m(i)-1}, y_{n(i)-1}) + d(y_{n(i)-1}, y_{n(i)}). \quad (14)$$

149 By adding (13) and (14), from (10), (11) and (12)

$$150 \quad d(x_{m(i)}, x_{n(i)}) + d(y_{m(i)}, y_{n(i)}) \leq [d(x_{m(i)}, x_{m(i)-1}) + d(y_{m(i)}, y_{m(i)-1})] + \\ 151 \quad [d(x_{m(i)-1}, x_{n(i)-1}) + d(y_{m(i)-1}, y_{n(i)-1})] + [d(x_{n(i)-1}, x_{n(i)}) + d(y_{n(i)-1}, y_{n(i)})]. \quad (15)$$

152 Getting the limit as $i \rightarrow \infty$ in (15), obtaining by (10), (11)

$$153 \quad \varepsilon \leq \lim_{i \rightarrow \infty} [d(x_{m(i)}, x_{n(i)}) + d(y_{m(i)}, y_{n(i)})] \leq \lim_{i \rightarrow \infty} [d(x_{m(i)-1}, x_{n(i)-1}) + d(y_{m(i)-1}, y_{n(i)-1})]. \quad (16)$$

154 Similarly from (RM3), we can obtain

$$155 \quad d(x_{m(i)-1}, x_{n(i)-1}) + d(y_{m(i)-1}, y_{n(i)-1}) \leq [d(x_{m(i)-1}, x_{m(i)}) + d(y_{m(i)-1}, y_{m(i)})] \\ 156 \quad + [d(x_{m(i)}, x_{n(i)}) + d(y_{m(i)}, y_{n(i)})] + [d(x_{n(i)}, x_{n(i)-1}) + d(y_{n(i)}, y_{n(i)-1})]. \quad (17)$$

157 Having the limit as $i \rightarrow \infty$ in (17), we get by (10), (11), (16)

$$158 \quad \lim_{i \rightarrow \infty} [d(x_{m(i)}, x_{n(i)}) + d(y_{m(i)}, y_{n(i)})] = \lim_{i \rightarrow \infty} [d(x_{m(i)-1}, x_{n(i)-1}) + d(y_{m(i)-1}, y_{n(i)-1})]. \quad (18)$$

159 Applying inequality (1) with $(x, y) = (x_{m(i)-1}, y_{m(i)-1})$, $(u, v) = (x_{n(i)-1}, y_{n(i)-1})$

$$160 \quad d(x_{m(i)}, x_{n(i)}) = d(T(x_{m(i)-1}, y_{m(i)-1}), T(x_{n(i)-1}, y_{n(i)-1})) \\ 161 \quad \leq \varphi(\rho((x_{m(i)-1}, y_{m(i)-1}), (x_{n(i)-1}, y_{n(i)-1}))) \\ 162 \quad < \rho((x_{m(i)-1}, y_{m(i)-1}), (x_{n(i)-1}, y_{n(i)-1}))). \quad (19)$$

163 Similarly we get

$$164 \quad d(y_{m(i)}, y_{n(i)}) < \rho((x_{m(i)-1}, y_{m(i)-1}), (x_{n(i)-1}, y_{n(i)-1}))). \quad (20)$$

165 Before by adding (19) and (20) and after taking the limit as $i \rightarrow \infty$, we get

$$166 \quad \lim_{i \rightarrow \infty} [d(x_{m(i)}, x_{n(i)}) + d(y_{m(i)}, y_{n(i)})] < \lim_{i \rightarrow \infty} [d(x_{m(i)-1}, x_{n(i)-1}) + d(y_{m(i)-1}, y_{n(i)-1})].$$

167 From (18), this is a contradiction. Then $\{x_n\}$, $\{y_n\}$ are RMS Cauchy sequences.

168 Because (X, d) is exact there subsist $x, y \in X$ with

$$\lim_{n \rightarrow \infty} x_n = x \text{ and } \lim_{n \rightarrow \infty} y_n = y. \quad (21)$$

From continuity of T and since X is Hausdorff, obtaining

$$x = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} T(x_n, y_n) = T(x, y),$$

and

$$y = \lim_{n \rightarrow \infty} y_{n+1} = \lim_{n \rightarrow \infty} T(y_n, x_n) = T(y, x).$$

Theorem 2.2 Suppose $(X, \circ)$ is partially ordered set and (X, d) Hausdorff and exact RMS. Assume $T : X \times X \rightarrow X$ is a mapping possessing mixed monotone property. Supposing there is a function φ, θ like in Theorem 2.1 and $\varphi(t) = 0 \Rightarrow t = 0$. Assuming X has property as below:

(a) for all n , if a non-decreasing sequence $\{x_n\} \rightarrow x$, at the time $x_n \circ x$

(b) for all n , if a non-increasing sequence $\{y_n\} \rightarrow y$, at the time $y_n \pm y$.

If there subsist $x_0, y_0 \in X$ with $x_0 \circ T(x_0, y_0)$ and $y_0 \pm T(y_0, x_0)$, at the time there subsist $x, y \in X$ with $x = T(x, y)$, $y = T(y, x)$, namely T has got an individual coupled fixed point.

Proof. Coming after the proof of Theorem 2.1, constructing a non-decreasing sequence $\{x_n\}$ in X and a non-increasing sequence $\{y_n\}$ in X with $x_{n+1} = T(x_n, y_n)$ and $y_{n+1} = T(y_n, x_n)$ for all $n \geq 0$ and verifying (21).

Thence by properties of X , obtaining $x_n \circ u$ and $y_n \pm v$ for all $n \geq 0$. By (1), having

$$\begin{aligned} d(x_{n+1}, T(x, y)) &= d(T(x_n, y_n), T(x, y)) \leq \varphi(\rho((x_n, y_n), (x, y))) - \theta(\rho((x_n, y_n), (x, y))) \\ &\leq \varphi(\rho((x_n, y_n), (x, y))). \end{aligned}$$

On letting $n \rightarrow \infty$, using (21) and properties of φ , we get that

$$\lim_{n \rightarrow \infty} d(x_{n+1}, T(x, y)) = 0. \quad (22)$$

Conversely, from (RM3) getting

$$d(x, T(x, y)) \leq d(x, x_{n+2}) + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, T(x, y)).$$

While $n \rightarrow \infty$ in the foregoing imparity, utilizing (21), (10) and (22), having $d(x, T(x, y)) = 0$ namely $x = T(x, y)$. Analogous, displaying $y = T(y, x)$.

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