

The Optimal homotopy analysis methods for solving the linear and non-linear Fokker-Planck equations

Original Article

Abstract

In this paper, linear and non-linear Fokker-Planck equations are solved by approximate method, namely Optimal homotopy analysis method (OHAM). The Optimal homotopy analysis method is a combination of the homotopy analysis method and optimizing the convergence control parameter by minimizing the square residual error. The mathematical calculation and graphics have been obtained by MATHEMATICA software. The comparisons are made between the results of OHAM with the exact results and other approximate method like, Adomian decomposition method, Variation iteration method (VIM) and Homotopy perturbation method (HPM).

Keywords: Optimal homotopy analysis method; Convergence control parameter; Fokker-Planck equations
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1 Introduction

Non-linear phenomena, that appear in many areas of scientific fields such as solid state physics, plasma physics, fluid mechanics, population models and chemical kinetics, can be modelled by non-linear differential equations. In many different fields of science and engineering, it is important to obtain exact or numerical solution of the non-linear partial differential equations, but it is still quite problematic that's need new methods for finding the approximate solutions. The Homotopy analysis method (HAM)[Abbasbandy and Shivanian (2011); Lan (2007); Liao (2012)] is based on the concept of Homotopy, a fundamental concept in topology and differential geometry, which was first time introduced by Shijun Liao in his PhD thesis (1992). The HAM is a general analytic approach to get series solutions of various types of non-linear equations, including algebraic equations, ordinary differential equations, partial differential equations, differential-integral equations, differential-difference equation, and coupled equations of them. Researches have introduce many other methods based on HAM like, Homotopy perturbation method (HPM)[Biazar et. al. (2008); Chowdhury et. al. (2009);

Lan (2007)], Optimal Homotopy analysis method (OHAM)[Liao (2012, 2010)] Optimal homotopy asymptotic method and Spectral homotopy analysis method to get the approximate solution of the non-linear PDE.

Inspired and motivated by the ongoing research in this area, we apply approximate method, namely Optimal Homotopy analysis method (OHAM) for solving the non-linear Fokker plank equation arising in the fluid flow through porous media and compare with other approximate methods like, Adomian decomposition method [Adomian (1994); Tatari (2007)], Variation iteration method (VIM)[Sadighi et. al. (2007)] and Homotopy perturbation method (HPM)[Biazar et. al. (2008)].

2 Fokker-Planck equation

Using a microscope, Robert Brown (1773-1858) observed and documented the motion of large pollen grains suspended in water known as Brownian motion. The Fokker-Planck equation was first introduced by Fokker and Planck to describe the Brownian motion of particles [Risken (1996)]. A general Fokker-Planck equation can be derived from the Chapman-Kolmogorov equation. This equation has been used in different fields in natural sciences such as quantum optics, solid state physics, chemical physics, theoretical biology, circuit theory and fluid flow through porous media. The general form of Fokker-Planck equation is

$$\frac{\partial U}{\partial t} = \left[-\frac{\partial}{\partial x} A(x, t) + \frac{\partial^2}{\partial x^2} B(x, t) \right] U, \quad (2.1)$$

with initial condition $U(x, 0) = f(x)$; $x \in \mathbb{R}$. Equation (2.1) is also well known as a forward Kolmogorov equation. There exists another type of this equation is called a backward one as

$$\frac{\partial U}{\partial t} = \left[-A(x, t) \frac{\partial}{\partial x} + B(x, t) \frac{\partial^2}{\partial x^2} \right] U. \quad (2.2)$$

The non-linear Fokker-Planck equation is a more general form of linear one which has also been applied in plasma physics, surface physics, astrophysics, the physics of polymer fluids and particle beams, non-linear hydrodynamics, population dynamics, theory of electronic-circuitry and laser arrays, engineering, biophysics, human movement sciences, psychology and marketing. The non-linear form of the Fokker-Planck equation can be expressed in the following way

$$\frac{\partial U}{\partial t} = \left[-\frac{\partial}{\partial x} A(x, t, U) + \frac{\partial^2}{\partial x^2} B(x, t, U) \right] U. \quad (2.3)$$

3 Basic idea of OHAM

Consider the following non-linear partial differential equation in a general form

$$\mathcal{N}[U(x, t)] = 0, \quad (3.1)$$

where, $\mathcal{N}$ is a non-linear operator, x and t denote the independent variables and U is an unknown function. For simplicity, we ignore all boundary or initial conditions. By means of the traditional HAM, we first construct the so-called zero-order deformation equation as

$$(1 - q) \mathcal{L}[\phi(x, t; q) - U_0(x, t)] = c_0 q \mathcal{H}(x, t) \mathcal{L}[\phi(x, t; q)], \quad (3.2)$$

where, $q \in [0, 1]$ is the embedding parameter, c_0 is an auxiliary parameter also know as convergence control parameter, $\mathcal{L}$ is an auxiliary linear operator, $U(x, t; q)$ is an unknown function, $U_0(x, t)$ is an initial guess of $U(x, t)$ and $\mathcal{H}(x, t)$ denote a non zero auxiliary function. According to HAM

we have great freedom to choose linear operator, auxiliary function and initial guess. Obviously, when the embedding parameter $q = 0$ and $q = 1$, it holds $\phi(x, t; 0) = U_0(x, t)$, $\phi(x, t; 1) = U(x, t)$ respectively. Thus, as q increases from 0 to 1, the solution $\phi(x, t; q)$ varies from the initial guess $U_0(x, t)$ to the solution $U(x, t)$. Expanding $\phi(x, t; q)$ in Maclaurin series with respect to q , we have

$$\phi(x, t; q) = U_0(x, t) + \sum_{m=1}^{\infty} U_m(x, t) q^m, \quad (3.3)$$

where

$$U_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, t; q)}{\partial q^m} \right|_{q=0}.$$

If the auxiliary linear operator, the initial guess, the convergence control parameter c_0 and the auxiliary function are properly chosen, then the series (3.3) converges at $q = 1$, then we have

$$U(x, t) = U_0(x, t) + \sum_{m=1}^{\infty} U_m(x, t) q^m, \quad (3.4)$$

which must be one of solutions of the original non-linear equation, as proved by Liao [Liao (2012)].

Define the vectors

$$U_m = \{U_0(x, t), U_1(x, t), \dots, U_m(x, t)\} \quad (3.5)$$

Differentiating the equation (3.2) m -times with respect to q , dividing by $m!$ and finally setting $q = 0$, we get the following m th-order deformation equation

$$\mathcal{L}[U(x, t) - \chi_m U_{m-1}(x, t)] = c_0 \mathcal{H}(x, t) \delta_m[U_{m-1}(x, t)], \quad (3.6)$$

where

$$\delta_m[U_{m-1}(x, t)] = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[\phi(x, t; q)]}{\partial q^{m-1}} \right|_{q=0}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

The Convergence control parameter c_0 play an important role in the OHAM. One can gain convergent series solution simply by choosing a proper auxiliary parameter c_0 . This is the reason why we call c_0 as the convergence-control parameter. The range of convergence R_c approximately determines by plotting curves $U'(0) \sim c_0$, $U''(0) \sim c_0$ and so on known as ' c_0 -curve'. However, such kind of c_0 -curves cannot tell us the best value of convergence-control parameter c_0 , which corresponds to the fastest convergent series. In 2007, Yabushita et al. [Yabushita et. al. (2007)] applied the HAM to solve two coupled non-linear ODEs. They suggested the so-called "optimization method" to find out the optimal convergence-control parameters by means of the minimum of the squared residual of governing equation as follow

$$E_m(c_0) = \iint_{\Omega} \left[\mathcal{N} \left\{ \sum_{n=0}^m U_n(x, t) \right\} \right]^2 d\Omega. \quad (3.7)$$

In real world problem the double integration of sum of square residual is very difficult, so we use it's approximate sum form as

$$E_m(c_0) = \frac{1}{(N+1)(M+1)} \sum_{i=0}^N \sum_{j=0}^M \left\{ \mathcal{N} \left[\sum_{n=0}^m H_n \left(\frac{i}{N}, \frac{j}{M} \right) \right]^2 \right\}. \quad (3.8)$$

Equation(3.8) gives the square residual error at m th-order approximation. It is obvious that we can find the optimal value of convergence-control parameter c_0 at the any order of approximation. If there exists convergence-control parameter, c_0 for which we get the minimum of the squared residua E_m , is so called the optimal convergence-control parameter c_0 .

4 Applications

We consider linear and non-linear Fokker-Planck equations, which arises in many problem of fluid flow through porous media such as, Pressure head in unsaturated soil during infiltration phenomenon, Boussinesq's equation for infiltration phenomenon in unsaturated porous media and Advection-diffusion equation for concentration distribution in fluid flow through porous media.

4.1 Example of Linear Fokker-Planck equation

Consider the following linear Fokker-Planck equation

$$U_t = U_x + U_{xx}, \quad (4.1)$$

subject to the initial condition

$$U(x, 0) = x. \quad (4.2)$$

According to the OHAM, we choose the initial guess as, $U_0(x, t) = x$. And the auxiliary linear operator

$$\mathcal{L}[U(x, t; q)] = \frac{\partial U(x, t; q)}{\partial t} - \frac{\partial U(x, t; q)}{\partial x}, \quad (4.3)$$

which posses the property $\mathcal{L}(c_1) = 0$; c_1 is an integral constant to be determine by initial condition (4.2). Also the equation (4.1) suggest to define the non-linear operator

$$\mathcal{N}[\phi(x, t; q)] = \frac{\partial \phi(x, t; q)}{\partial t} - \frac{\partial \phi(x, t; q)}{\partial x} - \frac{\partial^2 \phi(x, t; q)}{\partial x^2}.$$

The m th-order deformation equation is given by

$$\mathcal{L}[U(x, t) - \chi_m U_{m-1}(x, t)] = c_0 \mathcal{H}(x, t) \delta_m[U_{m-1}(x, t)], \quad (4.4)$$

where

$$\delta_m(U_{m-1}) = (U_{m-1})_t - (U_{m-1})_x - (U_{m-1})_{xx}.$$

And we choose $\mathcal{H}(x, t) = 1$, Applying the inverse operator on equation (4.4), we have

$$U(x, t) = \chi_m U_{m-1}(x, t) + c_0 \mathcal{L}^{-1}[\delta_m[U_{m-1}(x, t)]] \quad (4.5)$$

Solving equation (4.5), for $m = 1, 2, 3, \dots$ We obtain

$$\begin{aligned} U_1(x, t) &= -c_0 t, \\ U_2(x, t) &= -c_0(1 + c_0)t, \\ U_3(x, t) &= -c_0(1 + c_0)^2 t, \\ &\dots \end{aligned} \quad (4.6)$$

By using Yabushita's approach we can find the optimal value of convergence control parameter $c_0 = -1.0000249174897184$ at Minimum Square residual error $E_5 = 6.51E - 35$ (Figure 1).

By using this optimal value of c_0 we got the solution of (4.1) as

$$U(x, t) = \sum_{m=1}^{\infty} U_m(x, t) = x + t, \quad (4.7)$$

which is the exact solution and is same as obtained by ADM [Tatari (2007)], VIM [Sadighi et. al. (2007)] and HPM [Biazar et. al. (2008)].

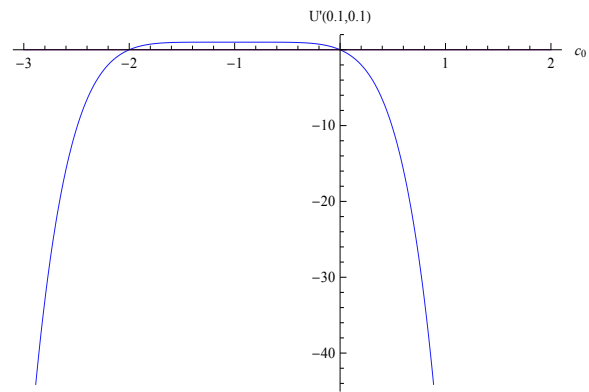


Figure 1: c_0 curve of $U'(x, t)$

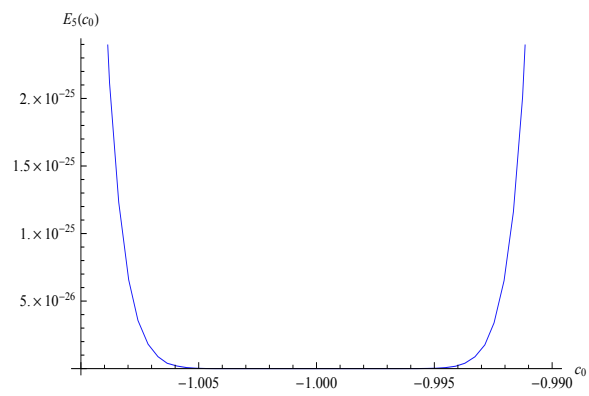


Figure 2: Square residual E_5 at 5th-order approximation

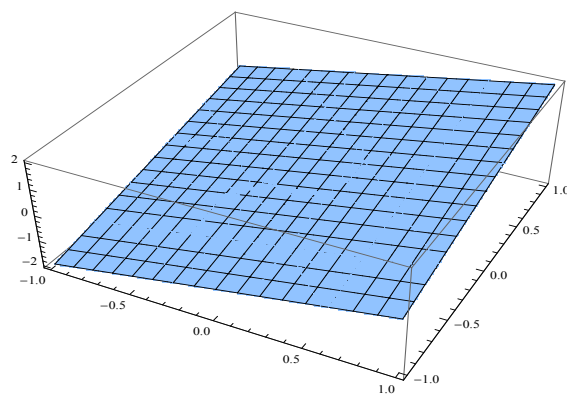


Figure 3: Numerical result for $U(x, t)$ by OHAM up to 5-term

4.2 Example of Non-linear Fokker-Planck equation

Consider the following non-linear Fokker-Planck equation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left(\frac{xU}{3} - \frac{4}{x} U^2 \right) + \frac{\partial^2}{\partial x^2} (U^2), \quad (4.8)$$

with the initial condition

$$U(x, 0) = x^2. \quad (4.9)$$

According to the OHAM, we choose the initial guess as, $U_0(x, t) = x^2$. and the auxiliary linear operator

$$\mathcal{L}[U(x, t; q)] = \frac{\partial U(x, t; q)}{\partial t}, \quad (4.10)$$

with the property that $\mathcal{L}(c_1) = 0$; c_1 is an integral constant to be determine by initial condition(4.9). Also define the non-linear operator using equation (4.8)

$$\mathcal{N}[\phi(x, t; q)] = \frac{\partial \phi(x, t; q)}{\partial t} - \frac{\partial}{\partial x} \left(\frac{x\phi(x, t; q)}{3} - \frac{4}{x} \phi^2(x, t; q) \right) - \frac{\partial^2 (\phi^2(x, t; q))}{\partial x^2}.$$

The m th-order deformation equation is given by

$$\mathcal{L}[U(x, t) - \chi_m U_{m-1}(x, t)] = c_0 \mathcal{H}(x, t) \delta_m [U_{m-1}(x, t)], \quad (4.11)$$

where

$$\delta_m(U_{m-1}) = (U_{m-1})_t - \left(\frac{x}{3} U_{m-1} - \frac{4}{x} \sum_{r=0}^{m-1} U_r U_{m-1-r} \right)_x - \left(\sum_{r=0}^{m-1} U_r U_{m-1-r} \right)_{xx}.$$

And we choose $\mathcal{H}(x, t) = 1$, Applying the inverse operator on equation (4.11), we have

$$U(x, t) = \chi_m U_{m-1}(x, t) + c_0 \mathcal{L}^{-1} [\delta_m [U_{m-1}(x, t)]] \quad (4.12)$$

Solving equation (4.11), for $m = 1, 2, 3, \dots$, we get

$$\begin{aligned} U_1(x, t) &= -c_0 x^2 t, \\ U_2(x, t) &= -c_0 (1 + c_0) x^2 t + \frac{c_0^2 x^2 t^2}{2}, \\ U_3(x, t) &= -c_0 (1 + c_0)^2 x^2 t + \frac{c_0^2 (1 + c_0) x^2 t^2}{2} - \frac{c_0^3 x^2 t^3}{6}, \\ &\dots \end{aligned} \quad (4.13)$$

From figure 4 we can observe that in the range $R_c = [-1.5, 0.1]$ of convergence control parameter the series solution is convergent. Using Yabushita's approach we can find the optimal value of convergence control parameter $c_0 = -1.0779113454061617$ at minimum square residual error $E_5 = 4.73E - 10$ By using this optimal value of c_0 we got the solution of (4.8) as

$$U(x, t) = \sum_{m=1}^{\infty} U_m(x, t) = e^t x^2, \quad (4.14)$$

which is the exact solution and is same as obtained by ADM [Tatari (2007)], VIM [Sadighi et. al. (2007)] and HPM [Biazar et. al. (2008)].

5 Conclusion

The results have been obtained by using the OHAM which are presented here agree well with the results obtained by ADM, VIM and HPM. The advantage of OHAM is that we can rapidly converges

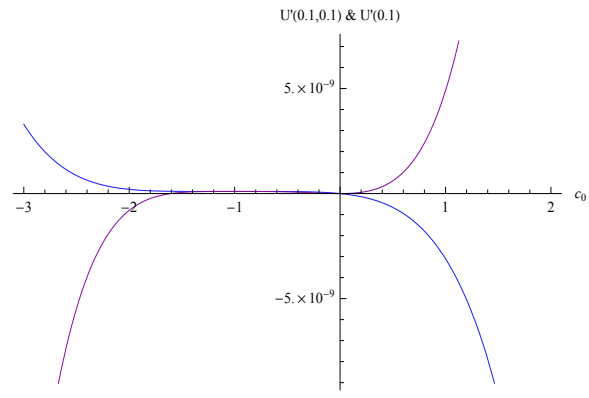


Figure 4: c_0 curves of $U'(x, t)$ & $U''(x, t)$

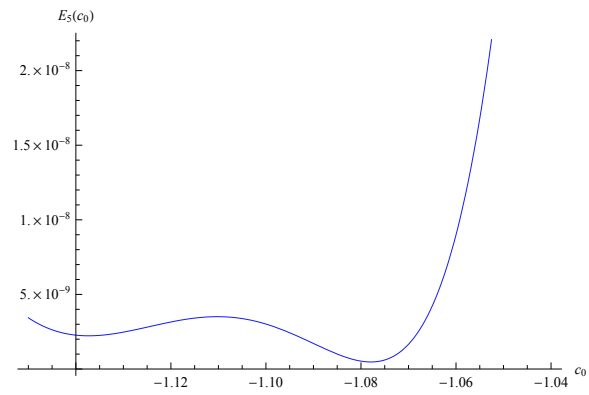


Figure 5: Square residual E_5 at 5th-order approximation

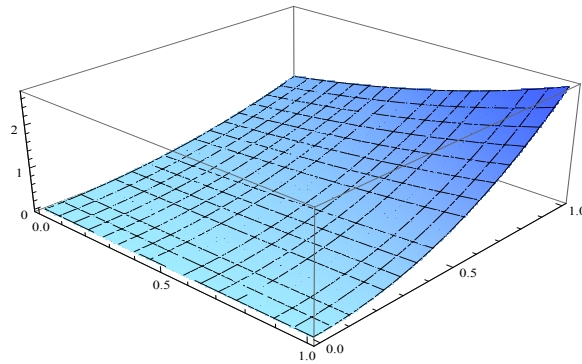


Figure 6: Numerical result for $U(x, t)$ by OHAM up to 5-term

the series solution by taking optimal value of c_0 , which is not possible in other analytic methods like ADM and VIM. Finally, we have concluded that the OHAM is very powerful and efficient analytical approximation method to solve such types of linear and non-linear partial differential equations arise in many problems of fluid flow through porous media and oil recovery processes.

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