

Demiclosed principle and Δ – convergence of fixed points for total asymptotically nonexpansive mappings in hyperbolic spaces*

Abstract: In this paper, we prove the existence of fixed points and demiclosed principle for total asymptotically nonexpansive mappings in hyperbolic spaces. As a consequence, we obtain a Δ –convergence theorem for such mappings in hyperbolic spaces. Our results improve and extend some results in the literature.

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Key Words: total asymptotically nonexpansive mappings; hyperbolic spaces; demiclosed principle

1. Introduction

In this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [1]. (X, d, W) is called a hyperbolic space if (X, d) is a metric space and $W : X \times X \times [0, 1] \rightarrow X$ is a function satisfying

$$(I) \quad \forall x, y, z \in X, \quad \forall \lambda \in [0, 1], \quad d(z, W(x, y, \lambda)) \leq (1 - \lambda)d(z, x) + \lambda d(z, y);$$

$$(II) \quad \forall x, y \in X, \quad \forall \lambda_1, \lambda_2 \in [0, 1], \quad d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2| \cdot d(x, y);$$

$$(III) \quad \forall x, y \in X, \quad \forall \lambda \in [0, 1], \quad W(x, y, \lambda) = W(y, x, (1 - \lambda));$$

$$(IV) \quad \forall x, y, z, w \in X, \quad \forall \lambda \in [0, 1], \quad d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1 - \lambda)d(x, y) + \lambda d(z, w).$$

If a metric space satisfies only (I), it coincides with the convex metric space introduced by Takahashi [2]. The concept of hyperbolic space in [1] is more restrictive than the hyperbolic type introduced by Goebel [3] since (I)–(III) are equivalent to (X, d, W) being a space of hyperbolic type in [3]. But it is slightly more general than the hyperbolic

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space defined by Reich [4] (see [1]). This class of metric spaces in [1] covers all normed linear spaces, $\mathbb{R}$ -trees in the sense of Tits, the Hilbert ball with the hyperbolic metric (see [5]), Cartesian products of Hilbert balls, Hadamard manifolds (see [4], [6]) and CAT(0) spaces in the sense of Gromov (see [7]). A thorough discussion of hyperbolic spaces and a detailed treatment of examples can be found in [1] (see also [3]–[5]).

A hyperbolic space X is *uniformly convex* [8] if for $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$ there exists a $\delta \in (0, 1]$ such that

$$d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1 - \delta)r,$$

provided that $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \varepsilon r$.

A map $\eta : (0, \infty) \times (0, 2] \rightarrow (0, 1]$ is called *modulus of uniform convexity* if $\delta = \eta(r, \varepsilon)$ for given $r > 0$. Moreover, η is *monotone* if it decreases with r (for a fixed ε), that is,

$$\eta(r_2, \varepsilon) \leq \eta(r_1, \varepsilon), \quad \forall r_2 \geq r_1 > 0.$$

A subset C of a hyperbolic space X is *convex* if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. For any $x \in X$, $r > 0$, the open (closed) ball with center x and radius r is denoted by $U(x, r)$ (respectively $\bar{U}(x, r)$).

Let (X, d) be a metric space and let C be a nonempty subset of X . Recall that a mapping $T : C \rightarrow C$ is said to be a $(\{\nu_n\}, \{\mu_n\}, \zeta)$ -*total asymptotically nonexpansive mapping* if there exist nonnegative sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \rightarrow 0$, $\mu_n \rightarrow 0$ and a strictly increasing continuous function $\zeta : [0, \infty) \rightarrow [0, \infty)$ with $\zeta(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + \nu_n \zeta(d(x, y)) + \mu_n, \quad \forall n \geq 1, \quad x, y \in C. \quad (1)$$

It is well known that each nonexpansive mapping is an asymptotically nonexpansive mapping and each asymptotically nonexpansive mapping is a $(\{\nu_n\}, \{\mu_n\}, \zeta)$ -total asymptotically nonexpansive mapping.

$T : C \rightarrow C$ is said to be *uniformly L -Lipschitzian* if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq Ld(x, y), \quad \forall n \geq 1, \quad x, y \in C.$$

Recently, Kohlenbach et al [9] proved the existence of fixed points and demiclosed principle for asymptotically nonexpansive mappings in hyperbolic spaces. Later, Zhang et al [10] obtained the existence of fixed points and demiclosed principle for mappings of asymptotically nonexpansive type in hyperbolic spaces. Motivated by the recent work, our purpose of this paper is to discuss the existence of fixed points and demiclosed principle for total asymptotically nonexpansive mappings in hyperbolic spaces.

2. Preliminaries

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X . For $x \in X$, we define

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}.$$

The asymptotic radius $r_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is given by

$$r_C(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

The asymptotic center $A_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is the set

$$A_C(\{x_n\}) = \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}.$$

In 1976, Lim [11] introduced the concept of Δ -convergence in a general metric space. Recall that a sequence $\{x_n\}$ in X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we call x the Δ -limit of $\{x_n\}$.

The following lemmas are important in our paper.

Lemma 1. [9] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then the intersection of any decreasing sequence of nonempty bounded closed convex subsets of X is nonempty.*

Lemma 2. [12] [13] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity and let C be a nonempty closed convex subset of X . Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to C .*

Lemma 3. [12] *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η . Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n \rightarrow \infty} d(x_n, x) \leq c$, $\limsup_{n \rightarrow \infty} d(y_n, x) \leq c$ and $\lim_{n \rightarrow \infty} d(W(x_n, y_n, \alpha_n), x) = c$ for some $c \geq 0$. Then*

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

Lemma 4. [14] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists.

3. Main results

In this section, we prove our main theorems.

Theorem 1. *(Existence of fixed points for total asymptotically nonexpansive mappings in hyperbolic spaces) Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η . Let C be a nonempty bounded closed convex subset of X . Then every continuous total asymptotically nonexpansive mapping $T : C \rightarrow C$ has a fixed point.*

Proof. For any $y \in C$, let

$$B_y := \{b \in \mathbb{R}^+ : \text{there exist } x \in C \text{ and } k \geq 1 \text{ such that } d(T^i y, x) \leq b \text{ for } i \geq k\}.$$

B_y is nonempty since $\text{diam}(C) \in B_y$. Define $\beta_y := \inf B_y$. For any $\theta > 0$, there exists $b_\theta \in B_y$ such that $b_\theta < \beta_y + \theta$. Then there exist $x \in C$ and $k \geq 1$ such that

$$d(T^i y, x) \leq b_\theta < \beta_y + \theta, \quad \forall i \geq k. \quad (2)$$

It is easy to see that $\beta_y \geq 0$. We consider the following two cases:

Case 1. $\beta_y = 0$. Let $\varepsilon > 0$ and apply (2) with $\theta = \frac{\varepsilon}{2}$. Then there exist $x \in C$ and $k \geq 1$ such that for all $i, j \geq k$

$$d(T^i y, T^j y) \leq d(T^i y, x) + d(T^j y, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which implies that $\{T^i y\}$ is a Cauchy sequence. Assume that $T^i y \rightarrow z$ as $i \rightarrow \infty$ for some $z \in C$. By the definition of T , we obtain

$$\begin{aligned} d(z, T^i z) &\leq d(z, T^{2i} y) + d(T^{2i} y, T^i z) \\ &= d(z, T^{2i} y) + d(T^i z, T^i T^i y) \\ &\leq d(z, T^{2i} y) + d(z, T^i y) + \nu_i \zeta(d(z, T^i y)) + \mu_i \rightarrow 0 \text{ as } i \rightarrow \infty. \end{aligned}$$

Thus $T^i z \rightarrow z$ as $i \rightarrow \infty$. By the continuity of T , we get

$$Tz = T\left(\lim_{i \rightarrow \infty} T^i z\right) = \lim_{i \rightarrow \infty} T^{i+1} z = z.$$

Hence, $z \in F(T)$.

Case 2. $\beta_y > 0$. For any $n \geq 1$, let

$$C_n := \bigcup_{k \geq 1} \bigcap_{i \geq k} \overline{U}\left(T^i y, \beta_y + \frac{1}{n}\right), \quad D_n := \overline{C_n} \cap C.$$

Taking $\theta = \frac{1}{n}$ in (2), there exist $x \in C$, $k \geq 1$ such that $x \in \bigcap_{i \geq k} \overline{U}(T^i y, \beta_y + \frac{1}{n})$. Thus $\{D_n\}$ is a decreasing sequence of nonempty bounded closed convex subsets of X . By Lemma 1, we have

$$D := \bigcap_{n \geq 1} D_n \neq \emptyset.$$

For any $x \in D$ and $\theta > 0$, let $N \geq 1$ be such that $\frac{2}{N} \leq \theta$. It follows that $x \in \overline{C_N}$ and there exists a sequence $\{x_n^N\} \subset C_N$ such that $\lim_{n \rightarrow \infty} x_n^N = x$. Let $P \geq 1$ be such that $d(x, x_n^N) \leq \frac{1}{N}$ for all $n \geq P$ and let $K \geq 1$ be such that $x_P^N \in \bigcap_{i \geq K} \overline{U}(T^i y, \beta_y + \frac{1}{N})$. Then for all $i \geq K$, we have

$$d(T^i y, x) \leq d(T^i y, x_P^N) + d(x_P^N, x) \leq \beta_y + \frac{1}{N} + \frac{1}{N} \leq \beta_y + \theta. \quad (3)$$

Now we are in the position to prove that any point of D is a fixed point of T . Let $x \in D$ and assume by contradiction that $Tx \neq x$. Then $\{T^i x\}$ does not converge to x as $i \rightarrow \infty$ and so we can find $\varepsilon > 0$, for any $m_0 \geq 1$, there exists $m \geq m_0$ such that

$$d(T^m x, x) \geq \varepsilon. \quad (4)$$

Without loss of generality, we assume that $\varepsilon \in (0, 2]$. Then $\frac{\varepsilon}{\beta_y + 1} \in (0, 2]$ and there exists $\theta_y \in (0, 1]$ such that

$$1 - \eta \left(\beta_y + 1, \frac{\varepsilon}{\beta_y + 1} \right) \leq \frac{\beta_y - \theta_y}{\beta_y + \theta_y}.$$

Taking $\theta = \frac{\theta_y}{2}$ in (3), there exists $K \geq 1$ such that

$$d(T^i y, x) \leq \beta_y + \frac{\theta_y}{2}, \quad \forall i \geq K. \quad (5)$$

By the definition of T , there exists M_0 such that if $i \geq M_0$, then we have

$$\begin{aligned} d(T^i x, T^i z) &\leq d(x, z) + \nu_i \zeta(d(x, z)) + \mu_i \\ &\leq d(x, z) + \frac{\theta_y}{2}, \quad \forall x, z \in C. \end{aligned} \quad (6)$$

By (4) with $m_0 = M_0$, there exists $M \geq M_0$ such that

$$d(T^M x, x) \geq \varepsilon. \quad (7)$$

Let $i \geq 1$ be such that $i \geq M + K$. It follows from (5), (6) and (7) that

$$d(x, T^i y) \leq \beta_y + \frac{\theta_y}{2} < \beta_y + \theta_y;$$

$$\begin{aligned} d(T^M x, T^i y) &= d(T^M x, T^M T^{i-M} y) \\ &\leq d(x, T^{i-M} y) + \frac{\theta_y}{2} \\ &\leq \beta_y + \theta_y; \end{aligned}$$

$$d(T^M x, x) \geq \varepsilon = \frac{\varepsilon}{\beta_y + \theta_y} \cdot (\beta_y + \theta_y) \geq \frac{\varepsilon}{\beta_y + 1} \cdot (\beta_y + \theta_y).$$

It follows from X is uniformly convex and η is monotone that

$$\begin{aligned} d(W(x, T^M x, \frac{1}{2}), T^i y) &\leq \left[1 - \eta \left(\beta_y + \theta_y, \frac{\varepsilon}{\beta_y + 1} \right) \right] (\beta_y + \theta_y) \\ &\leq \left[1 - \eta \left(\beta_y + 1, \frac{\varepsilon}{\beta_y + 1} \right) \right] (\beta_y + \theta_y) \\ &\leq \frac{\beta_y - \theta_y}{\beta_y + \theta_y} \cdot (\beta_y + \theta_y) \\ &= \beta_y - \theta_y. \end{aligned}$$

Hence, there exist $k := M + K$ and $z := W(x, T^M x, \frac{1}{2}) \in C$ such that for all $i \geq k$, $d(z, T^i y) \leq \beta_y - \theta_y$. It implies that $\beta_y - \theta_y \in B_y$, which contradicts with $\beta_y = \inf B_y$. Thus, $x \in F(T)$. $\square$

It is well known that one of the fundamental and celebrated results in the theory of nonexpansive mappings is Browder's *demiclosed principle* [15] which states that X is a uniformly convex Banach space, C is a nonempty closed convex subset of X , and $T : C \rightarrow X$ is a nonexpansive mapping, then $I - T$ is demiclosed at 0, i.e., for any sequence $\{x_n\}$ in C if $x_n \rightarrow x$ weakly and $\|(I - T)x_n\| \rightarrow 0$, then $x = Tx$. In the following, we shall prove that a total asymptotically nonexpansive mapping in a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity is demiclosed. Let X be a hyperbolic space and let C be a nonempty closed convex subset of X . Let $\{x_n\}$ be a bounded sequence in C . In what follows, we denote it by

$$\{x_n\} \rightharpoonup \omega \text{ if and only if } \Phi(\omega) = \inf_{x \in C} \Phi(x),$$

where $\Phi(x) := \limsup_{n \rightarrow \infty} d(x_n, x)$.

Theorem 2. (*Demiclosed principle for total asymptotically nonexpansive mappings in hyperbolic spaces*) Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η . Let C be a nonempty closed and convex subset of X . Let $T : C \rightarrow C$ be a uniformly L -Lipschitzian and $(\{\mu_n\}, \{\nu_n\}, \zeta)$ -total asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ and $\{x_n\} \rightharpoonup p$. Then we have $T(p) = p$.

Proof. Since $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$, by induction we can prove that

$$\lim_{n \rightarrow \infty} d(x_n, T^m x_n) = 0 \text{ for each } m \geq 1. \quad (8)$$

In fact, it is obvious that, the conclusion is true for $m = 1$. Suppose the conclusion holds for $m \geq 1$, now we prove that it is also true for $m + 1$. Indeed, since T is uniformly L -Lipschitzian, we have

$$\begin{aligned} d(x_n, T^{m+1} x_n) &\leq d(x_n, T^m x_n) + d(T^m x_n, T^{m+1} x_n) \\ &\leq d(x_n, T^m x_n) + Ld(x_n, Tx_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus (8) is proved. Now for each $x \in C$ and $m \geq 1$, from (8) we have

$$\Phi(x) := \limsup_{n \rightarrow \infty} d(x_n, x) = \limsup_{n \rightarrow \infty} d(T^m x_n, x). \quad (9)$$

In (9), taking $x = T^m p$, we get

$$\begin{aligned} \Phi(T^m p) &= \limsup_{n \rightarrow \infty} d(T^m x_n, T^m p) \\ &\leq \limsup_{n \rightarrow \infty} [d(x_n, p) + \nu_m \zeta(d(x_n, p)) + \mu_m]. \end{aligned}$$

Letting $m \rightarrow \infty$ and taking superior limit on the both sides, we have

$$\limsup_{m \rightarrow \infty} \Phi(T^m p) \leq \Phi(p). \quad (10)$$

We assume by contradiction that $Tp \neq p$. Then $\{T^m p\}$ does not converge to p as $m \rightarrow \infty$, so we can find $\varepsilon_0 > 0$, for any $k \geq 1$, there exists $m \geq k$ such that $d(T^m p, p) \geq \varepsilon_0$. We can assume $\varepsilon_0 \in (0, 2]$. Then $\frac{\varepsilon_0}{\Phi(p)+1} \in (0, 2]$ and there exists $\theta \in (0, 1]$ such that

$$1 - \eta \left(\Phi(p) + 1, \frac{\varepsilon_0}{\Phi(p) + 1} \right) \leq \frac{\Phi(p) - \theta}{\Phi(p) + \theta}. \quad (11)$$

By the definition of Φ and (10), there exist $N_1, M_1 \geq 1$ such that

$$\begin{aligned} d(p, x_n) &\leq \Phi(p) + \theta, \quad \forall n \geq N_1; \\ d(T^m p, x_n) &\leq \Phi(p) + \theta, \quad \forall n \geq N_1, \quad m \geq M_1. \end{aligned}$$

Besides, there exists $m \geq M_1$ such that

$$d(T^m p, p) \geq \varepsilon_0 = \frac{\varepsilon_0}{\Phi(p) + \theta} \cdot (\Phi(p) + \theta) \geq \frac{\varepsilon_0}{\Phi(p) + 1} \cdot (\Phi(p) + \theta).$$

Since X is uniformly convex and η is monotone, by (11) we get

$$\begin{aligned} d(W(p, T^m p, \frac{1}{2}), x_n) &\leq \left[1 - \eta \left(\Phi(p) + \theta, \frac{\varepsilon_0}{\Phi(p) + 1} \right) \right] \cdot (\Phi(p) + \theta) \\ &\leq \frac{\Phi(p) - \theta}{\Phi(p) + \theta} \cdot (\Phi(p) + \theta) \\ &= \Phi(p) - \theta. \end{aligned}$$

Hence $z := W(p, T^m p, \frac{1}{2}) \in C$ and $z \neq p$, which contradicts $\Phi(p) = \inf_{x \in C} \Phi(x)$. Thus $Tp = p$. $\square$

Theorem 3. Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η . Let $T_i : C \rightarrow C$, $i = 1, 2$, be uniformly L -Lipschitzian and $(\{\nu_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$ -total asymptotically nonexpansive mappings. Suppose that $F(T_1) \cap F(T_2) \neq \emptyset$. For arbitrarily chosen $x_1 \in C$, $\{x_n\}$ is defined as follows

$$\begin{cases} x_{n+1} = W(x_n, T_1^n y_n, \alpha_n), \\ y_n = W(x_n, T_2^n x_n, \beta_n), \end{cases} \quad (12)$$

where the following conditions are satisfied:

- (i) $\sum_{n=1}^{\infty} \nu_n^{(i)} < \infty$ and $\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$, $i = 1, 2$;
- (ii) there exist constants $a, b \in (0, 1)$ such that $\{\alpha_n\} \subset [a, b]$;
- (iii) there exists a constant $M^* > 0$ such that $\zeta^{(i)}(r) \leq M^*r$, $r \geq 0, i = 1, 2$.

Then the sequence $\{x_n\}$ defined by (12) $\triangle$ -converges to a common fixed point of T_1 and T_2 .

Proof. Without loss of generality, we can assume that $T_i : C \rightarrow C$ both are $(\{\nu_n\}, \{\mu_n\}, \zeta)$ -total asymptotically nonexpansive mappings, where $\nu_n = \max\{\nu_n^{(i)}, i = 1, 2\}$, $\mu_n = \max\{\mu_n^{(i)}, i = 1, 2\}$ and $\zeta = \max\{\zeta^{(i)}, i = 1, 2\}$. It is easy to see that conditions (i) and (iii) are still satisfied. Now we divide our proof into three steps.

Step 1. In the sequel, we shall show that

$$\lim_{n \rightarrow \infty} d(x_n, p) \text{ exists for each } p \in F(T_1) \cap F(T_2). \quad (13)$$

In fact, by conditions (1), (I) and (iii), one gets

$$\begin{aligned} d(y_n, p) &= d(W(x_n, T_2^n x_n, \beta_n), p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(T_2^n x_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n [d(x_n, p) + \nu_n \zeta(d(x_n, p)) + \mu_n] \\ &\leq (1 + \beta_n \nu_n M^*)d(x_n, p) + \beta_n \mu_n \end{aligned} \quad (14)$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, T_1^n y_n, \alpha_n), p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(T_1^n y_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n [d(y_n, p) + \nu_n \zeta(d(y_n, p)) + \mu_n] \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n [(1 + \nu_n M^*)d(y_n, p) + \mu_n]. \end{aligned} \quad (15)$$

Combining (14) and (15), we have

$$d(x_{n+1}, p) \leq (1 + \sigma_n)d(x_n, p) + \xi_n, \quad \forall n \geq 1, \quad (16)$$

where $\sigma_n = \alpha_n \nu_n M^* (1 + \beta_n + \beta_n \nu_n M^*)$ and $\xi_n = \alpha_n \mu_n (1 + \beta_n + \beta_n \nu_n M^*)$. Furthermore, using the condition (i), we get

$$\sum_{n=1}^{\infty} \nu_n < \infty \text{ and } \sum_{n=1}^{\infty} \mu_n < \infty, \quad (17)$$

a combination of (16), (17) and Lemma 4 shows that (13) is proved.

Step 2. We claim that

$$\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0, \quad i = 1, 2. \quad (18)$$

In fact, it follows from (13) that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each given $p \in F(T_1) \cap F(T_2)$. Without loss of generality, we assume that

$$\lim_{n \rightarrow \infty} d(x_n, p) = c \geq 0. \quad (19)$$

By (14) and (19), one has

$$\liminf_{n \rightarrow \infty} d(y_n, p) \leq \limsup_{n \rightarrow \infty} d(y_n, p) \leq \lim_{n \rightarrow \infty} [(1 + \beta_n \nu_n M^*)d(x_n, p) + \beta_n \mu_n] = c. \quad (20)$$

Noting

$$\begin{aligned} d(T_1^n y_n, p) &= d(T_1^n y_n, T_1^n p) \\ &\leq d(y_n, p) + \nu_n \zeta(d(y_n, p)) + \mu_n \\ &\leq (1 + \nu_n M^*)d(y_n, p) + \mu_n, \quad \forall n \geq 1, \end{aligned}$$

by (20) we obtain

$$\limsup_{n \rightarrow \infty} d(T_1^n y_n, p) \leq c. \quad (21)$$

Besides, by (16) we get

$$d(x_{n+1}, p) = d(W(x_n, T_1^n y_n, \alpha_n), p) \leq (1 + \sigma_n)d(x_n, p) + \xi_n,$$

which yields that

$$\lim_{n \rightarrow \infty} d(W(x_n, T_1^n y_n, \alpha_n), p) = c. \quad (22)$$

Now by (19), (21), (22) and Lemma 3, we have

$$\lim_{n \rightarrow \infty} d(x_n, T_1^n y_n) = 0. \quad (23)$$

On the other hand, we have

$$\begin{aligned} d(x_n, p) &\leq d(x_n, T_1^n y_n) + d(T_1^n y_n, p) \\ &\leq d(x_n, T_1^n y_n) + d(y_n, p) + \nu_n M^* d(y_n, p) + \mu_n \\ &= d(x_n, T_1^n y_n) + (1 + \nu_n M^*)d(y_n, p) + \mu_n, \end{aligned}$$

which implies that $\liminf_{n \rightarrow \infty} d(y_n, p) \geq c$. Combining with (20), it yields that

$$\lim_{n \rightarrow \infty} d(y_n, p) = c,$$

that is,

$$\lim_{n \rightarrow \infty} d(W(x_n, T_2^n x_n, \beta_n), p) = c.$$

By Lemma 3 we can also have that

$$\lim_{n \rightarrow \infty} d(x_n, T_2^n x_n) = 0. \quad (24)$$

By virtue of (24), we have

$$\begin{aligned} d(y_n, x_n) &= d(W(x_n, T_2^n x_n, \beta_n), x_n) \\ &\leq \beta_n d(T_2^n x_n, x_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (25)$$

Combining (23) and (25), one obtains

$$\begin{aligned}
 d(x_n, T_1^n x_n) &\leq d(x_n, T_1^n y_n) + d(T_1^n y_n, T_1^n x_n) \\
 &\leq d(x_n, T_1^n y_n) + d(x_n, y_n) + \nu_n \zeta(d(x_n, y_n)) + \mu_n \\
 &\leq d(x_n, T_1^n y_n) + (1 + \nu_n M^*)d(x_n, y_n) \\
 &\quad + \mu_n \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned} \tag{26}$$

This jointly with (23) yields that

$$\begin{aligned}
 d(x_{n+1}, x_n) &= d(W(x_n, T_1^n y_n, \alpha_n), x_n) \\
 &\leq \alpha_n d(T_1^n y_n, x_n) \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned} \tag{27}$$

Now by (24), (26) and (27), for each $i = 1, 2$, we get

$$\begin{aligned}
 d(x_n, T_i x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1} x_{n+1}) + d(T_i^{n+1} x_{n+1}, T_i^{n+1} x_n) \\
 &\quad + d(T_i^{n+1} x_n, T_i x_n) \\
 &\leq (1 + L)d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1} x_{n+1}) \\
 &\quad + Ld(T_i^n x_n, x_n) \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Therefore, (18) is proved.

Step 3. Now we are in a position to prove the Δ -convergence of $\{x_n\}$. Since $\{x_n\}$ is bounded, by Lemma 2, it has a unique asymptotic center $A_C(\{x_n\}) = \{x^*\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ with $A_C(\{u_n\}) = \{u\}$. Since $\lim_{n \rightarrow \infty} d(x_n, T_1 x_n) = \lim_{n \rightarrow \infty} d(x_n, T_2 x_n) = 0$, it follow from Theorem 2 that $u \in F(T_1) \cap F(T_2)$. By the uniqueness of asymptotic centers, we get that $x^* = u$. It implies that x^* is the unique asymptotic center of $\{u_n\}$ for each subsequence $\{u_n\}$ of $\{x_n\}$, that is, $\{x_n\}$ Δ -converges to $x^* \in F(T_1) \cap F(T_2)$. The proof is completed. $\square$

Example 1. Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $C = [-1, 1]$. Define two mappings $T_1, T_2 : C \rightarrow C$ by

$$T_1(x) = \begin{cases} -2 \sin \frac{x}{2}, & x \in [0, 1], \\ 2 \sin \frac{x}{2}, & x \in [-1, 0], \end{cases}$$

and

$$T_2(x) = \begin{cases} x, & x \in [0, 1], \\ -x, & x \in [-1, 0]. \end{cases}$$

It is proved in [16, Example 3.1] that both T_1 and T_2 are asymptotically nonexpansive mappings with $k_n = 1$, $\forall n \geq 1$. Therefore, they are total asymptotically nonexpansive mappings with $\nu_n = \mu_n = 0$, $\forall n \geq 1$, $\zeta(t) = t$, $\forall t \geq 0$. Moreover, they are uniformly L -Lipschitzian mappings with $L = 1$. $F(T_1) = \{0\}$ and $F(T_2) = \{x \in C : 0 \leq x \leq 1\}$. Let

$$\alpha_n = \frac{n}{2n+1}, \beta_n = \frac{n}{3n+1} \quad \forall n \geq 1.$$

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