

Original Research Article

A NUMERICAL METHOD BASED BOUNDARY INTEGRAL EQUATION TO SOLVE MULTIPHASE MOVING BOUNDARY PROBLEMS

Abstract

The new in the present paper, is to use the boundary integral formulation for *fixed boundary* to solve *moving boundary problem*. The basic idea is to track the moving boundary in advance by an assumption of the form $s(t) = 2\zeta^m \sqrt{t}$, $m \geq 2$, ζ is an unknown found from an iterative algorithm developed within the main algorithm then solve phases with moving boundaries as fixed boundaries. Two test problems solved to check the availability of the proposed algorithm. The results due to the present algorithm for the first test problem compared with previous results based on boundary integral formulation for moving boundaries, while the results due to the second test problem compared with available analytical solution. A good agreement for both problems is obtained within prescribed errors.

Keywords: Phase change problems, Moving boundary problems, Boundary elements Methods, Numerical Algorithms.

1. Introduction

Moving boundary problems frequently appear in physical, engineering, industrial and other problems of technological interests [1-2]. Phase change problem is a practical example of moving boundary problems and usually highly nonlinear due to the moving interface conditions [3]. Practical engineering problems are efficiently solved nowadays only by numerical methods, such as finite difference [4], finite element [5-7] and boundary elements [8-9]. Boundary integral method is very convenient to use for solution of Stefan problems in which, nodal points are located only on the boundaries and move together with the phase change interface, this means that, there is no need for any mesh adjustment [10-11]. The basic idea is to track the moving boundary in advance by an assumption of the form $s(t) = 2\zeta^m \sqrt{t}$, $m \geq 2$, ζ is an unknown found from an iterative algorithm developed within the main algorithm, then solve phases with moving boundaries as fixed domain. Two test problems are solved to check the

availability of the proposed algorithm. The results due to the present algorithm for the first test problem are compared with previous results based on boundary integral formulation for moving boundaries, while the results due to the second test problem are compared with available analytical solution. A good agreement for both two test problems is obtained within prescribed errors.

2. Mathematical formulation

A domain Ω consists of Ω_s -solid phase- and Ω_ℓ -liquid phase- representing the overall domain. The domain is bounded by a boundary Γ , while the two phases are separated by a moving interface Γ_m , see figure (1).

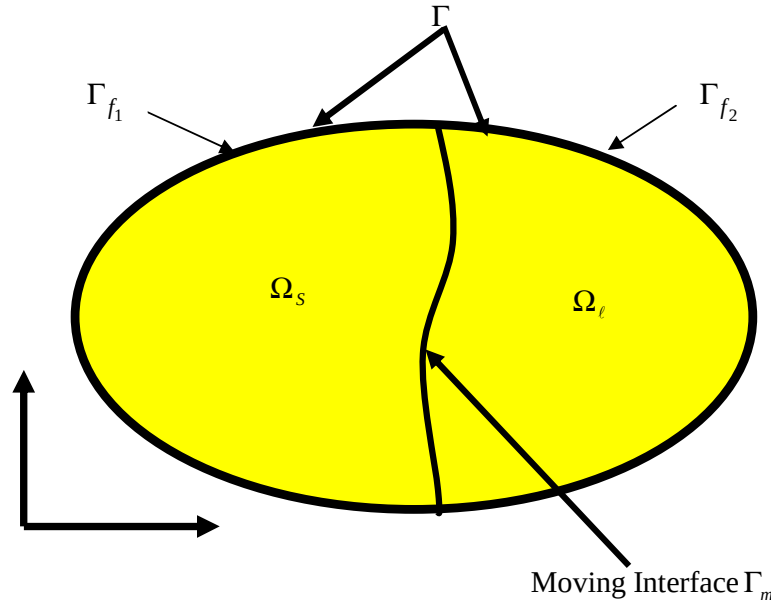


Figure 1: Problem domain

The mathematical formulation is as follows:

$$\alpha_i \nabla^2 u_i(x, t) = \frac{\partial u_i(x, t)}{\partial t}, \quad i = s, \ell \quad (1)$$

The subscripts $i = s, \ell$ refer to solid and liquid phases respectively. On the fixed boundary, two boundary conditions are prescribed:

$$u(x, t) = u_o \quad \forall x \in \Gamma_{f1} \quad (2)$$

63 In case of **solidification**, we have:

$$64 \quad K_s \left(\frac{\partial u(\mathbf{x}, t)}{\partial n} \right) = q(\mathbf{x}, t), \quad \forall \mathbf{x} \in \Gamma_{f_2} \quad (3)$$

65 Or in case of **melting**, we have:

$$66 \quad K_\ell \left(\frac{\partial u(\mathbf{x}, t)}{\partial n} \right) = q(\mathbf{x}, t), \quad \forall \mathbf{x} \in \Gamma_{f_2} \quad (4)$$

67 In equations (2-4) $\Gamma = \Gamma_{f_1} \cup \Gamma_{f_2}$ represent the boundary of the domain Ω . The
68 boundary condition (3) or (4) depends mainly on the type of the problem
69 under consideration. On the moving boundary, two boundary conditions are
70 prescribed:

$$71 \quad u(\mathbf{x}, t) = u_m \quad \text{or} \quad u_f, \quad \forall \mathbf{x} \in \Gamma_m \quad \text{or} \quad \Gamma_f \quad (5)$$

$$72 \quad \text{Input heat flux - output heat flux} = \pm \rho L V_n \quad (6)$$

$$73 \quad u(\mathbf{x}, 0) = u_i \quad (7)$$

74 Based on the boundary integral formulation for fixed boundary [8], for **any**
75 **point** the integral equation takes the following form:

$$76 \quad u(\xi, \tau) = \int_0^\ell u(x, 0) u^*(\xi, x; \tau, 0) dx + k \int_0^\tau \int_0^\ell u^*(\xi, x; \tau, t) q(x, t) dx dt - k \int_0^\tau \int_0^\ell q^*(\xi, x; \tau, t) u(x, t) dx dt \quad (8)$$

78 **4. Numerical iterative algorithm**

79 In this section, the suggested iterative algorithm starting from determination
80 the time at which phase change starts occurring up to the end of the process
81 is shown in figure (2). As it appears, the first part of the developed algorithm
82 is designed to determine the time at which phase change starts occurring
83 within a prescribed error to ensure high accuracy of the computed results.

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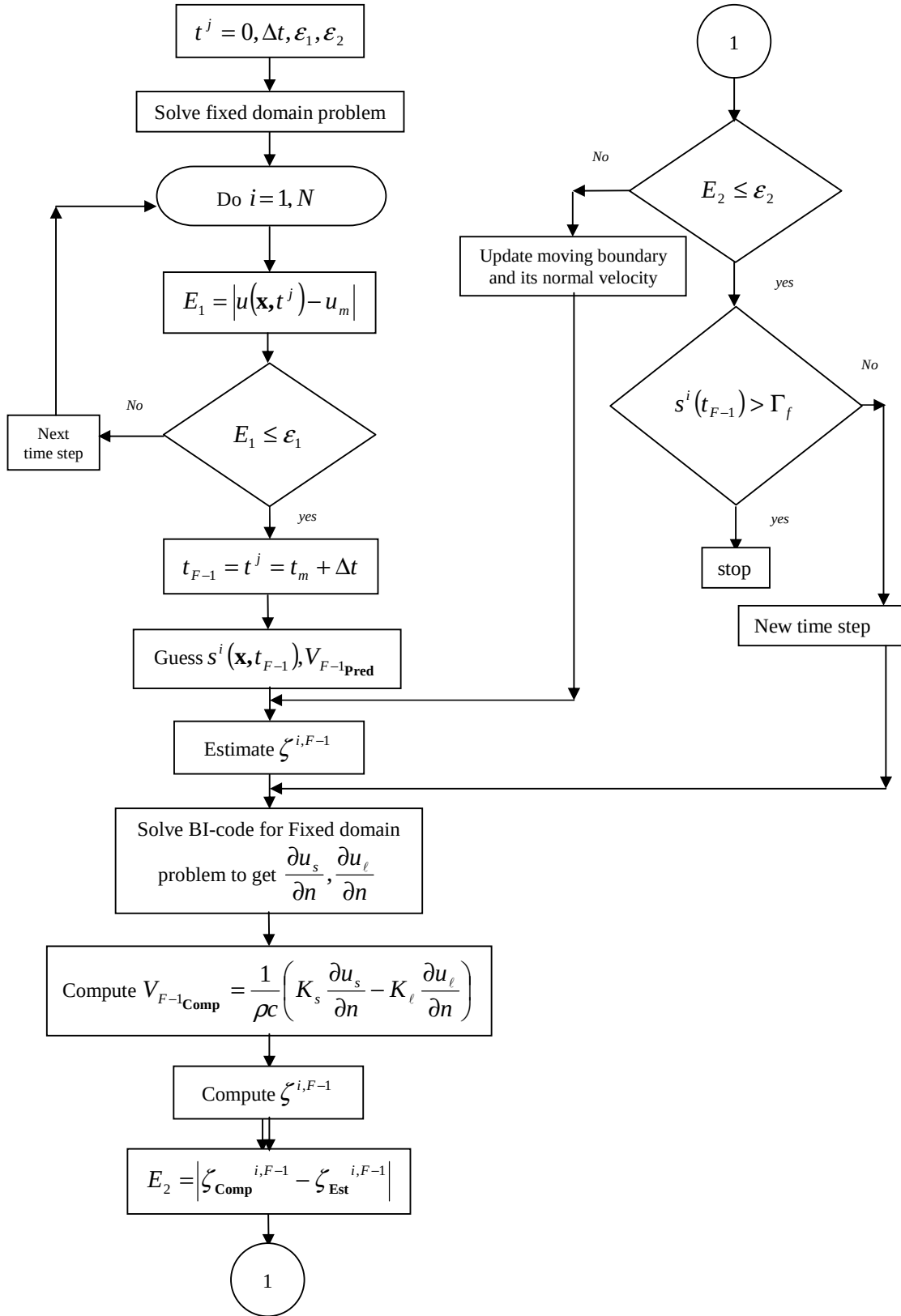


Figure 2: Flow chart of the suggested algorithm

5. Numerical results

Test problem (1)

A solid medium initially at uniform temperature, $U_i = 300^\circ K$, the boundary $x=0$ exposed to two different cases of input heat flux, constant, $Q(t) = 5 \times 10^6$ and linear, $Q(t) = 3 \times 10^4 t$ respectively. Figure (3) shows the movement of solid-liquid due to constant heat flux, and the resulting ablation surface due to the same heat flux is shown in figure (4). The same results due to linear case are plotted on the same plot as shown in figures (5-6). From the above figures, it is found that the solid-liquid interface has the same behavior in both constant and linear cases of heat flux that is concave upward. In case of linear heat flux input this concavity becomes more apparent than the constant case. In the contrary, the ablated interface behaves concave downward but in linear case this concavity increases.

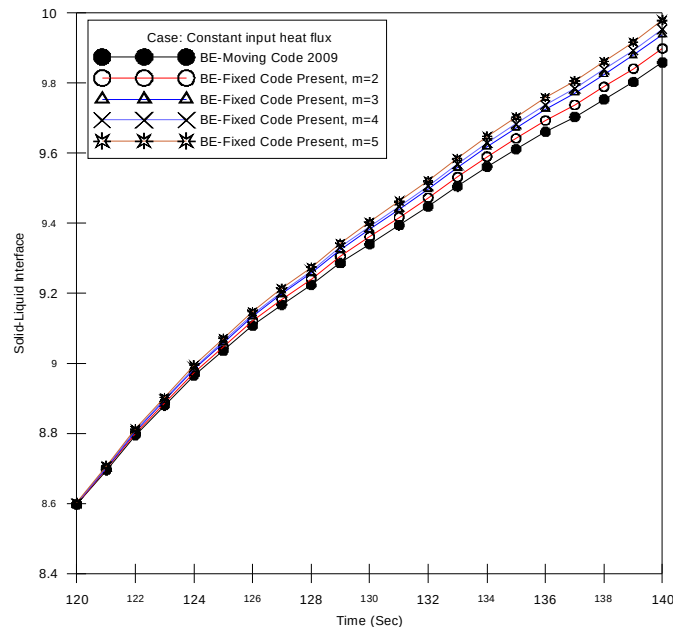


Figure (3): Solid-Liquid due to constant input heat flux

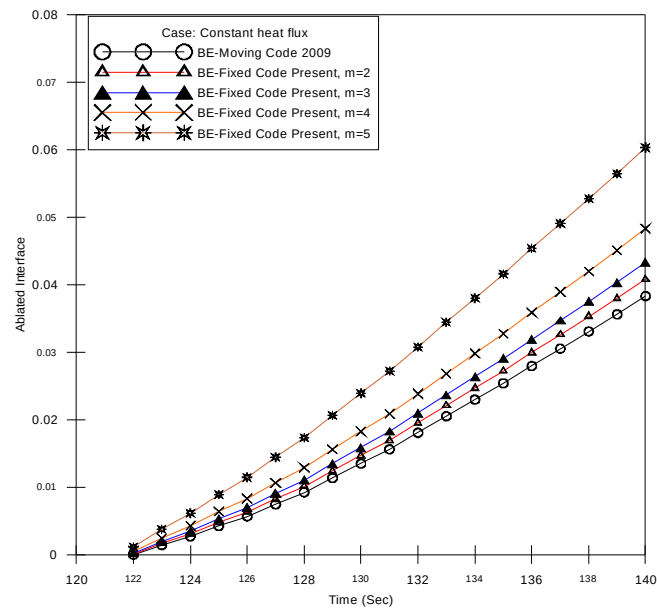


Figure (4): Ablation surface due to constant heat flux

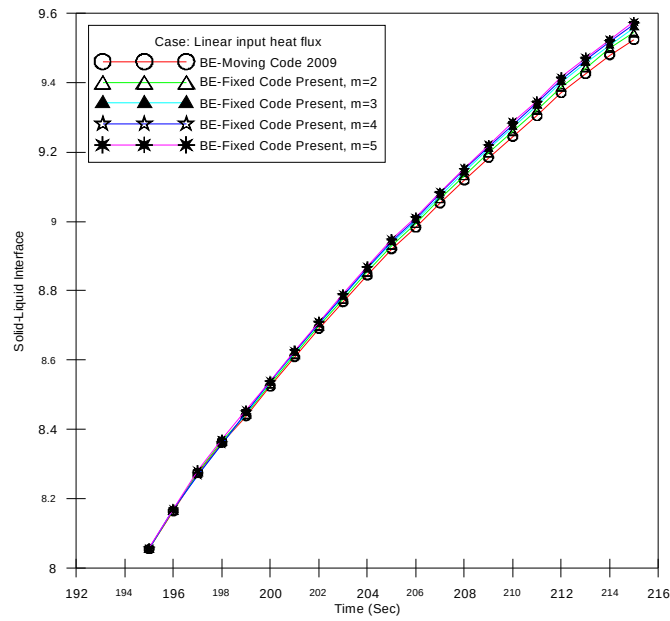


Figure (5): Solid-Liquid due to linear heat flux

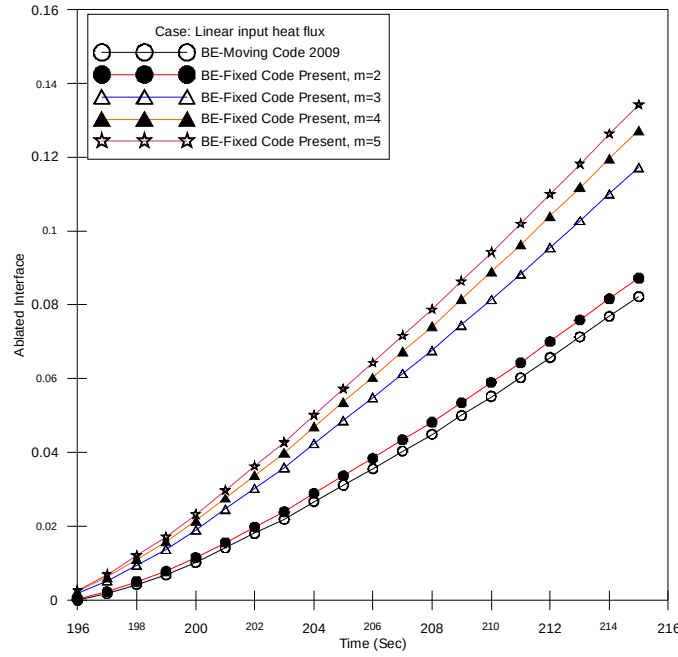


Figure (6): Ablation surface due to linear heat flux

Also, It is found from the computation that the exponent m has a direct effect in the obtained results where the deviation from the previous moving code results starts increasing by increasing m but still acceptable up to $m = 5$. But this conclusion is not acceptable when tracing ablation interface, see figures (4) and (6), respectively.

Test problem (2)

Suppose that a semi-infinite solid, be the physical domain. Assume that the free surface of the solid $x = 0$, is exposed to a constant oxygen concentration C_o , and initial oxygen concentration is zero. Assume that oxygen concentration in oxidized and metallic layers is denoted by C_1 and C_2 respectively, the state equations describing this process, see [10]. The following numerical data are used in the computations:

$$D_1 = 0.274 \mu m^2, D_2 = 0.166 \mu m^2, C_o = 1.0, C_{cr} = 0.65 \text{ and } [C] = 0.25.$$

Since the exact analytical solution to the planar oxidation problem is available, a comparison between exact solution and the present method is made as shown in figure (7). It is clear that there is a good agreement

160 between the two solutions with small acceptable error. Follow up the results
 161 of the present method, figures (8-9) show the oxygen concentration in both
 162 layers at different times $t = 0.5h$ and $t = 2.5h$. It is clear from these figures that
 163 the behavior is the same but the distance from the surface increases by
 164 growing up the time.

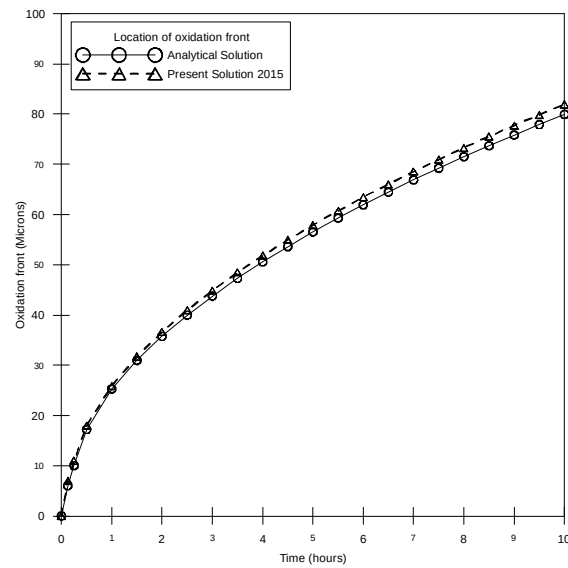


Figure (7). Location of oxidation front

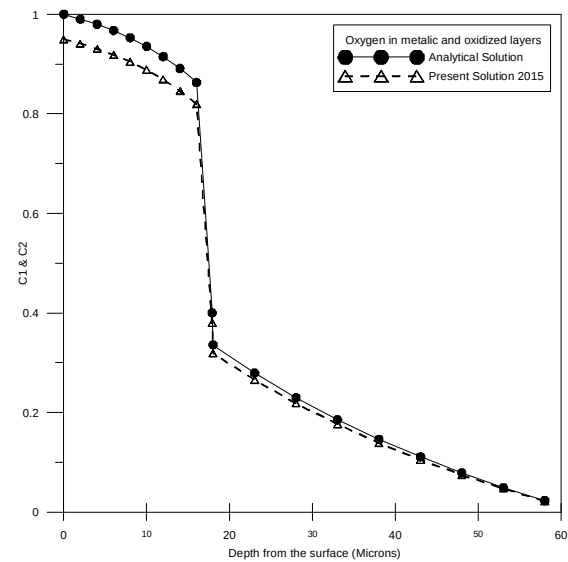


Figure (8): Oxygen concentrations in both layers at time = 0.5 h

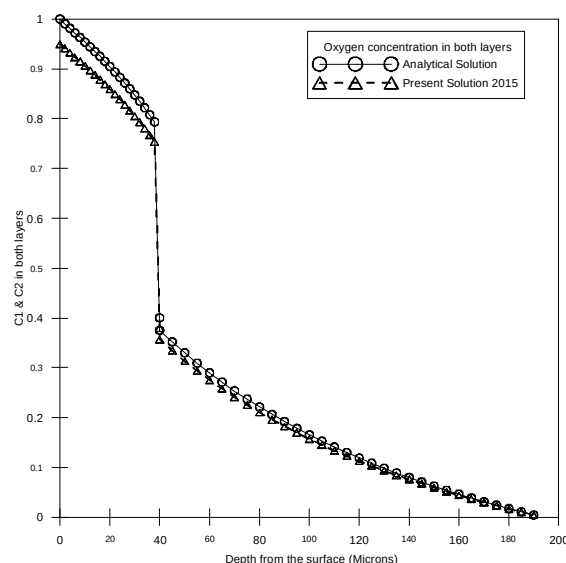


Figure (9): Oxygen concentrations in both layers at time = 2.5h

CONCLUSION

Numerical methods, techniques, and algorithms are long way and has no end as long as there exist life and scientific research. The present paper is just a trial in this long way. The main idea is in the following question: How to solve moving boundary problem as fixed boundary one based the integral equation method?. We solved this problem by fixing the moving interface within each time step by assumption for the moving boundary equation, then using an iterative scheme based the boundary integral equation the location of the moving boundary will be obtained within a prescribed error. It is found from the computations the following:

- 1- The mathematics of the proposed hybrid technique is so simple compared with the boundary integral equation for domain with moving boundary.
- 2- Global number of iterations ranges from 20 to 25 iteration per time step.
- 3- Prescribed errors are small and acceptable for the practical applications.
- 4- The proposed hybrid technique can be modified to cover higher dimensional problems.
- 5- All moving boundary problems of like similarity moving interface can be easily solved using the proposed hybrid technique.

REFERENCES

- (1) Crank, J., "Free and Moving Boundary Problems", Clarendon Press, Oxford, 1984.
- (2) Zerroukat, M. and Chatwin, C.R., Int. J. Num. Meth. Eng., Vol. 35, pp. 1503-1520, 1992.
- (3) Ahmed, S.G., Wrobel, L.C., 'Numerical solution of solidification problems using integral equations', Third Int. Conference of free and moving boundary problems, Bled, Slovenia, 1995.
- (4) Vildan Gu'lkac, 'On the finite differences schemes for the numerical solution of two-dimensional moving boundary problem', Applied Mathematics and Computations, vol. 168, pp. 549-556, 2005.
- (5) Rui Robalo, Carlos Sereno, Maria do Carmo, and Alirio Rodrigues, 'The numerical solution of moving boundary problems using the moving finite element

- method', European Symposium on Computer Aided Process Engineering-15, 2005.
- (6) Baines, M.J., Hubbard, M.E. and Jimack, P.K. (2005) A moving mesh finite element algorithm for the adaptive solution of time-dependent partial differential equations with moving boundaries. *Appl. Numer. Math.* **54** 450–469.
- (7) Miller, K. (1981) Moving finite elements. II. *SIAM J. Numer. Anal.* **18** 1033-1057
- (8) Ahmed, S. G., ' A new fixed-moving algorithm using BIM applied to phase change problems', Journal of Engineering and Applied Science, Cairo University, Vol. 5, no. 49, pp. 823-832, 2002.
- (9) Ahmed, S. G. and Intisar, A.R., ' New approach of BEM methods to solve solute diffusion problems with moving boundaries', Eighth International congress of Fluid Dynamics & Propulsion, Dec. 2006, Sharm El-Shiekh, Egypt.
- (10) Ahmed, S. G., ' A new algorithm for moving boundary problems subject to periodic boundary conditions', Int. J. of Numerical Methods for Heat and Fluid Flow, Vol. 16, no. 1, pp. 18-27, 2006.
- (11) Waldyr De Lima e Solva Jnr., ' A Front-Tracking Boundary Element Formulation for Heat Transfer Problems with Phase Changes', Ph.D., UK., 1994.
- (12) S. G. Ahmed, A Combined domain and integral techniques to solve planar oxidation problem', 6th International Conference of Free and Moving Boundaries Lemons, Greece, 9-11 May 2001