

NONLINEAR ELLIPTIC PROBLEMS INVOLVING THE ($p(x), q(x)$)-LAPLACIAN SYSTEM IN $\mathbb{R}^n$

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ABSTRACT. In this article, we study the elliptic problems involving ($p(x), q(x)$)-Laplacian in $\mathbb{R}^N$. We apply a result by Ricceri [22] to prove the existence of multiple nontrivial solutions.

1. INTRODUCTION

In this article, we study the following two nonlinear problems:

$$\begin{aligned} -\operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{p(x)-2}u &= \lambda f(x, u) + \mu g(x, u) \quad \text{in } \mathbb{R}^N \\ u &\in W_0^{1,p(x)}(\mathbb{R}^N) \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} -\operatorname{div}(a(x)\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{p(x)-2}u &= \lambda f_u(x, u, v) + \mu g_u(x, u, v) \quad \text{in } \mathbb{R}^N \\ -\operatorname{div}(a'(x)|\nabla v|^{q(x)-2}\nabla v) + b'(x)|u|^{q(x)-2}u &= \lambda f_v(x, u, v) + \mu g_v(x, u, v) \quad \text{in } \mathbb{R}^N \\ u &\in W_0^{1,p(x)}(\mathbb{R}^N), v \in W_0^{1,q(x)}(\mathbb{R}^N) \end{aligned} \quad (1.2)$$

For the one equation problem (1.1), we assume that $3 \leq N; p : \mathbb{R}^N \rightarrow \mathbb{R}$ is Lipschitz continuous with $2 \leq p^- \leq p(x) \leq p^+ < N$. λ and μ are positives parameters, and a is a measurable function such that $a \in L^\infty(\mathbb{R}^N)$ with the essential $\inf_{x \in \mathbb{R}^N} a(x) > 0$. The two functions $f(x, t)$ and $g(x, t)$ having subcritical growth with respect to t , more precisely, we assume that f is a Carathéodory function satisfying the following condition

$$|f(x, t)| \leq m(x)|t|^\gamma \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad \forall t \in \mathbb{R}, \quad (1.3)$$

where m is a positive function such that $m \in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^N) \cap L^{\frac{\nu}{\nu-1}(\frac{p^*}{p^*-(\gamma+1)}}(\mathbb{R}^N)$, with $p^+ < \gamma + 1 < \nu < p^*$, $p_-^* = \inf_{\mathbb{R}^N} (p^*(x))$, where $p^*(x)$ denotes the critical Sobolev exponent in [14], i.e., $p^*(x) = \frac{Np(x)}{N-p(x)}$.

The function $g = g(x, t)$, need to be a measurable function with respect to x in $\mathbb{R}^N$ for every t in $\mathbb{R}$, and is continuous with respect to t in $\mathbb{R}$ for almost every x in $\mathbb{R}^N$. Such that $g(x, 0) = 0$, and there exists a positive function $h(x) \in L^\infty(\mathbb{R}^N)$, satisfying

$$\sup_{(x,t) \in \mathbb{R}^N \times (\mathbb{R} \setminus \{0\})} \frac{|g(x, t)|}{h(x)|t|^{p_-^*}} < +\infty \quad (1.4)$$

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The function $b(x)$ needs to satisfy the next condition:

$$b(x) > b_0 > 0 \quad \forall x \in \mathbb{R}^N \quad (1.5)$$

Problems involving $p(x)$ -growth conditions, such as (1.1), have been given a special attention since they are important in modeling some physical phenomena. For example their applications to the study of electrorheological fluids and in elastic mechanics (see Diening [1], Halsey [2], Ruzicka [3], Zhikov [4]). The case of the degenerate equation $-\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x, u)$ in $\mathbb{R}^N$, was studied by many authors, we mention the work of P.Drabek [5], and others like [6],[7],[8]. Existence results for $p(x)$ -Laplacian Dirichlet problems on bounded domains we refer to [[9], [10], [11]] while for the study of $p(x)$ -Laplacian problems in $\mathbb{R}^N$ we refer to [[13],[14],[15],[16],[17][18], [12], [19] and [20]]. In both investigation, i.e., bounded and unbounded, most authors used a standard approach by applying the Mountain Pass Lemma for finding critical points of associated variational formulations of Ambrosetti and Rabinowitz [25].

Since the appearance of the abstract result proved by Ricceri in [21] and its revisited note established in [23] dealing with variational equations with both Dirichlet and Neumann conditions, they have extensively been investigated. In [22], Ricceri obtained a general three critical points theorem, that has been applied for a class of elliptic operators involving nonlinearities of polynomial growth. We mention that the result proved in [22], will be an essential tool for the study of the existence of at least three weak solutions for problems (1.1) and (1.2), when the nonlinearities have a maximal growth.

This article is organized as follows. In section 2 we introduce the generalized weighted Lebesgue-Sobolev spaces $L_{b(x)}^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x),b(x)}(\mathbb{R}^N)$, and some embedding results. In section 3 we treat the case of the one elliptic equation involving $p(x)$ -Laplacian (1.1) and we will prove multiple results by applying Ricceri's principle in [22]. Finally, in section 4, we extend the result of the last section to general elliptic systems of two nonlinear partial differential equations governed essentially by the $(p(x), q(x))$ -Laplacian (1.2). Let us first recall the crucial theorem

Theorem 1.1. [22] *Let X be a separable and reflexive real Banach space; $\Phi : X \rightarrow \mathbb{R}$, a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X , and whose derivative admits a continuous inverse on X^* ; $J : X \rightarrow \mathbb{R}$ a C^1 functional with compact derivative. Assume that Φ has a strict local minimum x_0 with $\Phi(x_0) = J(x_0) = 0$. Finally, set*

$$\alpha = \max \left\{ 0, \limsup_{\|x\| \rightarrow +\infty} \frac{J(x)}{\Phi(x)}, \limsup_{x \rightarrow x_0} \frac{J(x)}{\Phi(x)} \right\},$$

$$\beta = \sup_{x \in \Phi^{-1}(]0, +\infty[)} \frac{J(x)}{\Phi(x)},$$

and assume that $\alpha < \beta$. Then, for each compact interval $[a, b] \subset]1/\beta, 1/\alpha[$ (with the conventions $1/0 = +\infty$, $1/\infty = 0$) there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$, and every C^1 functional $\Psi : X \rightarrow \mathbb{R}$ with compact derivative, there exists $\sigma > 0$ such that for each $\mu \in [0, \sigma]$, the equation

$$\Phi'(x) = \lambda J'(x) + \mu \Psi'(x)$$

has at least three solutions in X whose norms are less than r .

2. ABSTRACT FRAMEWORK

for a $b(x) \in L_{loc}(\mathbb{R}^N)$ satisfying the condition (1.5) we introduce the generalized weighted Lebesgue-Sobolev spaces $L_{b(x)}^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x),b(x)}(\mathbb{R}^N)$

$$L_{b(x)}^{p(x)}(\mathbb{R}^N) = \{u, \int_{\mathbb{R}^N} b(x)|u(x)|^{p(x)} dx < +\infty\}$$

equipped with the so called Luxembourg norm

$$\|u\|_{b(x),p(x)} = \inf\{\nu > 0, \int_{\mathbb{R}^N} b(x)|\frac{u(x)}{\nu}|^{p(x)} dx < 1\}$$

and

$$W^{1,p(x),b(x)}(\mathbb{R}^N) = \{u \in L_{b(x)}^{p(x)}(\mathbb{R}^N), \nabla u \in L^{p(x)}(\mathbb{R}^N)\}$$

endowed with the norm

$$\|u\|_{1,p(x),b(x)} = \|u\|_{p(x),b(x)} + \|\nabla u\|_{a(x),p(x)}$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respect, they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- < p^+ < \infty$ with $p^- = \inf_{x \in \mathbb{R}^N} p(x)$ and $p^+ = \sup_{x \in \mathbb{R}^N} p(x)$ and continuous functions are dense if $p^+ < \infty$ ([26] Theorem 2.8). we denote by $L^{p'(x)}(\mathbb{R}^N)$ the conjugate space of $L^{p(x)}(\mathbb{R}^N)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\mathbb{R}^N), v \in L^{p'(x)}(\mathbb{R}^N)$ The holder type inequality

$$|\int_{\mathbb{R}^N} uv dx| \leq (\frac{1}{p^-} + \frac{1}{p'^-}) \|u\|_{L^{p(x)}} \|v\|_{L^{p'(x)}} \quad (2.1)$$

holds true.

Let E be the space defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{1,p(x),b(x)}$, the condition (1.5) on the function $b(x)$ implies that $E \subset W_0^{1,p}(\mathbb{R}^N)$. Also a simple calculation shows that the norm $\|u\|_{1,p(x),b(x)}$ is equivalent to the norm

$$\|u\| = \inf\{\nu > 0, \int_{\mathbb{R}^N} |\frac{a(x)\nabla u(x)}{\nu}|^{p(x)} dx + |\frac{b(x)u(x)}{\nu}|^{p(x)} dx < 1\}. \quad (2.2)$$

set

$$I(u) = \int_{\mathbb{R}^N} a(x)|\nabla u(x)|^{p(x)} dx + b(x)|u(x)|^{p(x)} dx$$

then for all $u \in E$

$$\|u\| > 1 \implies \|u\|^{p^-} \leq I(u) \leq \|u\|^{p^+} \quad (2.3)$$

$$\|u\| < 1 \implies \|u\|^{p^+} \leq I(u) \leq \|u\|^{p^-} \quad (2.4)$$

$u \in E$ is a weak solution of (1.1) if

$$\int_{\mathbb{R}^N} (a(x)|\nabla u|^{p(x)-2} \nabla u \nabla v + b(x)|u|^{p(x)-2} uv) dx = \int_{\mathbb{R}^N} \lambda f(x, u) v dx + \int_{\mathbb{R}^N} \mu g(x, u) v dx$$

$\forall v \in E$.

Let recall some embedding results concerning variable exponent Lebesgue-Sobolev spaces. If $p(x)$ is Lipschitz continuous and $p^+ < N$, then for any $q(x)$ satisfying $p(x) \leq q(x) \leq \frac{Np(x)}{N-p(x)}$, the embedding $W^{1,p(x)}(\mathbb{R}^n) \hookrightarrow L^{q(x)}$ is continuous ([14]

theorem 1.1). Also, the Sobolev inequality, $\|u\|_{L^{p^*(x)}} \leq C\|u\|$ holds for all $u \in E$ for some constant $C > 0$.

3. MAIN RESULT AND PROOF

We state and prove the following main result of this paper.

Theorem 3.1. *Let us suppose $2 < p(x) < N$ and (1.3). Furthermore, suppose for $1 < \tau < p^-$ and some positive function $\alpha(x) \in L^{(\frac{p^+}{\tau})'}(\mathbb{R}^N)$ that*

$$\limsup_{|u| \rightarrow +\infty} \frac{F(x, u)}{\alpha(x)|u|^\tau} \leq M < +\infty \quad \text{uniformly } x \in \mathbb{R}^n \quad (3.1)$$

and

$$\sup_{u \in E} \left\{ \int_{\mathbb{R}^N} F(x, u) dx \right\} > 0$$

where $F(x, t) = \int_0^t f(x, s) ds$

If we set

$$\omega = \frac{1}{p^+} \inf \left\{ \frac{\int_{\mathbb{R}^N} a(x)|\nabla u|^{p(x)} + \int_{\mathbb{R}^N} b(x)|u|^{p(x)} dx}{\int_{\mathbb{R}^N} F(x, u) dx} : u \in E, \int_{\mathbb{R}^N} F(x, u) dx > 0 \right\} \quad (3.2)$$

Then for each compact interval $[a, b] \subset [\omega, +\infty[$, there exists $r_1 > 0$ with the following property: for every $\lambda \in [a, b]$, and every function $g : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ which is measurable in Ω and continuous in $\mathbb{R}$ satisfying (1.4), there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the one equation problem (1.1) has at least three nonzero weak solutions in E whose norms are less than r_1 .

To prove our result we need to define the following functions $\Phi, J, \Psi : E \rightarrow \mathbb{R}$

$$\Phi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + b(x)|u|^{p(x)} \right) dx$$

and

$$J(u) = \int_{\mathbb{R}^N} F(x, u) dx$$

and

$$\Psi(u) = \int_{\mathbb{R}^N} G(x, u) dx$$

the function $\phi(u)$ satisfy the following inequalities

$$\frac{I(u)}{p^+} \leq \phi(u) \leq \frac{I(u)}{p^-}$$

then using (2.3) and (2.4),

$$\frac{\|u\|^{p^-}}{p^+} \leq \Phi(u) \leq \frac{\|u\|^{p^+}}{p^-} \quad \text{if } \|u\| \leq 1 \quad (3.3)$$

$$\frac{\|u\|^{p^+}}{p^+} \leq \Phi(u) \leq \frac{\|u\|^{p^-}}{p^-} \quad \text{if } 1 \leq \|u\| \quad (3.4)$$

For each $u \in E$ it is well known that Φ is well defined and continuously Gâteaux differentiable with

$$\Phi'(u)v = \int_{\mathbb{R}^N} a(x)|\nabla u|^{p(x)-2} \nabla u \nabla v + \int_{\mathbb{R}^N} b(x)|u|^{p(x)-2} uv dx$$

for all $u, v \in E$. Note also that Φ is clearly coercive, weakly lower semi-continuous and bounded on each bounded subset of E . By the standard uniform convexity algebraic inequality of the function $l(x) = |x|^{p(x)}$, $x \in \mathbb{R}^N$, we deduce that Φ is uniformly monotone operator in E . Moreover, a classical result on uniformly convex spaces ensures that Φ' admits a continuous inverse on $(E)^*$ (Theorem 26. A) of [24], i.e. $\Phi \in W_E$.

It follows from (1.3),

$$|F(x, u)| \leq \frac{1}{\gamma + 1} m(x) |u|^{\gamma+1} \quad (3.5)$$

for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$. Then by standard argument we have F is in $C^1(\mathbb{R}^N \times \mathbb{R})$, hence we see that J is well defined and continuously Gâteaux differentiable with

$$J'(u)v = \int_{\mathbb{R}^N} f(x, u)v \, dx$$

for all $u, v \in E$.

Lemma 3.2. *J' is a compact map from E to $(E)^*$.*

Proof.

Let $\{u_k\}$ be a sequence in E which converges weakly to u . On one hand, in view of Hölder's inequality and Sobolev embedding, we obtain for all $0 \leq R \leq +\infty$,

$$\int_{|x| \geq R} f(x, u)v \, dx \leq \left(\int_{|x| \geq R} |m|^{p_1} \, dx \right)^{\frac{1}{p_1}} (C_1 \|u\|)^\gamma \|v\|^{p_-^*},$$

for all $u, v \in E$, and $p_1 = \frac{p_-^*}{p_-^* - (\gamma+1)} \in \left[\frac{p_-^*}{p_-^* - 1}, \frac{\nu}{\nu-1} \left(\frac{p_-^*}{p_-^* - (\gamma+1)} \right) \right]$. Since $m \in L^{p_1}(\mathbb{R}^N)$, we have $\lim_{R \rightarrow +\infty} \int_{|x| \geq R} |m|^{p_1} \, dx = 0$. This implies with the fact that $\{u_k\}$ is a bounded sequence, for any ε , there exists $R_\varepsilon > 0$ such that

$$\int_{|x| \geq R_\varepsilon} f(x, u_k)v \, dx \leq \varepsilon \quad \text{and} \quad \int_{|x| \geq R_\varepsilon} f(x, u)v \, dx \leq \varepsilon \quad (3.6)$$

holds for all k .

On the other hand, applying Young's inequality, we get

$$\begin{aligned} f^{\frac{\nu}{\nu-1}}(x, t) &\leq m(x)^{\frac{\nu}{\nu-1}} t^{\frac{\gamma\nu}{\nu-1}} \\ &\leq \frac{p_-^* - (\gamma+1)}{p_-^*} m(x)^{\frac{\nu}{\nu-1} \left(\frac{p_-^*}{p_-^* - (\gamma+1)} \right)} + \frac{\gamma+1}{p_-^*} t^{\frac{\gamma\nu}{\nu-1} \frac{p_-^*}{\gamma+1}}, \end{aligned}$$

for all $t \in \mathbb{R}$ and a.e. $x \in B_\varepsilon = \{x \in \mathbb{R}^N; |x| < R_\varepsilon\}$. Let us remark that

$$\frac{\gamma\nu}{\nu-1} \frac{p_-^*}{\gamma+1} < p_-^*,$$

since $\gamma+1 < \nu$. Hence the Nemytskii operator $N_{f^{\frac{\nu}{\nu-1}}}$ associated with $f^{\frac{\nu}{\nu-1}}$ is continuous from $L^{\frac{\gamma\nu}{\nu-1} \frac{p_-^*}{\gamma+1}}(B_\varepsilon)$ in $L^1(B_\varepsilon)$. Then we conclude that

$$\int_{|x| < R_\varepsilon} f(x, u_k)^{\frac{\nu}{\nu-1}} \, dx \rightarrow \int_{|x| < R_\varepsilon} f(x, u)^{\frac{\nu}{\nu-1}} \, dx,$$

which implies that $f(x, u_k)$ converges to $f(x, u)$ in $L^{\frac{\nu}{\nu-1}}(B_\varepsilon)$. Hence since $L^{p^*}(B_\varepsilon) \subset L^\nu(B_\varepsilon)$, we have $f(x, u_k)v$ converges to $f(x, u)v$ in $L^1(B_\varepsilon)$, i.e.

$$\int_{|x| < R_\varepsilon} (f(x, u_k) - f(x, u))v \, dx \rightarrow 0, \quad (3.7)$$

for all $v \in L^\nu(B_\varepsilon)$. Finally, in view of (3.6) and (3.7), we get

$$\int_{\mathbb{R}^N} f(x, u_k)v \, dx \rightarrow \int_{\mathbb{R}^N} f(x, u)v \, dx.$$

This completes the proof of Lemma 2.1 and consequently J' is compact.

Now in order to prove our result, we prove that the conditions given in Theorem 1.1 are satisfied. Indeed, using Hölder's inequality and Sobolev embedding, we have in view of (3.5),

$$\int_{\mathbb{R}^N} F(x, u) \, dx \leq \frac{1}{\gamma + 1} \left(\int_{\mathbb{R}^N} |m|^{p_1} \, dx \right)^{\frac{1}{p_1}} (C_1 \|u\|)^{\gamma+1}$$

$$\forall (x, u) \in \mathbb{R}^N \times E \text{ with } p_1 = \frac{p_-^*}{p_-^* - (\gamma+1)}.$$

Then from (3.3) (3.4) we obtain

$$\frac{J(u)}{\Phi(u)} \leq \frac{p^+}{\gamma + 1} C_1^{\gamma+1} \|m\|_{L^{p_1}} \frac{\|u\|^{\gamma+1}}{\|u\|^{p^-}}, \quad \|u\| \leq 1$$

and

$$\frac{J(u)}{\Phi(u)} \leq \frac{p^+}{\gamma + 1} C_1^{\gamma+1} \|m\|_{L^{p_1}} \frac{\|u\|^{\gamma+1}}{\|u\|^{p^+}}, \quad 1 \leq \|u\|$$

for all $u \in E$. Hence since $p^- < p^+ < \gamma + 1$ we have for $\varepsilon > 0$ small enough

$$\limsup_{u \rightarrow 0} \frac{J(u)}{\Phi(u)} \leq \frac{p^+}{\gamma + 1} C_1^{\gamma+1} \|m\|_{L^{p_1}} \varepsilon. \quad (3.8)$$

On the other hand, using (3.1), there exists $A > 0$ such that

$$|F(x, u)| \leq M\alpha(x)|u|^\tau, \quad \forall |u| > A$$

where $\tau < p^- < p^+$ and $M > 0$. Then, using Hölder and Young inequalities and Sobolev embedding, we obtain for each $u \in E \setminus \{0\}$ for $\|u\| > 1$

$$\begin{aligned} \frac{J(u)}{\Phi(u)} &\leq \frac{p^+ \int_{\mathbb{R}^N} (|u| > A) F(x, u) \, dx}{\|u\|^{p^-}} + \frac{p^+ \int_{\mathbb{R}^N} (|u| \leq A) F(x, u) \, dx}{\|u\|^{p^-}} \\ &\leq \frac{p^+ M \int_{\mathbb{R}^N} (|u| > A) \alpha(x) |u|^\tau \, dx}{\|u\|^{p^-}} + \frac{p^+}{\gamma + 1} \frac{\int_{\mathbb{R}^N} (|u| \leq A) m(x) |u|^{\gamma+1} \, dx}{\|u\|^{p^-}} \\ &\leq \frac{p M C_1^\tau \|\alpha\|_{L(\frac{p_-^*}{\tau})'} \|u\|^\tau}{\|u\|^{p^-}} + \frac{p^+ A^{\gamma+1-\tau}}{\gamma + 1} \frac{\int_{\mathbb{R}^N} (|u| \leq A) m(x) |u|^\tau \, dx}{\|u\|^{p^-}} \end{aligned}$$

Now using the fact that $m \in L^{\frac{p_-^*}{p_-^* - \tau}}(\mathbb{R}^N)$ since $\frac{p_-^*}{p_-^* - \tau} \in [\frac{p_-^*}{p_-^* - 1}, \frac{p_-^*}{p_-^* - (\gamma+1)}]$, we get

$$\frac{J(u)}{\Phi(u)} \leq \frac{p^+ M C_1^\tau \|\alpha\|_{L(\frac{p_-^*}{\tau})'}}{\|u\|^{p^- - \tau}} + \frac{p^+ A^{\gamma+1-\tau} C_1^\tau}{\gamma + 1} \frac{\|m\|_{L^{\frac{p_-^*}{p_-^* - \tau}}} \|u\|^\tau}{\|u\|^{p^-}}$$

$$\leq \frac{p^+ MC_1^\tau \|\alpha\|_{L(\frac{p^*}{\tau})'}}{\|u\|^{p^--\tau}} + \frac{p^+ A^{\gamma+1-\tau} C_1^\tau}{\gamma+1} \frac{\|m\|_{L(\frac{p^*}{p^--\tau})}}{\|u\|^{p^--\tau}}$$

Therefore, since $\tau < p^-$ we obtain

$$\limsup_{\|u\| \rightarrow +\infty} \frac{J(u)}{\Phi(u)} \leq p^+ C_1^\tau \left(M \|\alpha\|_{L(\frac{p^*}{\tau})'} + \frac{p^+ A^{\gamma+1-\tau}}{\gamma+1} \|m\|_{L(\frac{p^*}{p^--\tau})} \right) \varepsilon \quad (3.9)$$

for $\|u\| > 1$ we get

$$\limsup_{\|u\| \rightarrow +\infty} \frac{J(u)}{\Phi(u)} \leq p^+ C_1^\tau \left(M \|\alpha\|_{L(\frac{p^*}{\tau})'} + \frac{p^+ A^{\gamma+1-\tau}}{\gamma+1} \|m\|_{L(\frac{p^*}{p^--\tau})} \right) \varepsilon \quad (3.10)$$

Finally, for an arbitrary ε and in view of (3.10) and (3.8), we get

$$\max \left\{ \limsup_{\|x\| \rightarrow +\infty} \frac{J(x)}{\Phi(x)}, \limsup_{\|x\| \rightarrow 0} \frac{J(x)}{\Phi(x)} \right\} \leq 0$$

Hence, all the assumptions of Theorem 1.1 are satisfied (with $x_0 = 0$).

Moreover, since the function $G : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable in Ω and C^1 in $\mathbb{R}$ such that $G_u = g$ satisfying (1.4). Then using standard arguments the functional $\Psi(u) = \int_{\mathbb{R}^N} G(x, u) dx$ is well defined and continuously Gâteaux differentiable on W , with compact derivative, and one has

$$\Psi'(u)v = \int_{\mathbb{R}^N} g(x, u)v dx$$

for all $u, v \in W$. So, by Theorem 1.1, the problem (1.1) has at least two nonzero weak solutions which are critical points of the functional $\Phi - \lambda J - \mu \Psi$.

Then the proof of Theorem 1.3 is achieved.

Example. In this example we consider the problem (1.1), where $N = 5$, $p(x) = \cos(|x|) + 3,001$ the one equation problem become,

$$\begin{aligned} -\operatorname{div}(a(x)|\nabla u|^{\cos(|x|)+1,001} \nabla u) + b(x)|u|^{\cos(|x|)+1,001} u &= \lambda f(x, u) + \mu g(x, u) \quad \text{in } \mathbb{R}^5 \\ u &\in W_0^{1, \cos(|x|)+3,001}(\mathbb{R}^5), \end{aligned} \quad (3.11)$$

$$2 < p^- = 2.001; p^+ = 4.001; 10 < p_-^* \simeq 10.01$$

The function $p(x)$ satisfy the conditions the of theorem 2.1, then for any function $m \in L^{\frac{p_-^*}{p_-^*-1}}(\mathbb{R}^5) \cap L^{\frac{\nu}{\nu-1}(\frac{p_-^*}{p_-^*-\gamma+1})}(\mathbb{R}^5)$, γ and ν satisfying $4,001 < \gamma+1 < \nu < 10 < p_-^*$. For the function $F(x, t)$, let choose τ such that $\frac{\gamma+1}{2} < \tau < p^-$, and let $0 < \varepsilon < \min(1, \tau - \frac{\gamma+1}{2})$ be small enough. Consider the function $F : \mathbb{R}^5 \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$F(x, u) = \frac{1}{\tau - \varepsilon} m(x) |u|^{\tau - \varepsilon} \ln |u| \text{ if } u \neq 0, \quad F(x, 0) = 0$$

Then, condition (1.3) is satisfied since $\ln |u| + \frac{1}{\tau - \varepsilon} < |u|^{\gamma - \tau + 1 + \varepsilon}$. It is not difficult to see that the remaining hypotheses of Theorem 1.2 are satisfied

4. (P(X), Q(X))-LAPLACIAN SYSTEM

In this section we consider the two-equations nonlinear problems involving the $(p(x), q(x))$ -laplacian :

$$\begin{aligned} -\operatorname{div} (a(x) \nabla |u|^{p(x)-2} \nabla u) + b(x) |u|^{p(x)-2} u &= \lambda f_u(x, u, v) + \mu g_u(x, u, v) \quad \text{in } \mathbb{R}^N \\ -\operatorname{div} (a'(x) \nabla |v|^{q(x)-2} \nabla v) + b'(x) |v|^{q(x)-2} v &= \lambda f_v(x, u, v) + \mu g_v(x, u, v) \quad \text{in } \mathbb{R}^N \\ u &\in W_0^{1,p(x)}(\mathbb{R}^N), v \in W_0^{1,q(x)}(\mathbb{R}^N) \end{aligned} \quad (4.1)$$

where $2 < p^- < p(x) < p^+ < N, 2 < q^- < q(x) < q^+ < N$. $\lambda, \mu > 0$ are parameters. The functions $a(x)$, $a'(x)$ are measurable in $L^\infty(\mathbb{R}^N)$ with the essential of $\inf_{(\mathbb{R}^N)} a'(x)$ and $\inf_{(\mathbb{R}^N)} a(x) > 0$. The functions $b(x)$, $b'(x)$ are measurable in $L_{loc}^\infty(\mathbb{R}^N)$ with $\inf_{(\mathbb{R}^N)} b'(x)$ and $\inf_{(\mathbb{R}^N)} b(x) > b_0 > 0$. The functions $f(x, t, t')$ and $g(x, t, t')$ having sub-critical growth with respect to t, t' More precisely, we assume that f is a Carathéodory function satisfying the following condition

$$|f(x, t, t')| \leq m(x) (|t|^\gamma + |t'|^\gamma) \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad \forall t \in \mathbb{R}, \quad (4.2)$$

where $m(x)$ is a positive function such that

$$m \in L^{\frac{s^-}{s^- - 1}}(\mathbb{R}^N) \cap L^{\frac{\nu}{\nu - 1}(\frac{s^-}{s^- - (\gamma + 1)})}(\mathbb{R}^N),$$

with

$$s^+ < \gamma + 1 < \nu < s_-^*$$

where

$$s^+ = \max(p^+, q^+), \quad s^- = \min(p^-, q^-), \quad s_-^* = \min(p_-^*, q_-^*)$$

with

$$\begin{aligned} p_-^* &= \frac{Np^-}{N - p^+}, \quad q_-^* = \frac{Nq^-}{N - q^+}. \\ \sup_{(x, t, t') \in \mathbb{R}^N \times (\mathbb{R} - \{0\})^2} \frac{|g(x, t, t')|}{h(x)(|t|^{p_-^*} + |t'|^{q_-^*})} &< +\infty, \end{aligned} \quad (4.3)$$

for some

$$h \in L^\infty(\mathbb{R}^N) \quad \text{and} \quad g(x, 0) = 0.$$

We shall look for a weak-solution of (4.1) in the space $W_0^{1,p(x)}(\mathbb{R}^N) \times W_0^{1,q(x)}(\mathbb{R}^N)$ which is endowed with the Cartesian norm $\|(u, v)\| = \|u\|_{W_0^{1,p(x)}(\mathbb{R}^N)} + \|v\|_{W_0^{1,q(x)}(\mathbb{R}^N)}$.

We set

$$E = W_0^{1,p(x)}(\mathbb{R}^N), \quad E' = W_0^{1,q(x)}(\mathbb{R}^N), \quad U = (u, v), \quad f(x, U) = f_u(x, u, v) + f_v(x, u, v), \\ g(x, U) = g_u(x, u, v) + g_v(x, u, v), \quad \text{and} \quad F(x, u, v) = \int_0^u f_u(x, s, v) ds + \int_0^v f_v(x, u, s) ds$$

Theorem 4.1. *Let us suppose for $1 < \tau < s^-$ and some positive function $\alpha(x) \in L^{(\frac{s_-^*}{\tau})'}(\mathbb{R}^N)$ that*

$$\limsup_{|U| \rightarrow +\infty} \frac{F(x, U)}{\alpha(x)|U|^\tau} \leq M < +\infty \quad \text{uniformly} \quad x \in \mathbb{R}^n, \quad (4.4)$$

and

$$\sup_{U \in E \times E'} \left(\int_{\mathbb{R}^N} F(x, U) dx \right) > 0$$

If we set

$$\omega = \frac{1}{s^+} \inf \left\{ \frac{\int_{\mathbb{R}^N} A(x, U) dx}{\int_{\mathbb{R}^N} F(x, U) dx} : U \in W, \right\}, \quad (4.5)$$

where $A(x, U) = (a(x)|\nabla u|^{p(x)} + a'(x)|\nabla u|^{q(x)} + b(x)|u|^{p(x)} dx + b'(x)|u|^{q(x)})$ and $W = \{U \in E \times E', \int_{\mathbb{R}^N} F(x, U) dx > 0\}$, then for each compact interval $[a, b] \subset [\omega, +\infty[$, there exists $r_1 > 0$ with the following property: for every $\lambda \in [a, b]$, and every function $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ which is measurable in Ω and continuous in $\mathbb{R}$ satisfying (4.3), there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the two- equation system (4.1) has at least two nonzero weak solutions in E whose norms are less than r_1 .

To prove this result we need to define the following functions

$$\Phi, J, \Psi : E \times E' \rightarrow \mathbb{R}$$

$$\Phi(U) = \int_{\mathbb{R}^N} \frac{1}{p(x)} (|\nabla u|^{p(x)} + b(x)|u|^{p(x)}) dx + \int_{\mathbb{R}^N} \frac{1}{q(x)} (|\nabla v|^{q(x)} + b'(x)|v|^{q(x)}) dx$$

and

$$J(U) = \int_{\mathbb{R}^N} F(x, U) dx$$

and

$$\Psi(U) = \int_{\mathbb{R}^N} G(x, U) dx$$

The functions $\Phi(U)$, $J(U)$, $\Psi(U)$ need to satisfy the conditions of the theorem 1.1, i.e

$\Phi : E \times E' \rightarrow \mathbb{R}$, a coercive, sequentially weakly lower semi continuous C^1 functional, belonging to $\mathcal{W}_{E \times E'}$, bounded on each bounded subset of $E \times E'$, and whose derivative admits a continuous inverse on X^* ; also it not difficult to see that $\phi(U)$ satisfy;

$$\frac{\|u\|_E^{p(x)}}{p^+} + \frac{\|v\|_{E'}^{q(x)}}{q^+} \leq \phi(U) \leq \frac{\|u\|_E^{p(x)}}{p^-} + \frac{\|v\|_{E'}^{q(x)}}{q^-}$$

then for $\|u\|_E \leq 1$ and $\|v\|_{E'} \leq 1$ we have:

$$\frac{\|u\|_E^{p^+} + \|v\|_{E'}^{q^+}}{s^+} \leq \Phi(U) \leq \frac{\|u\|_E^{p^-} + \|v\|_{E'}^{q^-}}{s^-}$$

then for a very small value of $\|u\|_E$, and, $\|v\|_{E'}$ we have

$$\frac{(\|u\|_E + \|v\|_{E'})^{s^+}}{s^+} \leq \Phi(U) \leq \frac{(\|u\|_E + \|v\|_{E'})^{s^-}}{s^-}$$

with leads us to an inequality similar to the one (3.3) in section 3,

$$\frac{(\|U\|_{E \times E'})^{s^+}}{s^+} \leq \Phi(U) \leq \frac{(\|U\|_{E \times E'})^{s^-}}{s^-} \quad (4.6)$$

Also for a very big value of $\|u\|_E$ and $\|v\|_{E'}$, we have an inequality similar (3.4) in section 3

$$\frac{(\|U\|_{E \times E'})^{s^-}}{s^+} \leq \Phi(U) \leq \frac{(\|U\|_{E \times E'})^{s^+}}{s^-} \quad (4.7)$$

Those inequalities (4.6), (4.7) will play a major role (similar to the one played by (3.3) and (3.4) in section 3) to prove

$$\alpha = \max \left\{ 0, \limsup_{\|U\| \rightarrow +\infty} \frac{J(U)}{\Phi(U)}, \limsup_{\|U\| \rightarrow 0} \frac{J(U)}{\Phi(U)} \right\},$$

$$\beta = \sup_{x \in \Phi^{-1}([0, +\infty])} \frac{J(U)}{\Phi(U)},$$

with $\alpha = 0 < \beta = \infty$ Then, for each compact interval $[a, b] \subset]0, \infty[$ there exists $r_1 > 0$ such that the problem (4.1) has at least three solutions in $E \times E'$ whose norms are less than r_1 .

REFERENCES

- [1] L. Diening, Theoretical and Numerical Results for Electrorheological Fluids, Ph.D. thesis, *University of Friburg, Germany*, 2002.
- [2] T. C. Halsey, Electrorheological fluids, *Science* 258 (1992), 761-766.
- [3] M. Ruzicka, Electrorheological Fluids: Modeling and Mathematical Theory, Springer-Verlag, Berlin, 2002.
- [4] V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, *Math. USSR Izv.* 29 (1987), 33-66.
- [5] P. Drábek. Nonlinear Eigenvalue for p-Laplacian in $\mathbb{R}^N$. *Math. Nach.*, 173 :131–139, 1995.
- [6] G. Anello G. Cordaro; Existence of solutions of the Neumann problem involving the p-Laplacian via variational principle of Ricceri. *Arch. Math. (Basel)*, 79: 274–287, 2002.
- [7] S. El Manouni A study of nonlinear problems for the p-Laplacian in $\mathbb{R}^N$ via Ricceri's principle. *Nonlinear Anal.* 74, No. 5-6: 4496–4502, 2011.
- [8] S. El Manouni, M. Kbiri Alaoui; A result on elliptic systems with Neumann conditions via Ricceri's three critical points theorem. *Nonlinear Anal.* 71, No. 5-6: 2343–2348, 2009.
- [9] X. L. Fan and Q. H. Zhang, Existence of solutions for p(x)-Laplacian Dirichlet problem, *Nonlinear Anal.* 52 (2003), 1843-1852.
- [10] M. Mihailescu and V. Radulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. Roy. Soc. London Ser. A* 462 (2006), 2625-2641.
- [11] M. Mihailescu and V. Radulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proceedings of the American Mathematical Society*, in press.
- [12] C. O. Alves and M. A. S. Souto; Existence of solutions for a class of problems in $\mathbb{R}^N$ involving the p(x)-Laplacian; *Progress in Nonlinear Differential Equations and Their Applications* 66,(2005), 17-32.
- [13] X. Fan and X. Han, Existence and multiplicity of solutions for p(x)-Laplacian equations in $\mathbb{R}^N$, *Nonlinear Anal.* 59 (2004), 173-188.
- [14] Xianling FAN, Jishen SHEN, Dun ZHAO. Sobolev embedding theorems for spaces $W^{k;p(x)}(\Omega)$ *J. Math. Anal. Appl.*, 2001, 262(2): 749-760.
- [15] Xianling FAN, Qihu ZHANG Existence of solutions for p(x) -Laplacian Dirichlet problems. *Nonlinear Anal.* 2003, 52(8): 1843-1852.
- [16] Xianling FAN, Qihu ZHANG, Dun ZHAO. Eigenvalues of p(x)-Laplacian Dirichlet problem. *J. Math. Anal. Appl.*, 2005, 302(2): 306-317.
- [17] Xianling FAN, Dun ZHAO. On the Spaces $L_p(x)$ and $W^{m;p(x)}$, *J. Math. Anal. Appl.*, 2001, 263(2): 424-446.
- [18] Xianling FAN, Yuanzhang ZHAO, Qihu ZHANG. A strong maximum principle for p(x)-Laplace equations. *Chinese J. Contemp. Math.*, 2003, 24(3): 277-282.
- [19] Maria-Magdalena Boureanu Existence Of Solutions For an elliptic equation involving the p(x)-laplace operateur *Electronic Journal of Differential Equations*, Vol. 2006(2006), No. 97, pp. 110.
- [20] Lao Sen Yu. Nonlinear p-Laplacian problems on unbounded domains. *Proceedings of the A.M.S.*, 115,4 : 1037–1045, 1992.

- [21] B. Ricceri; A three critical points theorem revisited. *Nonlinear Anal.*, 70, 9: 3084–3089, 2009.
- [22] B. Ricceri; A further three critical points theorem. *Nonlinear Anal.*, 71, 9: 4151–4157, 2009.
- [23] B. Ricceri; A note on the Neumann problem. *Complex Var. Elliptic Equ.*, 55, 5-6: 593–599, 2010.
- [24] E. Zeidler; Nonlinear functional analysis and its applications. *Vol. II/B. Berlin, Heidelberg, New York*, 1978.
- [25] A. Ambrosetti and P. H. Rabinowitz; Dual variational methods in critical point theory; *J.Funct. Anal.* 14 (1973); 349-381.
- [26] O. Kovacik and J. Rakosnk, On spaces $L_p(x)$ and $W_{1,p}(x)$, *Czechoslovak Math. J.* 41 (1991), 592-618

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